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Simultaneous Contractification

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Consider a finite family of continuous self-mappings of a topological space X, with a common fixed point. Suppose that for each member of the family, X has a metric for which that member is a contraction. It is shown that if the family is commutative, then X has a metric under which all members are (simultaneously) contractions. Additional hypotheses are given which ensure the same conclusion in the noncommutative case.

Key words: Contractions; functional analysis: metric spaces; topology.

1. Introduction

This paper deals with continuous self-mappings of a metrizable topological space X. Such a map f is a (ρ, λ)-*contraction* if $\lambda \epsilon (0, 1)$, ρ is some metric on X, and

$$\rho(fx, fy) \leq \lambda\rho(x, y) \qquad (\text{all } x, y \epsilon X). \tag{1.1}$$

We term f *contractifiable* if an appropriate metrization of X makes it a contraction, i.e., if there exists a pair (ρ, λ) such that f is a (ρ, λ)-contraction.

Banach's Contraction Theorem asserts that if f is a (ρ, λ)-contraction for some *complete* metric [2] ρ and some λ, then there is a point $\xi \epsilon X$ and an open neighborhood U of ξ such that [3]

$$f(\xi) = \xi, \tag{1.2}$$

$$f^n(x) \to \xi \qquad (\text{all } x \epsilon X), \tag{1.3}$$

$$f^n(U) \to \{\xi\}. \tag{1.4}$$

The explicit meaning of (1.4) is that for each neighborhood V of ξ, there is an $n(V) > 0$ such that $f^n(U) \subset V$ for all $n \geq n(V)$.

In a previous paper,[4] to be referred to as CONVERSE, the second author proved a converse result: If f satisfies (1.2)–(1.4) for some $\xi \epsilon X$ and some open neighborhood U of ξ, then f is contractifiable. [Moreover, the "contraction constant" λ can be specified to be any assigned member of (0, 1), and the "contractifying" metric ρ can be chosen complete if X admits a complete metric.] Interest in such converses stems from situations in which one would like to apply the Contraction Theorem to the study of some iterative numerical process, but encounters difficulty because for all $\lambda \epsilon (0, 1)$, the associated mapping fails to satisfy (1.1) for the metric initially considered.

Our purpose in the present paper is to extend this converse result to the "simultaneous contractification" of a *family* $\mathfrak{F}$ of maps. We will call such a family *simultaneously contractifiable*

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[2] We call a metric ρ *complete* if (X, ρ) is a complete metric space.
[3] Actually (1.4) is not usually cited as a conclusion of the Contraction Theorem, but is an easy consequence of (1.1) and (1.2).
[4] P. R. Meyers, A Converse to Banach's Contraction Theorem, J. Res. Nat. Bur. Stand. (U.S.), **71B**, (2&3) 73–76 (1967).

if there is a single metric that "works" for all of them, i.e., if X admits a metric ρ under which *all* $f \epsilon \mathfrak{F}$ are contractions. If the associated λ_f can be chosen independent of f, we term $\mathfrak{F}$ *uniformly contractifiable.*

It will be assumed throughout that the members of $\mathfrak{F}$ have a *common* fixed point ξ. This is automatically the case, for example, in the frequently-encountered situations in which the members of $\mathfrak{F}$ *commute.* For a proof, recall that a contractifiable map f with at least one fixed point has a *unique* fixed point ξ_f. For commuting maps f and g, with respective unique fixed points ξ_f and ξ_g, the relation

$$fg(\xi_f) = gf(\xi_f) = g(\xi_f)$$

identifies $g(\xi_f)$ as a fixed point of f, so that (by uniqueness) $g(\xi_f) = \xi_f$. Thus ξ_f is a fixed point of g, yielding the conclusion ($\xi_f = \xi_g$) of a common fixed point.

For the simplest case, in which $\mathfrak{F}$ is finite and commuting, we can draw the strong conclusion that *individual* contractifiability of the members of $\mathfrak{F}$ implies simultaneous contractifiability . . . in fact, *uniform* contractifiability . . . of the family. The following result will be proven in section 2:

THEOREM 1: *Let $\mathfrak{F}$ be a finite commuting family of continuous self-mappings of* X *which individually satisfy* (1.2)–(1.4). *Then $\mathfrak{F}$ is uniformly contractifiable.*

The following example will show *both* that the finiteness hypothesis cannot be omitted in Theorem 1, *and* that simultaneous contractifiability plus commutativity does not imply uniform contractifiability. Let X be the real line with the usual topology, and take $\mathfrak{F}$ to consist of the infinite family of maps

$$f_i(x) = (1-1/i)x \qquad (i = 1, 2, \ldots).$$

The members of this family commute, have $\xi = 0$ as fixed point, and are all contractions under the standard metric on X. It is easily verified, however, that for *no* open neighborhood U of ξ does

$$f^n(U) \to \{\xi\} \qquad \text{uniformly in } f \epsilon \mathfrak{F}$$

hold. Thus $\mathfrak{F}$ cannot be uniformly contractifiable, else the open unit ball around ξ in a suitable metric would have the property just displayed.

Preparing to drop the commutativity hypothesis in Theorem 1, we define $\mathfrak{F}^\circ$ to consist of the identity map of X, while $\mathfrak{F}^n$ $(n > 0)$ consists of all n-fold compositions of maps in $\mathfrak{F}$. If $\mathfrak{F}$, with common fixed point ξ, is to be uniformly contractifiable, then the following generalizations of (1.2)–(1.4) must hold:

$$\mathfrak{F}(\xi) = \cup \{f(\xi) : f \epsilon \mathfrak{F}\} = \{\xi\}, \tag{1.2'}$$

$$\mathfrak{F}^n(x) = \cup \{g(x) : g \epsilon \mathfrak{F}^n\} \to \{\xi\} \tag{1.3'}$$

for all $x \epsilon X$, and

$$\mathfrak{F}^n(U) = \cup \{g(U) : g \epsilon \mathfrak{F}^n\} \to \{\xi\} \tag{1.4'}$$

for some open neighborhood U of ξ. It will be shown in section 2 that these necessary conditions, for the uniform contractifiability of $\mathfrak{F}$, are also sufficient when $\mathfrak{F}$ is finite. In other words, the commutativity hypothesis in Theorem 1 can be dropped *if* the hypotheses (1.2)–(1.4), expressing the requirement that individual members of $\mathfrak{F}$ be contractifiable, are replaced by their "uniform in $\mathfrak{F}$" versions (1.2')–(1.4'). Thus we have:

THEOREM 2: *Let* $\mathfrak{F}$ *be a finite family of continuous self-mappings of* X, *which satisfies* (1.2′)–(1.4′). *Then* $\mathfrak{F}$ *is uniformly contractifiable.*

(Since (1.2′)–(1.4′) are obtained by systematically replacing f with $\mathfrak{F}$ in (1.2)–(1.4), one might expect that the same replacement would convert the proof of the main theorem in CONVERSE into a proof of Theorem 2. As will be seen, this is actually the case.)

The unresolved problem is that of finding appropriate additional hypotheses to ensure uniform contractifiability (or merely simultaneous contractifiability), with or without commutativity, when $\mathfrak{F}$ is *infinite*.[5]

2. Proofs of Theorems

We first show that Theorem 2 implies Theorem 1. Let $\mathfrak{F}=\{f_1, \ldots, f_m\}$ satisfy the hypotheses of Theorem 1. We shall prove that it also satisfies the hypotheses (1.3′) and (1.4′) of Theorem 2; satisfaction of (1.2′) is immediate since commutativity implies commonness of the fixed point.

As preparation, it will be shown that there is an open neighborhood U of the fixed point ξ such that

$$\mathfrak{F}(U) \subset U \quad \text{and} \quad \mathfrak{F}^n(U) \to \{\xi\}. \tag{2.1}$$

For this purpose, observe that for $1 \leq i \leq m$ there is an open neighborhood U_i of ξ such that $f_i^n(U_i) \to \{\xi\}$. Let $I = \cap_1^m U_i$; then there are integers $k(i) \geq 1$ such that $f_i^n(U_i) \subset I$ for all $n \geq k(i)$. Now set

$$U = \cap_{j(1) \ldots j(m)} f_1^{-j(1)} \ldots f_m^{-j(m)}(I),$$

where $j(1) \ldots j(m)$ ranges over all integer sequences with $0 \leq j(i) < k(i)$ for $1 \leq i \leq m$. Since $U \subset I$ and $\mathfrak{F}^n(I) \to \{\xi\}$, the second part of (2.1) holds. To prove that the first part also holds, e.g., that $f_1(U) \subset U$, note that

$$f_1(U) \subset f_1^{-j(1)} \ldots f_m^{-j(m)}(I)$$

follows when $j(1) < k(1)-1$ from the consequence

$$f_1(U) \subset f_1^{-[j(1)+1]} \ldots f_m^{-j(m)}(I)$$

of the definition of U, and follows when $j(1) = k(1)-1$ from the definition of $k(1)$ as justifying

$$U \subset f_2^{-j(2)} \ldots f_m^{-j(m)}(I) \subset f_2^{-j(2)} \ldots f_m^{-j(m)}(U_1)$$
$$\subset f_2^{-j(2)} \ldots f_m^{-j(m)} f_1^{-k(1)}(I) = f_1^{-k(1)} f_2^{-j(2)} \ldots f_m^{-j(m)}(I).$$

To demonstrate that $\mathfrak{F} = (f_1, \ldots f_m)$ satisfies (1.3′) and (1.4′), let V be any open neighborhood of ξ. By (2.1), there is an integer K such that $\mathfrak{F}^k(U) \subset V$ for all $k > K$. Let $N = mK+1$. For $n \geq N$, an arbitrary member g of $\mathfrak{F}^n$ can (by commutativity) be written in the form

$$g = f_1^{j(1)} \ldots f_m^{j(m)}, \tag{2.2}$$

where $\Sigma_i j(i) = n \geq N$ and thus $\max_i j(i) > K$. Since $\mathfrak{F}$ is commutative, there is no loss of generality in assuming that the maximum occurs for $i=1$. Then, since $f_i(U) \subset U$ for $i > 1$, we have

$$g(U) \subset f^{j(1)}(U) \subset V.$$

[5] Some partial results for the commutative case are given in the second author's manuscript, Contractive Semigroups and Uniform Asymptotic Stability, presented at the 6/67 National SIAM Meeting.

This shows that (1.4′) holds. To verify (1.3′), note that for any $x \epsilon X$ there is a $k(x)$ such that $\mathfrak{F}^{k(x)} \subset U$. Let $N' = m[K + k(x)] + 1$. Then for $n \geqslant N'$, an arbitrary $g \epsilon \mathfrak{F}^n$ can be written as (2.2) with $\max_i j(i) > K + k(x)$. Assuming the maximum occurs for $i = 1$, we have $j(1) - k(x) > K$ and thus

$$g(x) = f_1^{j(1)-k(x)} f_2^{j(2)} \ldots f_m^{j(m)} f_1^{k(x)}(x)$$
$$\epsilon f_1^{j(1)-k(x)} f_2^{j(2)} \ldots f_m^{j(m)}(U) \subset f_1^{j(1)-k(x)}(U) \subset V.$$

This shows that (1.3′) holds, completing the proof that Theorem 2 implies Theorem 1.

We turn now to the proof of Theorem 2. Let $\mathfrak{F}$ satisfy the hypotheses of this theorem. As a preliminary, it will be shown that the U in (1.4′) can be assumed to obey

$$\mathfrak{F}(U) \subset U. \tag{2.3}$$

For this purpose, begin with *any* U as in (1.4′). There is an N such that $\mathfrak{F}^n(U) \subset U$ for all $n \geqslant N$. Let

$$W = \cap_{t=0}^{N-1} \cap \{g^{-1}(U) : g \epsilon \mathfrak{F}^t\}.$$

Then W is an open neighborhood of ξ, and $W \subset U$ so that W satisfies (1.4′). Moreover $\mathfrak{F}(W) \subset W$, as can be seen by considering any $f \epsilon \mathfrak{F}$, any $t \epsilon [0, N-1]$, and any $g \epsilon \mathfrak{F}^t$: that $f(W) \subset g^{-1}(U)$, i.e., $gf(W) \subset U$, follows for $t < N-1$ because of $gf \epsilon \mathfrak{F}^{t+1}$, and for $t = N-1$ because $gf(W) \subset \mathfrak{F}^N(U) \subset U$. Replacing U by W, we can and will assume that the U of (1.4′) satisfies $\mathfrak{F}(U) \subset U$.

Choose any $\lambda \epsilon$ (0, 1) and let ρ_0 be any metric on X, complete if X admits a complete metric. We will construct a metric ρ^* on X, complete if ρ_0 is, such that each $f \epsilon \mathfrak{F}$ is a (ρ^*, λ)-contraction.[6] The construction, which follows closely that in CONVERSE, has three stages. First comes the construction of a metric ρ_M, complete if ρ_0 is, with respect to which each $f \epsilon \mathfrak{F}$ is *nonexpanding*, in the sense that

$$\rho_M(f(x), f(y)) \leqslant \rho_M(x, y) \qquad (\text{all } x, y \epsilon X).$$

The second stage yields a function d on $X \times X$ which has all the properties desired of ρ^* except perhaps for satisfying the triangle inequality. This is corrected in the third stage, in which $\rho^*(x, y)$ is introduced as what might be called the "d-geodesic distance" between x and y.

The first step is carried out by setting

$$\rho_M(x, y) = \max \{\rho_0(g(x), g(y)) : g \epsilon \cup_0^\infty \mathfrak{F}^n\}. \tag{2.4}$$

That ρ_M is a well-defined metric on X follows as in CONVERSE, and the nonexpansiveness assertion is obvious. Since $\rho_0 \leqslant \rho_M$, any ρ_M-convergent sequence is also ρ_0-convergent to the same limit, and a ρ_M-Cauchy sequence is also ρ_0-Cauchy. Thus the topological equivalence of the two metrics, as well as the completeness of ρ_M if ρ_0 is complete, will follow once it is shown that for each $x \epsilon X$ and each $\delta > 0$, there is an $\eta < 0$ such that

$$\rho_0(x, y) < \eta \qquad \text{implies} \qquad \rho_M(x, y) < \delta. \tag{2.5}$$

To prove this, observe that (1.3′) assures the finiteness of

$$\nu(x) = \min \{n \geqslant 0 : \mathfrak{F}^n(x) \subset U\}. \tag{2.6}$$

Since $\mathfrak{F}^{\nu(x)}(x) \subset U$, continuity and the finiteness of $\mathfrak{F}$ imply that for all small enough $\eta > 0$,

$$\rho_0(x, y) < \eta \qquad \text{implies} \qquad \mathfrak{F}^{\nu(x)}(y) \subset U. \tag{2.7}$$

[6] The arbitrariness of λ, and the assertion about completeness, make the results somewhat stronger than was stated in section 1.

Moreover, (1.4′) assures the existence of an N such that

$$\rho_0\text{-diam}\,[\mathfrak{F}^m(U)] < \delta \qquad \text{for} \qquad m > N, \tag{2.8}$$

and continuity and the finiteness of $\mathfrak{F}$ imply that for small enough η,

$$\rho_0(x, y) < \eta \qquad \text{implies} \qquad \rho_0(g(x), g(y)) < \delta \qquad \text{for} \qquad g \in \cup_{n=0}^{N+\nu(x)} \mathfrak{F}^n. \tag{2.9}$$

Suppose η is chosen so small that (2.7) and (2.9) both apply. Consider any n and any $g \in \mathfrak{F}^n$. It will be shown that

$$\rho_0(g(x), g(y)) < \delta, \tag{2.10}$$

thus establishing (2.5). If $n \leqslant N+\nu(x)$, then (2.10) follows from (2.9). If $n > N+\nu(x)$, write $g = g_2 g_1$ where $g_1 \in \mathfrak{F}^{\nu(x)}$ and $g_2 \in \mathfrak{F}^m$ with $m > N$. By (2.6) and (2.7), $g_1(x)$ and $g_1(y)$ both lie in U, so that $g(x)$ and $g(y)$ both lie in $g_2(U) \subset \mathfrak{F}^m(U)$; now (2.10) follows from (2.8).

To begin the second stage of the construction, let K_n be the closure of $\mathfrak{F}^n(U)$ for $n \geqslant 0$, and let

$$K_{(-n)} = \mathfrak{F}^{-n}(K_0) = \cap \{g^{-1}(K_0) : g \in \mathfrak{F}^n\}.$$

Then $\{K_n\} \to \{\xi\}$, and since $\mathfrak{F}(U) \subset U$, the sequence $\{K_n\}$ is nonascending. Let $n(\xi) = \infty$, and for $x \neq \xi$ set

$$n(x) = \max\{n : x \in K_n\} < \infty;$$

then $n(x) \geqslant 0$ for $x \in K_0$, while for $x \in X - K_0$,

$$n(x) = -\min\{k : \mathfrak{F}^k(x) \subset K_0\} < 0.$$

It is easily checked that

$$n(f(x)) \geqslant n(x) + 1 \qquad \text{for } f \in \mathfrak{F},$$

so that the definitions

$$c(x, y) = \min\{n(x), n(y)\}; \; d(x, y) = \lambda^{c(x, y)} \rho_M(x, y)$$

imply that d has the desired property

$$d(f(x), f(y)) \leqslant \lambda d(x, y) \qquad (x, y \in X; f \in \mathfrak{F}).$$

For the third stage, denote by Σ_{xy} the set of finite chains $\sigma_{xy} = [x = x_0, \ldots, x_t = y]$ between x and y, with associated lengths

$$L(\sigma_{xy}) = \sum_1^t d(x_{i-1}, x_i),$$

and put

$$\rho^*(x, y) = \inf\{L(\sigma_{xy}) : \sigma_{xy} \in \Sigma_{xy}\}.$$

That ρ^* has all the desired properties follows exactly as in CONVERSE, with sets $B_\nu = X - f^{-\nu}(U)$ of CONVERSE replaced by

$$B_\nu = X - \mathfrak{F}^{-\nu}(U) = X \cap \{g^{-1}(U) : g \in \mathfrak{F}^\nu\}.$$

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