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Commutator Groups and Algebras*

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Let H and K be connected, Lie subgroups of a Lie group G. The group $[H, K]$, generated by all commutators $hkh^{-1}k^{-1}(h\epsilon H, k\epsilon K)$ is arcwise connected. Therefore, by a theorem of Yamabe, $[H, K]$ is a Lie subgroup. If $\mathfrak{H}$, $\mathfrak{K}$ denote the Lie algebras of H and K, respectively, then the Lie algebra of $[H, K]$ is the smallest algebra containing $[\mathfrak{H}, \mathfrak{K}]$, which is invariant under $ad\mathfrak{H}$ and $ad\mathfrak{K}$. An immediate consequence is that if H and K are complex Lie subgroups, then $[H, K]$ is also complex.

Key words: Adjoint representation; commutator; Lie algebra; Lie group.

Let G be a real Lie group, and let H and K be connected, Lie subgroups, with Lie algebras $\mathfrak{h}$ and $\mathfrak{k}$. The group $[H, K]$ generated by commutators $\{hkh^{-1}k^{-1}|h\epsilon H, k\epsilon K\}$ is arc-wise connected. This implies, by a theorem of Yamabe [1],[1] that $[H, K]$ is a Lie subgroup. The question we raise in this note is: What is the Lie algebra of $[H, K]$? We shall prove that this is the smallest algebra which contains $[\mathfrak{h}, \mathfrak{k}]$ and is invariant under $ad\mathfrak{h}$ and $ad\mathfrak{k}$ (i.e., it is the smallest ideal in the algebra generated by $\mathfrak{h}$ and $\mathfrak{k}$ which contains $[\mathfrak{h}, \mathfrak{k}]$.) An immediate consequence is that if H and K are complex groups, then $[H, K]$ is also complex. Of course, one obtains as a special case the known fact that if H and K are normal, then $[\mathfrak{h}, \mathfrak{k}]$ is the Lie algebra of $[H, K]$. More generally, the Lie algebra of $[H, K]$ is the smallest algebra $\mathfrak{m}$ containing $[\mathfrak{h}, \mathfrak{k}]$, if and only if $\mathfrak{m}$ is invariant under $ad\mathfrak{h}$ and $ad\mathfrak{k}$.

We shall first give a list of notation and terminology. In the following, G is a real Lie group, with Lie algebra $\mathfrak{g}$.

(1) If $g\epsilon G$ and X is a tangent vector or vector field on G, then gX and Xg denote the left and right translation of X. (i.e., if l_g, r_g are the left and right translations of G, then $gX = dl_g(X)$ and $Xg = dr_g(X)$.)

(2) We denote by ad and Ad the adjoint representations of $\mathfrak{g}$ and G, respectively. Thus $ad(X)Y = [X, Y]$ and $Ad(g)Y = gYg^{-1}$. $\theta(g)$ denotes the inner automorphism $x \to gxg^{-1}$. Note that

$$Ad\,(\exp X) = \exp\,(adX) \text{ and } \theta(g)\,\exp X = \exp\,[Ad(g)X].$$

(3) If X is a tangent vector at some point of G, $\hat{X}$ denotes the corresponding left-invariant vector field. Of course, $\mathfrak{g}$ is the algebra of left-invariant vector fields on G.

(4) If X is a vector field, $g\epsilon G$, then X_g is the value of X at g. If $\mathfrak{h} \subset \mathfrak{g}$ is a subalgebra, $\mathfrak{h}_g = \{X_g | X\epsilon\, \mathfrak{h}\}$.

(5) If $\mathfrak{h}$ and $\mathfrak{k}$ are subalgebras of $\mathfrak{g}$, $[\mathfrak{h}, \mathfrak{k}]$ denotes the linear span of the commutators $\{[X, Y] | X\epsilon\, \mathfrak{h}, Y\epsilon\, \mathfrak{k}\}$.

(6) If $\mathfrak{m}$ is a subspace of $\mathfrak{g}$, $\mathscr{A}\mathfrak{m}$ denotes the subalgebra generated by $\mathfrak{m}$.

(7) If $\mathfrak{m}$ is a subspace, $\mathfrak{h}$ a subalgebra, with $\mathfrak{m} \subset \mathfrak{h}$, then $\mathfrak{J}(\mathfrak{m}, \mathfrak{h})$ denotes the smallest ideal in $\mathfrak{h}$ which contains $\mathfrak{m}$. Thus $\mathfrak{J}(\mathfrak{m}, \mathfrak{h})$ is the linear span of $\mathfrak{m}$ and elements of the form $(adX_1)(adX_2)\ .\ .\ .\ (adX_r)Y$, where $X_i\epsilon\, \mathfrak{h}$, $Y\epsilon\, \mathfrak{m}$. We write $\mathfrak{J}[\mathfrak{h}, \mathfrak{k}]$ for $\mathfrak{J}([\mathfrak{h}, \mathfrak{k}], \mathscr{A}(\mathfrak{h}+\mathfrak{k}))$.

(8) If $g: I \to G$ is a differentiable curve (where I is an interval), we write $\left.\frac{dg}{dt}\right|_{t_0}$ for the tangent vector $\left.dg\right|_{t_0}\left(\frac{d}{dt}\right)$. We shall write $0((t-t_0)^k)$ for a function $X(t)$ (with values in $\mathfrak{g}$) if there exist numbers $M > 0$, $\epsilon > 0$ so that $||X(t)|| < M|t-t_0|^k$ when $|t-t_0| < \epsilon$. (Here $||X||$ is some norm in $\mathfrak{g}$.)

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[1] H. Yamabe, On an arcwise connected subgroup of a Lie group, Osaka Math. J. **2**, 13–14 (1950).

For example, if $g(t)$ is a differentiable curve in G, with $g(t_0)=1$ and $\left.\frac{dg}{dt}\right|_{t_0}=X$, then (for t near t_0) $\log[g(t)]=(t-t_0)X+0((t-t_0)^2)$, or equivalently, $g(t)=\exp[(t-t_0)X+0((t-t_0)^2)]$. The Campbell-Baker-Hausdorff formula implies.

$$\exp(tX)\exp(tY)=\exp\left(t(X+Y)+\frac{t^2}{2}[X,Y]+0(t^3)\right).$$

LEMMA 1: *Let* H *and* K *be connected Lie subgroups of* G, *with Lie algebras* $\mathfrak{h}$ *and* $\mathfrak{k}$.
(a) *If* $h\in H$ *and* $Y\in\mathfrak{k}$, *then* $(\mathrm{Ad}(h)-1)Y\in\mathfrak{I}[\mathfrak{h},\mathfrak{k}]$.
(b) *If* $k\in K$ *and* $X\in\mathfrak{h}$, *then* $(\mathrm{Ad}(k)-1)X\in\mathfrak{I}[\mathfrak{h},\mathfrak{k}]$.
PROOF: We shall prove (a). Note that $(Ad(h_1h_2)-1)Y=Ad(h_1)(Ad(h_2)-1)Y+(Ad(h_1)-1)Y$. Thus, if the statement is true for h_1 and h_2, it is true for h_1h_2. Therefore it suffices to prove it for a set of generators. We shall show that it is true for $h=\exp(X), X\in\mathfrak{h}$.

$$Ad(\exp X)-1=\exp(adX)-1=\sum_{n=1}^{\infty}\frac{(adX)^n}{n!},$$

while $(adX)^nY\in\mathfrak{I}[\mathfrak{h},\mathfrak{k}]$. *Q.E.D.*

LEMMA 2: *Let* H *and* K *be connected, Lie subgroups of* G, *with Lie algebras* $\mathfrak{h}$ *and* $\mathfrak{k}$. *Let* $h(t)$ (*respectively* $k(t)$) *be a differentiable curve in* H (*respectively* K) *with* $h(0)=1$ (*respectively* $k(0)=1$). *Let* $g(t)=h(\sqrt{t})k(\sqrt{t})h(\sqrt{t})^{-1}k(\sqrt{t})^{-1}$. *Then for each* $t_0\geqslant 0$ *at which* $g(t)$ *is defined,*

$$\left.\frac{dg}{dt}\right|_{t_0}\in\mathfrak{I}[\mathfrak{h},\mathfrak{k}]_{g(t_0)}.$$

PROOF: For $t_0=0$, $\left.\frac{dg}{dt}\right|_0=[\hat{X},\hat{Y}]_1$, where $X=\left.\frac{dh}{dt}\right|_0$ and $Y=\left.\frac{dk}{dt}\right|_0$. We must find out what happens when $t_0>0$.

Let $u=\sqrt{t}$, $u_0=\sqrt{t_0}$, $h_0=h(u_0)$, $k_0=k(u_0)$,

$$g_0=g(t_0)=h_0k_0h_0^{-1}k_0^{-1},$$

$$X=\left.\frac{dh(u)}{du}\right|_{u_0}, Y=\left.\frac{dk(u)}{du}\right|_{u_0} \text{ and } Z=\left.\frac{dg(t)}{dt}\right|_{t_0}.$$

The translated curve $g_0^{-1}g(t)$ has tangent vector $g_0^{-1}Z$ at 1. We shall use properties of exp to compute this tangent vector.
Noting that

$$\left.\frac{d}{du}(h_0^{-1}h(u))\right|_{u_0}=h_0^{-1}X,$$

and

$$\left.\frac{d}{du}(h_0h(u)^{-1})\right|_{u_0}=-\left.\frac{d}{du}(h(u)ho^{-1})\right|_{u_0}=-Xh_0^{-1}.$$

We obtain

(1) $h_0^{-1}h(u)=\exp\{(u-u_0)\hat{X}+0((u-u_0)^2)\}$,
(2) $h_0h(u)^{-1}=\exp\{-(u-u_0)Ad(h_0)\hat{X}+0((u-u_0)^2)\}$.
Similarly,
(3) $k(u)k_0^{-1}=\exp\{(u-u_0)Ad(k_0)\hat{Y}+0((u-u_0)^2)\}$,
(4) $k(u)^{-1}k_0=\exp\{-(u-u_0)\hat{Y}+0((u-u_0)^2)\}$.
Next, we see that

$$\begin{aligned}g_0^{-1}g(t)&=k_0h_0k_0^{-1}h_0^{-1}h(u)k(u)h(u)^{-1}k(u)^{-1}\\&=\theta(k_0)\{[\theta(h_0k_0^{-1})(h_0^{-1}h(u)k(u)k_0^{-1})][h_0h(u)^{-1}k(u)^{-1}k_0]\}.\end{aligned}$$

Now by using equations (1)–(4),

$$\text{(5)}\quad \exp(sU+0(s^2))\exp(sV+0(s^2))=\exp\{s(U+V)+0(s^2)\}$$

and

$$\text{(6)}\quad \theta(g)\exp(U)=\exp[Ad(g)U],$$

a straightforward computation shows that $g_0^{-1}g(t)=\exp[W(t)]$, where

$$W(t)=(u-u_0)\{Ad(k_0h_0)(Ad(k_0^{-1})-1)\hat{X}+Ad(k_0)(Ad(h_0)-1)\hat{Y}\}+0((u-u_0)^2).$$

It follows that the tangent vector $g_0^{-1}Z$ is the value of the vector field

$$\left.\frac{dW}{dt}\right|_{t_0}=\frac{1}{\sqrt{t_0}}\{Ad(k_0h_0)(Ad(k_0^{-1})-1)\hat{X}+Ad(k_0)(Ad(h_0)-1)\hat{Y}\}$$

at the unit element. Thus, by Lemma 1, $g_0^{-1}Z\in\mathfrak{I}[\mathfrak{h},\mathfrak{k}]_1$ and $Z\in\mathfrak{I}[\mathfrak{h},\mathfrak{k}]_{g_0}$.

Q.E.D.

LEMMA 3: [H, K] *is normalized by* H *and* K.

PROOF: Let h and $h_1\in H$, $k\in K$. Then

$$h_1(hkh^{-1}k^{-1})h_1^{-1}=(h_1hkh^{-1}h_1^{-1}k^{-1})(kh_1k^{-1}h_1^{-1}).$$

This shows that $[H, K]$ is normalized by H. Since $[H, K]=[K, H]$, the same is true for K.

Q.E.D.

THEOREM: *Let* G *be a real Lie group, and let* H *and* K *be connected, Lie subgroups with Lie algebras* $\mathfrak{h}$ *and* $\mathfrak{k}$. *Then* [H, K] *is a Lie subgroup with Lie algebra* $\mathfrak{I}[\mathfrak{h},\mathfrak{k}]$.

PROOF. We have already remarked that Yamabe's theorem implies that $[H, K]$ is a Lie subgroup. Let $\mathfrak{m}$ denote its Lie algebra. We first show that $\mathfrak{I}[\mathfrak{h},\mathfrak{k}]\subset\mathfrak{m}$. If $X\in\mathfrak{h}$, $Y\in\mathfrak{k}$, then the curve

$$g(t)=\exp(\sqrt{t}X)\exp(\sqrt{t}Y)\exp(-\sqrt{t}X)\exp(-\sqrt{t}Y)$$

lies in $[H, K]$. Its tangent vector at 1 is $[X, Y]_1$. This shows that $[\mathfrak{h},\mathfrak{k}]\subset\mathfrak{m}$. Lemma 3 implies that $\mathfrak{m}$ is invariant under $ad(\mathfrak{h})$ and $ad(\mathfrak{k})$. Therefore $\mathfrak{I}[\mathfrak{h},\mathfrak{k}]\subset\mathfrak{m}$.

Now we show that $\mathfrak{m}\subset\mathfrak{I}[\mathfrak{h},\mathfrak{k}]$. Equivalently, we show that $[H, K]\subset L$, where L is the connected Lie subgroup whose Lie algebra is $\mathfrak{I}[\mathfrak{h},\mathfrak{k}]$. Let $h\in H$, $k\in K$, and let $h(t)$ and $k(t)$ be differentiable curves in H and K, respectively, so that $h(0)=1=k(0)$, $h(1)=h$ and $k(1)=k$. Let

$$g(t)=h(\sqrt{t})k(\sqrt{t})h(\sqrt{t})^{-1}k(\sqrt{t})^{-1}.$$

Lemma 2 says that

$$\left.\frac{dg}{dt}\right|_{t_0}\in\mathfrak{I}[\mathfrak{h},\mathfrak{k}]_{g(t_0)}\qquad(0\leq t_0\leq 1).$$

Therefore, the curve $g(t)$ lies in the maximal connected integral manifold (through 1) of the differential system $\mathfrak{I}[\mathfrak{h},\mathfrak{k}]$. In other words, the curve $g(t)$ lies in L. Thus, the commutators $hkh^{-1}k^{-1}\in L$, and consequently $[H, K]\subset L$.

Q.E.D.

COROLLARY 1: *Let* G *be a complex Lie group, and let* H *and* K *be connected (complex) Lie subgroups. Then* [H, K] *is a (complex) Lie subgroup.*

PROOF: Considering G with its real structure, we see that $[H, K]$ is a real Lie subgroup whose Lie algebra is $\mathfrak{I}[\mathfrak{h},\mathfrak{k}]$ (where $\mathfrak{h}$ and $\mathfrak{k}$ are the Lie algebras of H and K). Since $\mathfrak{h}$ and $\mathfrak{k}$ are complex, so is $\mathfrak{I}[\mathfrak{h},\mathfrak{k}]$ and $[H, K]$.

Q.E.D.

COROLLARY 2: *The Lie algebra of* [H, K] *is* $\mathscr{A}[\mathfrak{h},\mathfrak{k}]$, *if and only if* $\mathscr{A}[\mathfrak{h},\mathfrak{k}]$ *is invariant under* ad($\mathfrak{h}$) *and* ad($\mathfrak{k}$).

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