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Orthogonal Decompositions of Tensor Spaces *

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Let V be an n-dimensional vector space over the complex numbers. Let H be a subgroup of S_m, the symmetric group on $\{1, \ldots, m\}$, and let $W = \bigotimes_1^m V$ be the tensor product of V with itself m times. In this note we give an orthogonal direct sum decomposition of W in terms of the system of inequivalent irreducible characters of H.

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1. Introduction

Let V be an n-dimensional vector space over the field of complex numbers. Let $W = \bigotimes_1^m V$ be the tensor product of V with itself m times. For $\sigma \epsilon S_m$, the symmetric group on $\{1, \ldots, m\}$, define the permutation operator $P(\sigma): W \rightarrow W$ by $P(\sigma)v_1 \otimes \ldots \otimes v_m = v_{\theta(1)} \otimes \ldots \otimes v_{\theta(m)}$, $v_1, \ldots, v_m \epsilon V$, where $\theta = \sigma^{-1}$. It is easy to check [1] that $P(\sigma)$ is linear and that $P(\sigma)P(\tau) = P(\sigma\tau)$, $\sigma, \tau \epsilon S_m$. Any linear combination $T = \sum_{\sigma \epsilon S_m} a(\sigma)P(\sigma)$ is called a *symmetry operator* and the range of T is called a *symmetry class* of tensors.

In [4],[1] Weyl expressed W as a direct sum of symmetry classes. The corresponding symmetry operators were determined from the idempotent generators of the minimal right ideals of the group ring of S_m. In this paper we obtain an orthogonal direct sum decomposition of W into symmetry classes with respect to the inner product defined below.

Let $(\ ,\ )$ be an inner product on V. Define an inner product on W by

$$(x_1 \otimes \ldots \otimes x_m, y_1 \otimes \ldots \otimes y_m) = \prod_{i=1}^m (x_i, y_i), \quad x_i, y_i \epsilon V. \tag{1}$$

THEOREM 1: *Let* H *be a subgroup of* S_m *of order* h. *Let* $\chi_1, \ldots, \chi_k$ *be the complete system of inequivalent irreducible characters on* H, *with* χ_i *having degree* r_i, $i = 1, \ldots, k$. *Define* $T_{\chi_i}: W \rightarrow W$ by

$$T_{\chi_i} = \frac{r_i}{h} \sum_{\sigma \epsilon H} \chi_i(\sigma)P(\sigma), \qquad i = 1, \ldots, k.$$

Let $V_{\chi_i}^m(H)$ *be the range of* T_{χ_i}. *Then with respect to the inner product* (1), W *is the orthogonal direct sum of the symmetry classes* $V_{\chi_i}^m(H)$:

$$W = \perp_{i=1}^{k} V_{\chi_i}^m(H). \tag{2}$$

*This work was done (1968–1969) while the author was a National Academy of Sciences-National Research Council Postdoctoral Research Associate at the National Bureau of Standards, Washington, D.C.

[1] Figures in brackets indicate the literature references at the end of this paper.

In section 3, we discuss some of the difficulties involved in constructing a suitable basis for $V_\chi^m(H)$. This problem has been dealt with [1] when χ is linear, but little has been done otherwise. We attempt to show the extent to which the methods of [1] will apply when χ is of higher degree.

2. Proof of Theorem 1

To establish (2), it suffices to show that

$$\begin{aligned} &\text{a. } T_{\chi_i} \text{ is hermitian,} \\ &\text{b. } T_{\chi_i}T_{\chi_j} = \delta_{ij}T_{\chi_i}, \\ &\text{c. } \sum_{i=1}^{k} T_{\chi_i} = \text{identity.} \end{aligned} \tag{3}$$

If "*" denotes the adjoint with respect to the inner product (1), then $(P(\sigma))^* = P(\sigma^{-1})$. Since $\overline{\chi_i(\sigma)} = \chi_i(\sigma^{-1})$, $T_{\chi_i}^* = T_{\chi_i}$. We now compute

$$T_{\chi_i}T_{\chi_j} = \frac{r_i r_j}{h^2} \sum_{\sigma,\, \tau \in H} \chi_i(\sigma)\chi_j(\tau)P(\sigma\tau) = \frac{r_i r_j}{h^2} \sum_{\mu \in H} P(\mu) \sum_{\sigma \in H} \chi_i(\sigma)\chi_j(\sigma^{-1}\mu).$$

The orthogonality relations for characters [3, p. 16] now imply that

$$T_{\chi_i}T_{\chi_j} = \frac{r_i r_j}{h^2} \sum_{\mu \in H} P(\mu)\, \frac{\delta_{ij} h \chi_j(\mu)}{r_j} = \delta_{ij}T_{\chi_i}.$$

Let e be the identity in H. Then

$$\begin{aligned} \sum_{i=1}^{k} T_{\chi_i} &= \sum_{i=1}^{k} \frac{r_i}{h} \sum_{\sigma \in H} \chi_i(\sigma)P(\sigma) \\ &= \frac{1}{h} \sum_{\sigma \in H} P(\sigma) \sum_{i=1}^{k} \chi_i(e)\chi_i(\sigma). \end{aligned}$$

Again, the orthogonality relations imply that

$$\begin{aligned} \sum_{i=1}^{k} T_{\chi_i} &= \frac{1}{h} \sum_{\sigma \in H} P(\sigma)\delta_{e,\sigma} h \\ &= P(e), \end{aligned}$$

the identity transformation on W. This proves (3).

We note that Weyl's decomposition of W is orthogonal with respect to the above inner product only when $m = 2$.

3. Bases for Symmetry Classes

As above, H is a subgroup of S_m, χ is an irreducible character on H of degree r, and T_χ and $V_\chi^m(H)$ are the associated symmetry operator and symmetry class. If $x_1 \otimes \ldots \otimes x_m$ is a decomposable tensor in W, set

$$T_\chi x_1 \otimes \ldots \otimes x_m = x_1 * \ldots * x_m.$$

The tensor $x_1 * \ldots * x_m$ is called a decomposable element of $V_\chi^m(H)$. Now if $v_1, \ldots, v_n$ is a basis of V, the set of n^m tensors $v_{\alpha_1} \otimes \ldots \otimes v_{\alpha_m}$, $1 \leq \alpha_i \leq n$ is a basis of $W = \bigotimes_1^m V$. Thus there is a basis of $V_\chi^m(H)$ consisting of decomposable elements. In [1], Marcus and Minc give a construction for such a basis when χ is linear, i.e., $r=1$. Let $\Gamma_{m,n}$ be the set of n^m sequences $\alpha=(\alpha_1, \ldots, \alpha_m)$, $1 \leq \alpha_i \leq n$. We write $\alpha \sim \beta$ if $\alpha^\sigma=(\alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(m)})=\beta$ for some $\sigma \epsilon H$. Clearly "$\sim$" is an equivalence relation on $\Gamma_{m,n}$. Choose lexicographically the lowest representative from each class determined by $\sim$. Call this set of representatives Δ. For each $\alpha \epsilon \Delta$, let H_α be the subgroup of all $\sigma \epsilon H$ such that $\alpha^\sigma=\alpha$. Let $\bar{\Delta}$ be the set of all $\alpha \epsilon \Delta$ such that $\sum_{\sigma \epsilon H_\alpha} \chi(\sigma) \neq 0$. If $v_1, \ldots, v_t$ is a basis of V, then the tensors

$$v_\alpha^* = v_{\alpha_1} * \ldots * v_{\alpha_m}, \; \alpha \epsilon \bar{\Delta} \tag{4}$$

are a linearly independent set in $V_\chi^m(H)$. Moreover, if χ is linear, the tensors (4) are a basis of $V_\chi^m(H)$. (For details, see [1].) Note that $\bar{\Delta}$ depends only on H and χ, and if χ is linear $v_{\alpha\sigma}^*=\chi(\sigma)v_\alpha^*$ for all $\sigma \epsilon H$, $\alpha \epsilon \bar{\Delta}$ (see [1]). Thus in the linear case one can conveniently determine matrix representations of certain linear transformations on $V_\chi^m(H)$. Using these properties, Marcus, Minc and Newman (see, e.g., [1], [2]) have been able to prove a large class of inequalities for determinants, permanents, and other multilinear matrix functions. Thus it would be useful to find a basis of $V_\chi^m(H)$ when degree $\chi > 1$. The following result demonstrates some obstacles.

THEOREM 2: *If* $v_1, \ldots, v_n$ *is a basis of* V, *the tensors*

$$v_\alpha^*, \; \alpha \epsilon \bar{\Delta} \tag{5}$$

are a basis of $V_\chi^m(H)$ *if and only if* χ *is linear. Moreover, if* $|\bar{\Delta}|$ *is the cardinality of* $\bar{\Delta}$ *then*

$$\dim V_\chi^m(H) \geqslant 2|\bar{\Delta}| \tag{6}$$

if χ *is not linear.*

PROOF. We may assume $v_1, \ldots, v_n$ is an orthonormal basis of V. The procedure in [1] still applies to show that the tensors (5) are linearly independent. For σ, $\tau \epsilon H$ and α, $\beta \epsilon \bar{\Delta}$, we compute

$$\begin{aligned}(v_{\alpha\sigma}^*, v_{\beta\tau}^*) &= (T_\chi v_{\alpha_{\sigma^{-1}(1)}} \otimes \ldots \otimes v_{\alpha_{\sigma^{-1}(m)}}, T_\chi v_{\beta_{\tau^{-1}(1)}} \otimes \ldots \otimes v_{\beta_{\tau^{-1}(m)}}) \\ &= (T_\chi v_{\alpha_{\sigma(1)}} \otimes \ldots \otimes v_{\alpha_{\sigma(m)}}, v_{\beta_{\tau(1)}} \otimes \ldots \otimes v_{\beta_{\tau(m)}}),\end{aligned}$$

because T_χ is idempotent hermitian. Hence

$$\begin{aligned}(v_{\alpha\sigma}^*, v_{\beta\tau}^*) &= \frac{r}{h} \sum_{\rho \epsilon H} \chi(\rho) \prod_{t=1}^m (v_{\alpha_{\sigma\rho^{-1}(t)}}, v_{\beta_{\tau(t)}}) \\ &= \frac{r}{h} \sum_{\rho \epsilon H} \chi(\rho) \prod_{t=1}^m (v_{\alpha_{\sigma\rho^{-1}\tau^{-1}(t)}}, v_{\beta_t}).\end{aligned} \tag{7}$$

Since α, $\beta \epsilon \bar{\Delta}$, the product in (7) is zero unless $\alpha=\beta$ and $\sigma\rho^{-1}\tau^{-1} \epsilon H_\alpha$.

Thus

$$\begin{aligned}(v_{\alpha\sigma}^*, v_{\beta\tau}^*) &= \delta_{\alpha,\beta} \frac{r}{h} \sum_{\substack{\rho \epsilon H \\ \sigma\rho^{-1}\tau^{-1} \epsilon H_\alpha}} \chi(\rho) \\ &= \delta_{\alpha,\beta} \frac{r}{h} \sum_{\mu \epsilon H_\alpha} \chi(\sigma\tau^{-1}\mu).\end{aligned} \tag{8}$$

Thus we are finished if we can show that for each $\alpha \in \bar{\Delta}$ there is a $\sigma \in H$ such that v_{α}^{*} and $v_{\alpha\sigma}^{*}$ are linearly independent. Suppose this were false for some $\alpha \in \bar{\Delta}$. Then

$$v_{\alpha\sigma}^{*}=\eta(\sigma)v_{\alpha}^{*},\ \sigma\epsilon H, \tag{9}$$

where $\eta(\sigma)$ is a scalar. From (8) and (9).

$$\begin{aligned}\eta(\sigma)\,\overline{\eta(\tau)}\,(v_{\alpha}^{*},\ v_{\alpha}^{*}) &= (v_{\alpha\sigma}^{*},\ v_{\alpha\tau}^{*})\\ &=\frac{r}{h}\sum_{\mu\epsilon H_{\alpha}}\chi(\sigma\tau^{-1}\mu)\\ &=(v_{\alpha\sigma\tau^{-1}}^{*},\ v_{\alpha}^{*})\\ &=\eta(\sigma\tau^{-1})\,(v_{\alpha}^{*},\ v_{\alpha}^{*}).\end{aligned} \tag{10}$$

Since $v_{\alpha}^{*}\neq 0$, (10) implies that η is a character on H of degree 1. Thus for all $\sigma\epsilon H$

$$\begin{aligned}\sum_{\mu\epsilon H_{\alpha}}\chi(\sigma\mu) &=\frac{h}{r}\,(v_{\alpha\sigma}^{*},\ v_{\alpha}^{*})\\ &=\frac{h}{r}\,\eta(\sigma)\,(v_{\alpha}^{*},\ v_{\alpha}^{*})\\ &=\eta(\sigma)\sum_{\mu\epsilon H_{\alpha}}\chi(u).\end{aligned} \tag{11}$$

Multiply both sides of (11) by $\eta(\sigma^{-1})$ and sum on σ, obtaining

$$\begin{aligned}0 &\neq \sum_{\sigma\epsilon H}\sum_{\mu\epsilon H_{\alpha}}\chi(\mu)\\ &=\sum_{\sigma\epsilon H}\eta(\sigma^{-1})\sum_{\mu\epsilon H_{\alpha}}\chi(\sigma\mu)\\ &=\sum_{\mu\epsilon H_{\alpha}}\sum_{\sigma\epsilon H}\eta(\sigma^{-1})\chi(\sigma\mu).\end{aligned} \tag{12}$$

As long as the degree of χ is greater than 1, the orthogonality relations imply that the right side of (12) is zero. This establishes (6) and thus Theorem 2 is proved.

4. References

[1] Marcus, M., and Minc, H., Generalized matrix functions, Trans. AMS, **4**, 316–329 (1965).
[2] Marcus, M., and Newman, M., Inequalities for the permanent function, Ann. of Math. **75**, 47–62 (1962).
[3] Newman, M., Matrix representations of groups, Nat. Bur. Stand. (U.S.), Appl. Math. Ser. 83 pages (July 1968).
[4] Weyl, H., Der Zusammenhang zwischen der symmetrischen und der linearen Gruppe, Ann. of Math. **30**, 499–516 (1929).

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