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Minimax Adjustment of a Univariate Distribution to Satisfy Componentwise Bounds and/or Ranking

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Consider a discrete probability distribution, represented by an n-vector **a**. This paper treats the problem of adjusting **a** as little as possible, in the sense of minimizing $\max_i|x_i - a_i|$, to obtain a distribution **x** which satisfies given componentwise bounds $\mathbf{L} \leq \mathbf{x} \leq \mathbf{U}$, or a given componentwise ranking, or both. The resulting linear programs are shown to admit special explicit solution algorithms.

Key words: Linear programs; mathematical models; minimax estimation; operations research; probability distribution.

1. Introduction

An n-vector **x** will be called a *probability vector* if its components x_i are nonnegative and sum to unity. This paper deals first with the following problem: given a probability n-vector **a**, and n-vectors **L** and **U**, to find a probability n-vector **x** which minimizes

$$F(\mathbf{x}) = \max_i|x_i - a_i| \tag{1.1}$$

subject to the constraints

$$\mathbf{O} \leq \mathbf{L} \leq \mathbf{x} \leq \mathbf{U}. \tag{1.2}$$

For the problem to be feasible, it is obviously necessary that

$$L_i \leq U_i \qquad \text{(all } i\text{)}, \tag{1.3}$$

$$\Sigma_i L_i \leq 1 \leq \Sigma_i U_i, \tag{1.4}$$

and so these conditions are imposed at the outset.

The motivating situation is one in which values must be attributed to the components x_i of an unknown discrete probability distribution. One type of information, e.g., data on some previous analogous situation, suggests the estimate **a**. Another, e.g., theoretical analyses or subjective opinions on the "present" situation, imposes the componentwise bounds (1.2). If **a** does not satisfy (1.2), a common procedure is to "adjust **a** as little as possible" so as to satisfy (1.2). Here the minimization of $F(\mathbf{x})$ is taken to express the "as little as possible" criterion in replacing **a** by a suitable **x**.

After some preliminaries are disposed of in section 2, a solution method for this problem is presented in section 3. In fact, the method is developed for the more general version in which (1.1) is replaced by

$$F(\mathbf{x}) = \max_i \{w_i|x_i = a_i|\}, \tag{1.5}$$

where the w_i are prescribed positive "weights." This corresponds to the case in which the accuracy, with which **x** approximates **a**, is more important for some components than for others.

Section 4 considers the analogous problem in which the componentwise-bound constraints (1.2) are replaced by a given componentwise ranking

$$x_1 \leq x_2 \leq \ldots \leq x_n. \tag{1.6}$$

Then, in section 5, the problem with *both* bounds *and* ranking is treated. (Both of these sections deal only with the "unweighted" objective function (1.1); the weighted version can be handled as a linear program, but our concern is with more explicit methods.)

Related work is found in [1, 2, 3][1]. The present paper, though self-contained, isolates in a form more convenient for reference some material appearing in [1].

2. Preliminaries

This section contains solution methods for four subproblems which will arise later. The material is presented for the sake of completeness; the same or similar problems have surely arisen in the literature as phases in other optimization analyses. In each case below, the solution method provides a constructive proof that certain obviously necessary conditions, for the existence of solutions, are also sufficient.

PROBLEM I. Given n-vectors $\mathbf{A}$ and $\mathbf{B}$, and number S, find an n-vector $\mathbf{y}$ such that

$$\mathbf{A} \leq \mathbf{y} \leq \mathbf{B}, \tag{2.1}$$

$$\Sigma_i y_i = S. \tag{2.2}$$

The conditions

$$\mathbf{A} \leq \mathbf{B}, \tag{2.3}$$

$$\Sigma_i A_i \leq S \leq \Sigma_i B_i, \tag{2.4}$$

which are clearly necessary for Problem I to have a solution, will be assumed to hold. If $S = \Sigma_i B_i$ then $\mathbf{y} = \mathbf{B}$ is clearly a solution, so we assume $S < \Sigma_i B_i$. Then $k = n$ has the property

$$\Sigma_{j \leq k} B_j + \Sigma_{j > k} A_j > S, \tag{2.5}$$

but $k = 0$ does not, and so there is a smallest $k \epsilon \{1, 2, \ldots, n\}$ with this property. For that k, not only (2.5) but also

$$\Sigma_{j < k} B_j + \Sigma_{j \geq k} A_j \leq S \tag{2.6}$$

holds. Now set

$$y_j = B_j \qquad \text{for } j < k,$$

$$y_j = A_j \qquad \text{for } j > k,$$

$$y_k = S - \Sigma_{j \neq k} y_j;$$

that $A_k \leq y_k \leq B_k$ follows from (2.5) and (2.6).

PROBLEM II. Given n-vector $\mathbf{Z}$, positive n-vector $\mathbf{w}$, and number S, find the minimum value z^* of z such that $z \geq 0$ and

$$\Sigma_1^n \max\{0, Z_i - z/w_i\} \leq S. \tag{2.7}$$

The condition $S \geq 0$, which is clearly necessary for Problem II to have a solution, will be assumed. Moreover, if $S = 0$, then each of the nonnegative summands in (2.7) would have to vanish, yielding

$$z^* = \max\{0, \max_i w_i Z_i\}$$

as the solution; thus from now on we assume $S > 0$. Furthermore if $Z_i \leq 0$, then the ith summand in (2.7) will vanish for *every* $z \geq 0$, and so such Z_i can be deleted from the problem in advance; if none are left then clearly $z^* = 0$ is the solution. So $Z_i > 0$ will be assumed.

[1]Figures in brackets indicate the literature references at the end of the paper.

Now choose $Z_{n+1}=0$ and any $w_{n+1}>0$, renumber so that

$$w_1 Z_1 \geqslant w_2 Z_2 \geqslant \cdot\ \cdot\ \cdot \geqslant w_n Z_n > w_{n+1} Z_{n+1} = 0,$$

and set

$$Z_j^* = \Sigma_1^j Z_i - w_j Z_j \Sigma_1^j (1/w_i).$$

The sequence $\{Z_j^*\}_1^{n+1}$ is given by the recursion

$$Z_{j+1}^* = Z_j^* + (w_j Z_j - w_{j+1} Z_{j+1})\ \Sigma_1^j (1/w_i),$$

which shows it to be nondecreasing. And *unless* $\Sigma_1^n Z_i < S$ (in which case $z^*=0$ is the solution), we have

$$Z_1^* = 0 < S \leqslant \Sigma_1^n Z_i = Z_{n+1}^*.$$

Thus there is a unique $J \epsilon \{1, 2, \ldots, n\}$ such that

$$0 = Z_1^* \leqslant Z_2^* \leqslant \ldots \leqslant Z_J^* < S \leqslant Z_{J+1}^* \leqslant \ldots \leqslant Z_{n+1}^*. \tag{2.8}$$

If $0 \leqslant z < w_{J+1} Z_{J+1}$, then

$$\Sigma_i \max\{0, Z_i - z/w_i\} \geqslant \Sigma_1^{J+1} Z_j - z\ \Sigma_1^{J+1} (1/w_j) > Z_{J+1}^* \geqslant S,$$

so that z does not satisfy (2.7). But if $w_{J+1} Z_{J+1} \leqslant z \leqslant w_J Z_J$, then (2.7) becomes

$$\Sigma_1^J Z_i - z\ \Sigma_1^J (1/w_i) \leqslant S$$

which is equivalent to

$$z \geqslant z^* = (\ \Sigma_1^J Z_j - S)/\Sigma_1^J (1/w_i). \tag{2.9}$$

By use of (2.8), the value of z^* proposed in (2.9) is easily verified to satisfy $w_{J+1} Z_{J+1} \leqslant z \leqslant w_J Z_J$, and so is indeed the smallest $z \geqslant 0$ obeying (2.7).

PROBLEM III. Given n-vector $\mathbf{Z}$, positive n-vector $\mathbf{w}$ and number S, find the minimum value z^{**} of z such that $z \geqslant 0$ and

$$\Sigma_1^n \min\{Z_i, z/w_i\} \geqslant S. \tag{2.10}$$

The condition

$$\Sigma_1^n Z_i \geqslant S, \tag{2.11}$$

which is obviously necessary if (2.10) is to have a solution, will be assumed. If equality holds in (2.11), then for each i the ith summand in (2.10) must equal Z_i, so that the solution is

$$z^{**} = \max\{0, \max_i w_i Z_i\};$$

thus from now on we assume strict inequality in (2.11). Moreover, if $Z_i < 0$ then for any $z \geqslant 0$, Z_i could be replaced by 0 on the left-hand side of (2.10) without change in value; hence it can be assumed that all $Z_i \geqslant 0$. Now $z^{**}=0$ will give the solution if $S \leqslant 0$, so we also assume $S>0$.

Choose $Z_{n+1}=0$ and any $w_{n+1}>0$, renumber so that

$$w_1 Z_1 \geqslant w_2 Z_2 \geqslant \ldots \geqslant w_n Z_n \geqslant w_{n+1} Z_{n+1} = 0,$$

and set

$$Z_j^{**} = w_j Z_j \Sigma_1^j (1/w_i) + \Sigma_{j+1}^n Z_i.$$

The sequence $\{Z_j^{**}\}_1^{n+1}$ obeys the recursion

$$Z_{j+1}^{**}=Z_j^{**}+(w_{j+1}Z_{j+1}-w_jZ_j)\ \Sigma_1^j\ (1/w_i)$$

and so is nonincreasing. Since

$$Z_{n+1}^{**}=0<S<\Sigma_1^nZ_i=Z_1^{**},$$

there is a unique $J\epsilon\{1, 2, \ldots, n\}$ such that

$$Z_1^*\geqslant Z_2^*\geqslant\ldots\geqslant Z_J^*\ \ \geqslant S>Z_{J+1}^*\geqslant\ldots\geqslant Z_{n+1}^*=0. \tag{2.12}$$

Now if $z\leqslant w_{J+1}$, then

$$\Sigma_i\ \min\{Z_i, z/w_i\}\leqslant z\Sigma_1^{J+1}(1/w_i)+\Sigma_{J+2}^nZ_1\leqslant Z_{J+1}^{**}<S$$

so that z does not satisfy (2.10). But if $w_{J+1}Z_{J+1}\leqslant z\leqslant w_JZ_J$, then (2.10) becomes

$$z\Sigma_1^J(1/w_i)+\Sigma_{J+1}^n Z_j\leqslant S$$

which is equivalent to

$$z\geqslant z^{**}=(S-\Sigma_{J+1}^nZ_j)/\Sigma_1^J(1/w_i). \tag{2.13}$$

By use of (2.12), the value of z^{**} proposed in (2.13) is easily verified to satisfy $w_{J+1}Z_{J+1}\leqslant z\leqslant w_JZ_J$, and so is indeed the smallest $z\geqslant 0$ obeying (2.10).

PROBLEM IV. Given n-vectors $\mathbf{A}$ and $\mathbf{B}$, and number S, find an n-vector $\mathbf{y}$ such that

$$\mathbf{A}\leqslant\mathbf{y}\leqslant\mathbf{B}, \tag{2.14}$$

$$\Sigma_iy_i=S, \tag{2.15}$$

$$y_1\leqslant y_2\leqslant\ldots\leqslant y_n. \tag{2.16}$$

Here it is convenient to define nondecreasing sequences $\{A_i'\}_1^n$ and $\{B_i'\}_1^n$, forming the components of respective vectors $\mathbf{A}'$ and $\mathbf{B}'$, by

$$A_i'=\max_{j\leqslant i}A_j,\qquad B_i'=\min_{j\geqslant i}B_j. \tag{2.17}$$

Then (2.16) and (2.14) are readily proved equivalent to (2.16) and

$$\mathbf{A}'\leqslant\mathbf{y}\leqslant\mathbf{B}'. \tag{2.14'}$$

Thus necessary conditions, for Problem IV to have a solution, are

$$\mathbf{A}'\leqslant\mathbf{B}'\qquad(\text{i.e.}, A_i\leqslant B_j \text{ for } i\leqslant j), \tag{2.18}$$

$$\Sigma_iA_i'\leqslant S\leqslant\Sigma_iB_i'\ . \tag{2.19}$$

These will be assumed to hold.

If $\mathbf{A}'=\mathbf{B}'$, then $\mathbf{y}=\mathbf{A}'=\mathbf{B}'$ is the solution. For $\mathbf{A}'\neq\mathbf{B}'$, define

$$\theta=[S-\Sigma_iA_i']/[\Sigma_iB_i'-\Sigma_iA_i']$$

and set

$$\mathbf{y}=\mathbf{A}'+\theta(\mathbf{B}'-\mathbf{A}').$$

Then (2.15) follows from the choice of θ, and (2.14') becomes $0 \leq \theta \leq 1$, which follows from (2.19). As for (2.16), $i \leq j$ implies that $A_i' \leq A_j'$ and $B_i' \leq B_j'$, so that

$$y_i = (1-\theta)A_i' + \theta B_i' \leq (1-\theta)A_j' + \theta B_j' = y_j .$$

(The same approach yields a simpler solution method for Problem I than the one given above.)

3. Solution for Componentwise Bounds

We now return to the problem posed at the beginning of the paper, with objective function (1.5). It can be rephrased as the following linear program: choose a number z and a probability n-vector $\mathbf{x}$, to minimize z subject to the conditions

$$\mathbf{0} \leq \mathbf{L} \leq \mathbf{x} \leq \mathbf{U}, \tag{3.1}$$

$$z \geq w_i(x_i - a_i) \qquad (\text{all } i), \tag{3.2}$$

$$z \geq w_i(a_i - x_i) \qquad (\text{all } i). \tag{3.3}$$

The constraints of the linear program, including the requirement that $\mathbf{x}$ be a probability vector, can be written as follows:

$$\max\{L_i,\ a_i - z/w_i\} \leq x_i \leq \min\{U_i,\ a_i + z/w_i\} \qquad (\text{all } i), \tag{3.3a}$$

$$\Sigma_i x_i = 1. \tag{3.3b}$$

A redundant constraint $z \geq 0$ can also be imposed. Thus the aim is to determine the smallest $z \geq 0$ for which the system (3.3a), (3.3b) has a solution $\mathbf{x}$.

For any fixed z, the system is an instance of Problem I in Section 2, with $S=1$ and

$$A_i = \max\{L_i,\ a_i - z/w_i\}, \qquad B_i = \min\{U_i,\ a_i + z/w_i\}.$$

By the analysis in Section 2, a solution $\mathbf{x}$ exists if and only if

$$\max\{L_i,\ a_i - z/w_i\} \leq \min\{U_i,\ a_i + z/w_i\} \qquad (\text{all } i), \tag{3.4}$$

$$\Sigma_i \max\{L_i,\ a_i - z/w_i\} \leq 1, \tag{3.5}$$

$$\Sigma_i \min\{U_i,\ a_i + z/w_i\} \geq 1. \tag{3.6}$$

So the objective is to determine the smallest value $z_{\min}$ of z which will satisfy (3.4), (3.5) and (3.6).

Now the left-hand side in (3.4) is nonincreasing in z, while the right-hand side is nondecreasing. The left-hand sides of (3.5) and (3.6) are respectively nonincreasing and nondecreasing in z. It follows that, if

$$z^\circ = \text{least } z \text{ obeying (3.4)},$$

$$z^* = \text{least nonnegative } z \text{ obeying (3.5)},$$

$$z^{**} = \text{least nonnegative } z \text{ obeying (3.6)},$$

then

$$z_{\min} = \max\{z^\circ,\ z^*,\ z^{**}\}. \tag{3.7}$$

Since $L_i \leq U_i$, (3.4) reduces to

$$a_i - z/w_i \leq U_i,\ L_i \leq a_i + z/w_i \qquad (\text{all } i),$$

and so z° is readily determined as

$$z^\circ = \max\{\max_i w_i(a_i - U_i),\ \max_i w_i(L_i - a_i)\}. \tag{3.8}$$

Next, (3.5) can be rewritten

$$\Sigma_i \max\{0, (a_i - L_i) - z/w_i\} \leq 1 - \Sigma_i L_i, \tag{3.9}$$

so that determining z^* is an instance of Problem II in Section 2, with

$$\mathbf{Z} = \mathbf{a} - \mathbf{L}, \qquad S = 1 - \Sigma_i L_i .$$

The feasibility condition $S \geq 0$ is satisfied by virtue of the first part of (1.4).

Finally, (3.6) is equivalent to

$$\Sigma_i \min\{U_i - a_i, z/w_i\} \geq 0, \tag{3.10}$$

so that the determination of z^{**} is an instance of Problem III in Section 2, with

$$\mathbf{Z} = \mathbf{U} - \mathbf{a}, \qquad S = 0.$$

The feasibility condition $\Sigma_1^n Z_i \geq S$ is satisfied by virtue of the second part of (1.4).

With z°, z^*, and z^{**} determined, $z_{\min}$ can be found from (3.7). Then a single optimizing $\mathbf{x}$ can be found by applying, to the previously-mentioned instance of Problem I with $z = z_{\min}$, the solution method given in Section 2. Concerning the nonuniqueness of $\mathbf{x}$, compare Section 5 of [1].

4. Solution for Componentwise Ranking

The next problem to be considered is the determination of a probability n-vector $\mathbf{x}$, among those which obey the componentwise ranking

$$x_1 \leq x_2 \leq \ldots \leq x_n, \tag{4.1}$$

which minimizes

$$F(\mathbf{x}) = \max_i |x_i - a_i| .$$

This too can be reformulated as a linear program, namely to select a number z and a vector $\mathbf{x} \geq 0$, so as to minimize z subject to

$$\Sigma_i x_i = 1, \tag{4.2}$$

$$z \geq x_i - a_i \qquad \text{(all } i\text{)}, \tag{4.3}$$

$$z \geq a_i - x_i \qquad \text{(all } i\text{)}. \tag{4.4}$$

Conditions (4.3) and (4.4), together with $\mathbf{x} \geq 0$, can be abbreviated to

$$\max\{0, a_i - z\} \leq x_i \leq a_i + z. \tag{4.5}$$

A redundant constraint $z \geq 0$ can also be imposed. Thus the aim is to determine the smallest $z \geq 0$ for which the system (4.1), (4.2), (4.5) has a solution $\mathbf{x}$.

For any fixed $z \geq 0$, the system is an instance of Problem IV in Section 2, with $S = 1$ and

$$A_i = \max\{0, a_i - z\}, B_i = a_i + z.$$

It is convenient to define vectors $\mathbf{a}^*$ and $\mathbf{a}^{**}$, with nondecreasing component sequences given by

$$a_i^* = \max_{j \leq i} a_i,\ a_i^{**} = \min_{j \geq i} a_i.$$

Then the vectors $\mathbf{A}'$ and $\mathbf{B}'$, described in Section 2's analysis of Problem 4, are given by

$$A_i' = \max\{0, a_i^* - z\}, B_i' = a_i^{**} + z.$$

The conditions (2.18) and (2.19), for Problem IV to be feasible, become

$$\max\{0,\ a_i - z\} \leq a_j + z \text{ for } i \leq j, \tag{4.6}$$

$$\Sigma_i \max\{0,\ a_i^* - z\} \leq 1 \leq \Sigma_i\,(a_i^{**} + z). \tag{4.7}$$

Now the objective is to find $z_{\min}$, the smallest $z \geq 0$ satisfying (4.6) and (4.7). Arguing as in Section 3, one finds that if

$z^0 =$ least nonnegative z obeying (4.6),

$z^* =$ least nonnegative z obeying first part of (4.7),

$z^{**} =$ least nonnegative z obeying second part of (4.7),

then

$$z_{\min} = \max\{z^0, z^*, z^{**}\}. \tag{4.8}$$

Since $z \geq 0$ and each $a_j \geq 0$, z^0 is readily determined from (4.6) as

$$z^0 = \max\{0,\ \max_{i \leq j}(a_i - a_j)/2\}. \tag{4.9}$$

Since $a_i^{**} \leq a_i$, implying

$$\Sigma_i a_i^{**} \leq \Sigma_i a_i = 1,$$

z^{**} is readily determined from (4.7) as

$$z^{**} = (1 - \Sigma_i a_i^{**})/n. \tag{4.10}$$

Finally, the determination of z^* is an instance of Problem II, with $S=1$, $w_i = 1$, and $Z_i = a_i^*$.

5. Solution for Componentwise Bounds and Ranking

The final version to be treated is the determination of a probability n-vector $\mathbf{x}$ which minimizes $\max_i|x_i - a_i|$, subject *both* to the ranking condition

$$x_1 \leq x_2 \leq \ .\ .\ .\ \leq x_n,$$

and to the componentwise bounds

$$\mathbf{O} \leq \mathbf{L} \leq \mathbf{x} \leq \mathbf{U}.$$

The analysis proceeds much as in the preceding Section, with (4.5) replaced by

$$\max\{L_i,\ a_i - z\} \leq x_i \leq \min\{U_i,\ a_i + z\}. \tag{5.1}$$

Again we have an instance of Problem IV, with $S=1$, and with

$$A_i = \max\{L_i,\ a_i - z\}, \qquad B_i = \min\{U_i,\ a_i + z\}.$$

As before, set

$$a_i^* = \max_{j \leq i} a_j,\ a_i^{**} = \min_{j \geq i} a_j,$$

and also put

$$L_i' = \max_{j \leq i} L_j, \qquad U_i' = \min_{j \geq i} U_j.$$

Then the vectors $\mathbf{A}'$ and $\mathbf{B}'$ are now given by

$$A_i' = \max\{L_i', a_i^* - z\}, \qquad B_i' = \min\{U_i', a_i^{**} + z\}. \tag{5.2}$$

We continue to have

$$z_{\min} = \max\{z^0, z^*, z^{**}\}, \tag{5.3}$$

where

$$z^0 = \text{least } z \text{ for which } \mathbf{A}' \leqslant \mathbf{B}',$$

$$z^* = \text{least nonnegative } z \text{ for which } \Sigma_i A_i' \leqslant 1,$$

$$z^{**} = \text{least nonnegative } z \text{ for which } \Sigma_i B_i' \geqslant 1.$$

The problem defining z^0, and hence the overall problem, has a solution only if

$$L_i \leqslant U_j \qquad \text{for } i \leqslant j. \tag{5.4}$$

Assuming this holds, the remaining conditions defining z^0 are (for $i \leqslant j$)

$$L_i \leqslant a_j + z,\ a_i - z \leqslant U_j,\ a_i - z \leqslant a_j + z,$$

and so

$$z^0 = \max_{i \leqslant j} \max\{L_i - a_j,\ a_i - U_j,\ (a_i - a_j)/2\}. \tag{5.5}$$

The condition defining z^* reads

$$\Sigma_i \max\{L_i', a_i^* - z\} \leqslant 1,$$

which can be rewritten

$$\Sigma_i \max\{0, (a_i^* - L_i') - z\} \leqslant 1 - \Sigma_i L_i'.$$

This is an instance of Problem II, with $w_i = 1$,

$$Z_i = a_i^* - L_i', \qquad S = 1 - \Sigma_i L_i'.$$

The consistency condition $S \geqslant 0$, i.e.,

$$\Sigma_i L_i' \leqslant 1, \tag{5.6}$$

is required for a solution to exist.

The condition defining z^{**} reads

$$\Sigma_i \min\{U_i', a_i^{**} + z\} \geqslant 1,$$

which can be rewritten

$$\Sigma_i \min\{U_i' - a_i^{**}, z\} \geqslant 1 - \Sigma_i a_i^{**}.$$

This is an instance of Problem III, with $w_i = 1$,

$$Z_i = U_i' - a_i^{**}, \qquad S = 1 - \Sigma_i a_i^{**}.$$

The consistency condition $\Sigma_i Z_i \geqslant S$, i.e.,

$$\Sigma_i U_i' \geqslant 1 \tag{5.7}$$

is required for a solution to exist.

Thus the conditions on the data $\mathbf{L}$ and $\mathbf{U}$, for the componentwise bounds and ranking to be consistent, are given by (5.4), (5.6) and (5.7).

6. References

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