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A Theorem on Convex Hulls

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Let p be a point in the interior K° of the convex hull $K=K(S)$ of a bounded point-set S in a real Hilbert space. A quantity $R(p)$ is determined such that every closed ball of radius $>R(p)$, if it contains p, must also meet S.

Key words: Convex geometry; convex sets.

1. Introduction

Let $K=K(S)$ be the convex hull of a bounded point-set S in a real Hilbert space E. This note is concerned with the following questions, which arise for example [1] in connection with the theory of "ϵ-convex sets."

Consider a point $p\epsilon K$. Is it true that every large enough closed ball, which contains p, must also meet S? How large is "large enough"?

It is readily seen that the answer to the first question is in general negative if p lies in the boundary ∂K of K (unless, of course, $p\epsilon S$). Thus it will be assumed throughout that p lies in the interior K° of $K=K(S)$.

With the notation

$$\delta=\text{diam } (S)=\text{diam } (K)<\infty,$$

$$\Delta=d(p,\,\partial K)\leqslant\delta,$$

define $R^*(p)$ and $R(p)$ by

$$R^*(p)=\delta^2/2\Delta,$$

$$R(p)=R^*(p)+\delta.$$

For $p\epsilon K^\circ$, with K, S, E, $R^*(p)$ and $R(p)$ as above, the following results will be proved:

THEOREM 1: *Every closed ball* B *with radius* $>$ R*(p) *for which* p$\epsilon\partial$B *meets* S.

THEOREM 2: *Every closed ball* B *with radius* $>$ R(p), *which contains* p, *meets* S.

2. Proof of Theorem 1

Let B be a closed ball of radius $r>R^*(p)$, such that $p\epsilon\partial B$. We wish to show that B meets S.

Let c be the center of B, and u the unit vector in the direction of $p-c$. Thus $p=c+ru$. Let H_1 be the closed hyperplane tangent to B at p, i.e.

$$H_1=p+\{y:(y,\,u)=0\}.$$

[1] J. Perkal, Sur les ensembles ϵ-convexes, Colloq. Math. 4 (1956), pp. 1–10.

Let E_1 be that one of the closed halfspaces of E determined by H_1 which contains c and thus contains B.

Since $r > R^*(p) = \delta^2/2\Delta$, we can choose $\eta < \Delta$ such that $r\eta > \delta^2/2$. Let H_2 be the hyperplane parallel to H_1 which passes through the point $q = c + (r-\eta)u$. Then H_2 is at distance η from H_1, and lies in E_1.

Let E_2 be that one of the closed halfspaces of E determined by H_2 which does not contain H_1. Then $E_2 - H_2$ is met by the open ball of radius Δ centered at p, e.g. at the midpoint of q and $c + (r-\Delta)u$. This ball lies wholly in K, so $E_2 - H_2$ meets K. Thus the closed convex set $E - (E_2 - H_2)$ does not contain all of the convex hull $K = K(S)$, and so does not contain all of S. Hence $E_2 \cap S$ is nonempty; to complete the proof of theorem 1, it suffices to prove that $E_2 \cap S \subset B \cap S$.

Consider then any $x \epsilon E_2 \cap S$. We need to show that

$$d(x, c) = \|x - c\| \leq r. \tag{2.1}$$

Note that $x - c$ admits a unique orthogonal decomposition relative to u, i.e.,

$$x - c = y + \theta u, \qquad (y, u) = 0. \tag{2.2}$$

This yields

$$\|x - c\|^2 = \|y\|^2 + \theta^2. \tag{2.3}$$

Because $x \epsilon E_2$, we have

$$\theta \leq r - \eta. \tag{2.4}$$

Because $p \epsilon K$ and $x \epsilon S \subset K$, we have

$$\begin{aligned} \delta^2 = [\mathrm{diam}(K)]^2 &\geq \|x - p\|^2 \\ &= \|(x - c) - (p - c)\|^2 = \|y - (r - \theta)u\|^2 \\ &= \|y\|^2 + (r - \theta)^2 = \|y\|^2 + \theta^2 + r^2 - 2r\theta. \end{aligned}$$

Combining this with (2.3) gives

$$\|x - c\|^2 \leq \delta^2 + 2r\theta - r^2,$$

and application of (2.4) then yields

$$\|x - c\|^2 \leq \delta^2 + 2r(r - \eta) - r^2 = r^2 - (2r\eta - \delta^2).$$

Since η was so chosen that $2r\eta > \delta^2$, (2.1) is proved.

3. Proof of Theorem 2

We shall show that theorem 2 follows from theorem 1. Let B be a closed ball of radius $r - R(p)$ such that $p \epsilon B$. There are two cases to be considered, depending on the distance of p from the center c of B.

If $d(c, p) \leq R^*(p) = R(p) - \delta$, then B contains the closed ball B' of radius δ centered at p. Since

$$\delta = \mathrm{diam}\ (S) = \mathrm{diam}\ (K),$$

while $p \epsilon K$, it follows that $K \subset B' \subset B$ and thus that $S \subset B$. So B certainly meets S.

Now suppose $d(c, p) > R^*(p)$. Let B^* be the closed ball of radius $d(c, p)$ centered at c. Then B^* has radius $> R^*(p)$, and $p \epsilon \partial B^*$. By theorem 1, B^* meets S. Since $B^* \subset B$, B meets S.

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