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Partitioned Hermitian Matrices*

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A class of Cauchy-Schwarz type inequalities for partitioned hermitian matrices is presented.

Key words: Generalized matrix function; positive semi-definite hermitian matrix.

Let M_n^r denote the set of $r\times n$ matrices over the complex numbers. Write M_n for M_n^n. Let G be a subgroup of S_{ml}, the symmetric group on ml symbols. Suppose λ is a character of degree 1 on G. If $X=(x_{ij})\epsilon M_{ml}$, the *generalized matrix function* of X is

$$d(X)=\sum_{\sigma\epsilon G}\lambda(\sigma)\prod_{t=1}^{ml}x_{t\sigma(t)}.$$

Let $f:M_p\to M_m$ be any function. Then f induces a function $\varphi_f:M_{pq}\to M_{mq}$ as follows:

$$\varphi_f(X)=(f(X_{st}))$$

where $X=(X_{st})$ is a block matrix in which X_{st} is a $p\times p$ submatrix of X, $1\leq s,\ t\leq q$.

Let $A_1, \ldots, A_k \epsilon M_{pl}^r$. Let H be the block matrix

$$H=(A_i^*A_j)\epsilon M_{pkl}.$$

(A_i^* is the conjugate transpose of A_i.)

LEMMA. *The matrix* H *is positive semi-definite hermitian* (psdh).

PROOF:

$$H=\begin{bmatrix} A_1^* & & 0 \\ & \ddots & \\ 0 & & A_k^* \end{bmatrix}\begin{bmatrix} I_r & \ldots & I_r \\ \vdots & & \vdots \\ I_r & \ldots & I_r \end{bmatrix}\begin{bmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_k \end{bmatrix},$$

where $I_r\epsilon M_r$ is the identity matrix.

(Observe that in general the block matrix whose i, jth block is $A_j^*A_i$ is not *psdh*. Take

$$A_1=\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},\ A_2=I_2.)$$

Suppose f is such that $\varphi_f(H)$ is *psdh*. We may write

$$\varphi_f(H)=(\varphi_f(A_i^*A_j)).$$

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Now, by a trivial consequence of a result of Marcus and Katz [4, Theorem 3],[1] the block matrix

$$(d(\varphi_f(A_i^*A_j)))$$

is *psdh*. Taking the determinant of the leading 2×2 principal submatrix, one obtains the main result:

THEOREM.

$$d\varphi_f(A_1^*A_1)\; d\varphi_f(A_2^*A_2) \geqslant d\varphi_f(A_1^*A_2)\; d\varphi_f(A_2^*A_1).$$

One easily sees that a string of further inequalities is available by taking any generalized matrix function of any of the leading principal submatrices.

Some functions, f, which have the property that φ_f sends psdh matrices to psdh matrices have been discovered. A partial list follows. Let $X \epsilon M_p$.

1. Let $f(X)=X$. Then the theorem reduces to the well known result ([9, p. 168] or [5, p. 323])

$$d(A^*A)d(B^*B) \geqslant |d(A^*B)|^2.$$

When $p=l=1$, this is the Cauchy-Schwarz inequality.

2. If $p=1$, let $f(x)=x^r$, where r is a positive integer. Then Schur proved f has the required property [8]. Löwner [2] extended this to cover the case for any real number at least 1.

3. If $p=1$, let $f(x)=|x|^2$ [3].

4. Let S be a subgroup of S_p and χ a character on S, let

$$f(X)=\sum_{\sigma \epsilon S} \chi(\sigma) \sum_{t=1}^{p} x_{t\sigma(t)} \qquad [7].$$

5. For $l=1$, $k=2$, let $f(X)=\text{trace }(X^2)$, [6].

6. For a given symmetry class of tensors arising from a group and a character of degree 1, let $f(X)=K(X)$ be the associated matrix, or let $f(X)$ be any generalized matrix function of X [4, Theorem 3]. (For a discussion of these terms, see [9], [6], [10].)

7. Let $\alpha_1^2(X), \ldots, \alpha_p^2(X)$ be the squares of the singular values of X. Let

$$f(X)=\text{trace } K(\text{diag }(\alpha_1^2(X), \ldots, \alpha_p^2(X)))$$

be the Schur function of the squares of the singular values of the matrix X [6]. We may take

$$f(X)=E_s(\lambda_1(X), \ldots, \lambda_p(X))$$

to be the sth elementary symmetric function of the eigenvalues of X [1].

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References

[1] de Pillis, J., Transformations on partioned matrices, Duke Math. J. **36,** 511–515 (1969).
[2] Löwner, Charles, Some Theorems on Positive Matrices (unpublished).
[3] Marcus, Marvin, and Khan, Nisar A., A note on the Hadamard product, Canad. Math. Bull. **2,** 81–83 (1959).
[4] Marcus, Marvin, and Katz, Susan M., Matrices of Schur functions, Duke Math. J. **36,** 343–352 (1969).
[5] Marcus, Marvin, and Minc, Henryk, Generalized matrix functions, Trans. Amer. Math. Soc. **116,** 316–329 (1965).
[6] Marcus, Marvin, and Watkins, William, Partitioned hermitian matrices, Duke Math. J. (submitted).
[7] Merris, Russell, Trace functions 1, Duke Math. J. (submitted).
[8] Mirsky, L., An Introduction to Linear Algebra, 421 (Clarendon Press, Oxford, England 1955).
[9] Shisha, Oved (ed.), Inequalities, 163–176 (Academic Press, 1967).
[10] Wedderburn, J. H. M., Lectures on Matrices, Amer. Math. Soc., Colloq. Publ. vol. 17, New York (1934).

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[1] Figures in brackets indicate the literature references at the end of this paper.