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Minimax Error Selection of a Univariate Distribution With Prescribed Componentwise Bounds and Ranking

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The topic treated is that of finding a reproducible, plausible and computationally simple method of selecting a discrete frequency distribution with a prescribed ranking of its components and prescribed upper and lower bounds on these components. The problem is shown to be tractable when a minimax error selection criterion is employed, and "error" is measured by maximum absolute deviation among components. In this case one obtains a linear program of a special form admitting explicit solution.

Key words: Linear programs; mathematical models; minimax estimation; operations research; probability distribution.

1. Introduction

A discrete univariate probability distribution will be represented here as a probability n-vector, i.e., an n-vector $\mathbf{x}$ whose components x_i are nonnegative and sum to unity. Let P be some class of such distributions, describable as a closed subset of $\mathbf{x}$-space. Then one can pose the problem of *minimax error selection* of a member of P, i.e. of choosing $\mathbf{x}\epsilon P$ to minimize

$$F(\mathbf{x})=\max\{d(\mathbf{x},\mathbf{y}):\mathbf{y}\epsilon P\} \tag{1.1}$$

where d is some metric on n-space. The particular metric

$$d(\mathbf{x},\mathbf{y})=\max_i|x_i-y_i| \tag{1.2}$$

will be employed, essentially because of its tractability for what follows.

In [1],[1] this selection problem was solved for the case in which the class P of admissable distributions was specified by componentwise bounds, i.e.,

$$\mathbf{O}\leq\mathbf{L}\leq\mathbf{x}\leq\mathbf{U} \tag{1.3}$$

where $\mathbf{L}$ and $\mathbf{U}$ are given n-vectors. (It was also solved for a more general "weighted" version of the metric d.) In [2] it was solved for the case in which P was described by a given componentwise ranking,

$$x_1\leq x_2\leq\ .\ .\ .\ \leq x_n. \tag{1.4}$$

The present note gives a solution method for the case in which both types of constraint are imposed, i.e., P is specified by (1.3) and (1.4) *together.*

Some preliminaries are presented in section 2. The formulation of the solution method is then begun in section 3, and completed in section 4.

[1] Figures in brackets indicate the literature references at the end of the paper.

2. Preliminaries

This section considers three subproblems which will arise later.

PROBLEM I: Given n-vectors $\mathbf{A}$ and $\mathbf{B}$, and number S, find an n-vector $\mathbf{y}$ such that

$$\mathbf{A} \leqslant \mathbf{y} \leqslant \mathbf{B}, \tag{2.1}$$

$$\Sigma_i y_i = S, \tag{2.2}$$

$$y_1 \leqslant y_2 \leqslant \ldots \leqslant y_n. \tag{2.3}$$

This problem is treated in [3] (see "Problem IV" in section 2 of that paper), where a solution method is given, and the following two conditions are shown to be jointly necessary and sufficient for the existence of a solution. First,

$$A_i \leqslant B_j \quad \text{for} \quad i \leqslant j. \tag{2.4}$$

In terms of the nondecreasing sequences $\{A_i'\}_1^n$ and $\{B_i'\}_1^n$ defined by

$$A_i' = \max_{j \leqslant i} A_j, \qquad B_i' = \min_{j \geqslant i} B_j, \tag{2.5}$$

the second condition is

$$\Sigma_i A_i' \leqslant S \leqslant \Sigma_i B_i'. \tag{2.6}$$

PROBLEM II: Given n-vector $\mathbf{Z}$ and number S, find the minimum value z^* of z such that $z \geqslant 0$ and

$$\Sigma_i \max\{0, Z_i - z\} \leqslant S. \tag{2.7}$$

This problem is a special case of one treated in [3] (see "Problem II" in section 2 of that paper), where a solution method is given, and

$$S \geqslant 0 \tag{2.8}$$

is shown to be a necessary and sufficient condition for a solution to exist.

PROBLEM III: Given n-vector $\mathbf{Z}$ and number S, find the minimum value z^{**} of z such that $z \geqslant 0$ and

$$\Sigma_i \min\{Z_i, z\} \geqslant S. \tag{2.9}$$

This problem too is a special case of one treated in [3] (see "Problem III" in section 2 of that paper), where a solution method is given, and

$$\Sigma_i Z_i \geqslant S \tag{2.10}$$

is shown to be a necessary and sufficient condition for a solution to exist.

3. Analysis

We turn now to the original problem, concerning the constraint set

$$P = \{\mathbf{x}: 0 \leqslant \mathbf{L} \leqslant \mathbf{x} \leqslant \mathbf{U}, \Sigma_i x_i = 1, \tag{3.1}$$
$$x_1 \leqslant x_2 \leqslant \ldots \leqslant x_n\}.$$

Recall that the problem is to find $\mathbf{x} \epsilon P$ to minimize

$$F(\mathbf{x}) = \max\{\max_i |x_i - y_i| : \mathbf{y} \epsilon P\}. \tag{3.2}$$

As a first application of the material in the preceding section, consider the question of whether the constraints are consistent, i.e., whether P is nonempty. The existence of a $\mathbf{y} \epsilon P$ is an instance of Problem I, with

$$\mathbf{A} = \mathbf{L},\ \mathbf{B} = \mathbf{U},\ S = 1.$$

Thus the necessary and sufficient conditions for consistency are that

$$L_i \leqslant U_j \qquad \text{for } i \leqslant j, \tag{3.3}$$

and, in terms of the quantities

$$L_i' = \max_{j \leqslant i} L_j,\ \ U_i' = \min_{j \geqslant i} U_j, \tag{3.4}$$

that

$$\Sigma_i L_i' \leqslant 1 \leqslant \Sigma_i U_i'. \tag{3.5}$$

These conditions will be assumed satisfied from here on. Note that P can be rewritten

$$P = \{\mathbf{x} : 0 \leqslant \mathbf{L}' \leqslant \mathbf{x} \leqslant \mathbf{U}',\ \ \Sigma_i x_i = 1, \tag{3.6}$$

$$x_1 \leqslant x_2 \leqslant \ .\ .\ .\ \leqslant x_n\}$$

where $\mathbf{L}'$ and $\mathbf{U}'$ are the vectors with respective components L_i' and U_i'.

Next, set

$$M_i^+ = \max\ \{x_i : \mathbf{x} \epsilon P\}, \tag{3.7}$$

$$M_i^- = \min\ \{x_i : \mathbf{x} \epsilon P\}; \tag{3.8}$$

the determination of these quantities will be discussed later. As in [1] and [2], we have

$$\begin{aligned} F(\mathbf{x}) &= \max_y \max_i \max\ \{y_i - x_i,\ x_i - y_i\} \\ &= \max_i \max\ \{\max_y (y_i - x_i),\ \max_y (x_i - y_i)\} \\ &= \max_i \max\ \{M_i^+ - x_i,\ x_i - M_i^-\}. \end{aligned} \tag{3.9}$$

By (3.9), the minimization of $F(\mathbf{x})$ is equivalent to the following linear program; choose number z and vector $\mathbf{x} \epsilon P$, to minimize z subject to the further conditions

$$z \geqslant M_i^+ - x_i, \qquad z \geqslant x_i - M_i^- \qquad (\text{all } i).$$

For fixed z, the conditions on $\mathbf{x}$ read

$$\max\ \{L_i',\ M_i^+ - z\} \leqslant x_i \leqslant \min\ \{U_i',\ M_i^- + z\} \qquad (\text{all } i), \tag{3.10}$$

$$\Sigma_i x_i = 1, \tag{3.11}$$

$$x_i \leqslant x_2 \leqslant \ .\ .\ .\ \leqslant x_n. \tag{3.12}$$

Thus the objective is to find the smallest z for which such an $\mathbf{x}$ exists (and to find one).

Now finding an $\mathbf{x}$ to satisfy (3.10) through (3.12) is an instance of Problem I, with $S = 1$ and

$$A_i = \max\ \{L_i',\ M_i^+ - z\},\ B_i = \min\ \{U_i',\ M_i^- + z\}.$$

Since sequences $\{L_i'\}_1^n$ and $\{U_i'\}_1^n$ are nondecreasing, while definitions (3.7–8) and the definition of P imply that $\{M_i^+\}_1^n$ and $\{M_i^-\}_1^n$ are nondecreasing, it follows that $\{A_i\}_1^n$ and $\{B_i\}_1^n$ are nondecreasing. Hence in this case, the conditions for the existence of a solution to Problem I become

$$\max\ \{L_i',\ M_i^+ - z\} \leqslant \min\ \{U_i',\ M_i^- + z\} \qquad (\text{all } i), \tag{3.13}$$

$$\Sigma_i \max \{L_i', M_i^+ - z\} \leqslant 1, \tag{3.14}$$

$$\Sigma_i \min \{U_i', M_i^- + z\} \geqslant 1. \tag{3.15}$$

We seek $z_{\min}$, the smallest value of z satisfying these three conditions. The redundant condition $z \geqslant 0$ can also be imposed.

Let us set

$$z^\circ = \text{least } z \text{ satisfying (3.13)},$$
$$z^* = \text{least nonnegative } z \text{ obeying (3.14)},$$
$$z^{**} = \text{least nonnegative } z \text{ obeying (3.15)}.$$

Then we have

$$z_{\min} = \max \{z^\circ, z^*, z^{**}\}; \tag{3.16}$$

this is the case since the left-hand and right-hand sides in (3.13) are respectively nonincreasing and nondecreasing functions of z, while the sums in (3.14) and (3.15) are respectively nonincreasing and nondecreasing.

Since $L_i' \leqslant U_i'$, the determination of z^0 involves the conditions

$$L_i' \leqslant M_i^- + z, \; M_i^+ - z \leqslant U_i', \; M_i^+ - z \leqslant M_i^- + z,$$

and leads to

$$z^0 = \max_i \max \{L_i' - M_i^-, M_i^+ - U_i', (M_i^+ - M_i^-)/2\}. \tag{3.17}$$

Next, (3.14) is equivalent to

$$\Sigma_i \max \{0, (M_i^+ - L_i') - z\} \leqslant 1 - \Sigma_i L_i',$$

so that determining z^* is an instance of Problem II with

$$Z_i = M_i^+ - L_i', \qquad S = 1 - \Sigma_i L_i'.$$

Finally, (3.15) is equivalent to

$$\Sigma_i \min \{U_i' - M_i^-, z\} \geqslant 1 - \Sigma_i M_i^-,$$

so that determining z^{**} is an instance of Problem III, with

$$Z_i = U_i' - M_i^-, \qquad S = 1 - \Sigma_i M_i^-.$$

4. Determination of M_i^+ and M_i^-

It only remains to discuss the determination of the quantities M_i^+ and M_i^- defined by (3.7) and (3.8). For this purpose, consider the conditions under which

$$\mathbf{x} = (y_1, \ldots, y_{i-1}, x_i, y_i, \ldots, y_{n-1}) \epsilon P.$$

These conditions are

$$L_i' \leqslant x_i \leqslant U_i', \tag{4.1}$$

$$L_j' \leqslant y_j \leqslant \min \{U_j', x_i\} \qquad \text{for } j < i, \tag{4.2}$$

$$\max \{L_{j+1}', x_i\} \leqslant y_j \leqslant U_{j+1}' \qquad \text{for } j \geqslant i, \tag{4.3}$$

$$\Sigma_j y_j = 1 - x_i, \tag{4.4}$$

$$y_1 \leq y_2 \leq \ . \ . \ . \leq y_{n-1}. \tag{4.5}$$

Now the determination of an $(n-1)$-vector $\mathbf{y}$, satisfying (4.2–5), is an instance of Problem I with $n-1$ replacing n, with $S=1-x_i$, and with

$$A_j = L'_j \text{ and } B_j = \min\{U'_j, x_i\} \qquad \text{for } j < i,$$

$$A_j = \max\{L'_{j+1}, x_i\} \text{ and } B_j = U'_{j+1} \text{ for } j \geq i.$$

Since $\{L'_j\}_1^n$ and $\{U'_j\}_1^n$ are nondecreasing, and (4.1) holds, the sequences $\{A_j\}_1^{n-1}$ and $\{B_j\}_1^{n-1}$ are also nondecreasing. Thus the conditions in this case, for Problem I to have a solution $\mathbf{y}$, are

$$L'_j \leq \min\{U'_j, x_i\} \qquad \text{for } j < i, \tag{4.6}$$

$$\max\{L'_{j+1}, x_i\} \leq U'_{j+1} \qquad \text{for } j \geq i, \tag{4.7}$$

$$\Sigma_{j<i} L'_j + \Sigma_{j>i} \max\{L'_j, x_i\} \leq 1 - x_i, \tag{4.8}$$

$$\Sigma_{j<i} \min\{U'_j, x_i\} + \Sigma_{j>i} U'_j \geq 1 - x_i. \tag{4.9}$$

Conditions (4.6) and (4.7) are automatically satisfied. Condition (4.8) can be rewritten

$$x_i + \Sigma_{j>i} \max\{L'_j, x_i\} \leq 1 - \Sigma_{j<i} L'_j; \tag{4.10}$$

the left-hand side is a continuous increasing function of x_i, bounded neither above nor below, which therefore equals the right-hand side for a unique value x_i^+ of x_i. Condition (4.9) can be rewritten

$$x_i + \Sigma_{j<i} \min\{U'_j, x_i\} \geq 1 - \Sigma_{j>i} U'_j; \tag{4.11}$$

again the left-hand side is an increasing function of x_i, which equals the right-hand side for a unique value x_i^- of x_i. It follows that (4.8) and (4.9) are equivalent to

$$x_i^- \leq x_i \leq x_i^+.$$

By combination with (4.1), this yields

$$M_i^+ = \min\{U'_i, x_i^+\}, \tag{4.12}$$

$$M_i^- = \max\{L'_i, x_i^-\}. \tag{4.13}$$

Now the determination of x_i^+ and x_i^- must be discussed.

To determine x_i^+, form the quantities

$$L_k^* = (k-i) L'_k + \Sigma_{j \geq k} L'_j \qquad (k \geq i).$$

The sequence $\{L_k^*\}_i^n$ is nondecreasing, since

$$L_{k+1}^* - L_k^* = (k+1-i)(L'_{k+1} - L'_k) \geq 0.$$

We have

$$1 - \Sigma_{j<i} L'_j \geq \Sigma_{j \geq i} L'_j = L_i^*,$$

and so there is a last $k \in \{i, i+1, \ . \ . \ ., n\}$ for which

$$1 - \Sigma_{j<i} L'_j \geq L_k^*.$$

For this k, it is readily verified that

$$x_i^+ = [1 - \Sigma_{j<i} L'_j - \Sigma_{j>k} L'_j]/(k+1-i) \tag{4.14}$$

satisfies $x_i^+ \epsilon [L'_k, L'_{k+1}]$ . . . or $x_i^+ \geq L'_n$ if $k=n$ . . . and also satisfies (4.10) as an equation.
To determine x_i^-, form the quantities

$$U_k^* = (i-k)\, U'_k + \Sigma_{j \leq k} U'_j \qquad (k \leq i).$$

The sequence $\{U_k^*\}_1^i$ is nondecreasing, since

$$U_k^* - U_{k-1}^* = (i-k+1)\,(U'_k - U'_{k-1}) \geq 0.$$

We have

$$1 - \Sigma_{j>i}\; U'_j \leq \Sigma_{j \leq i}\; U'_j = U_i^*,$$

and so there is a *least* $k \epsilon \{1, 2, \ldots, i\}$ for which

$$1 - \Sigma_{j>i}\; U'_j \leq U_k^*.$$

For this k, it is readily verified that

$$x_i^- = \left[1 - \Sigma_{j>i}\; U'_j - \Sigma_{j<k}\; U'_j\right] / (i+1-k) \qquad (4.15)$$

satisfies $x_i^- \epsilon [U'_{k-1}, U'_k]$ . . . or $x_i^- \leq U'_1$ if $k=1$ . . . and also satisfies (4.11) as an equation.

5. References

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