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Principal Ideals in Matrix Rings

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It is shown that every left ideal of the complete matrix ring of a given order over a principal ideal ring is principal, and a partial converse is proven.

Key words: Dedekind ring; matrix ring; non-Noetherian ring; principal ideal ring.

1. Introduction

Let R be a ring with a unity 1, and let n be a positive integer. It is well-known [3, p. 37][1] that every two-sided ideal of R_n (the complete matrix ring of order n over R) is necessarily of the form M_n, where M is a two-sided ideal of R. Simple examples show that this result no longer holds for one-sided ideals. In this note we investigate the left ideals of R_n in the case when R is a principal ideal ring (an integral domain in which every ideal is principal). We shall prove

THEOREM 1: *If* R *is a principal ideal ring, then every left ideal of* $\mathrm{R_n}$ *is principal.*

The proof of Theorem 1 depends upon the fact that if A is any $p \times q$ matrix over R, then a unit matrix U of R_p exists such that the $p \times q$ matrix UA is upper triangular [2, p. 32].

We also establish the following partial converse to Theorem 1:

THEOREM 2: *If* R *is not Noetherian or if* R *is a Dedekind ring but not a principal ideal ring, then* $\mathrm{R_n}$ *contains a nonprincipal left ideal.*

For general information on rings, see [3]. For information on Dedekind rings, see [1, p. 101].

2. Proofs

We denote the matrix of R_n which has 1 in position (i, j) and 0 elsewhere by E_{ij}. We first prove

LEMMA 1: *Suppose that every left ideal of* R *has a finite* R*-basis. Then so has every left ideal of* $\mathrm{R_n}$.

PROOF: Let a be a left ideal of R_n. Let a_k, $2 \leq k \leq n$, be the subset of a consisting of all matrices of a whose first $k-1$ columns are 0; and set $a_1 = a$. Then, as is easily verified, a_k is a left ideal of R_n for $1 \leq k \leq n$.

Let M_{ik}, $1 \leq i \leq n$, be the set of elements of R occurring in the (i, k) position of all matrices of a_k, $1 \leq k \leq n$. Then M_{ik} is a left ideal of R (since a_k is a left ideal of R_n) and so has a finite R-basis, say

$$m_{ik}^{l},\ 1 \leq l \leq l_{ik}.$$

Hence we can find l_{ik} matrices of a_k, say A_{ik}^{l}, such that the (i, k)th entry of A_{ik}^{l} is m_{ik}^{l}. It follows that the l_{ik} matrices

$$B_{ik}^{l} = E_{ii}A_{ik}^{l},\ 1 \leq i,\ k \leq n,\ 1 \leq l \leq l_{ik},$$

also belong to a_k, have m_{ik}^{l} as their (i, k)th entry, but have nonzero entries in the ith row only. These

[1] Figures in brackets indicate the literature at the end of this paper.

matrices constitute a finite R-basis for a. For suppose that A is any element of a. We first find elements r_{i1}^l of R such that

$$A - \sum_{i=1}^{n} \sum_{l=1}^{l_{i1}} r_{i1}^l B_{i1}^l = A_2 \epsilon a_2;$$

we then find elements r_{i2}^l of R such that

$$A_2 - \sum_{i=1}^{n} \sum_{l=1}^{l_{i2}} r_{i2}^l B_{i2}^l = A_3 \epsilon a_3;$$

and continuing in this manner, we determine elements r_{ik}^l of R such that

$$A = \sum_{k=1}^{n} \sum_{i=1}^{n} \sum_{l=1}^{l_{ik}} r_{ik}^l B_{ik}^l.$$

This completes the proof.

We now prove Theorem 1. Let a be a left ideal of R_n. By Lemma 1, a possesses a finite R-basis, say $B_1, B_2, \ldots, B_t$. Let B be the $nt \times t$ matrix

$$B = \begin{pmatrix} B_1 \\ B_2 \\ \cdot \\ \cdot \\ \cdot \\ B_t \end{pmatrix}$$

Let U be a unit matrix of R_{nt} such that $UB = T$ is upper triangular. Thus

$$UB = T = \begin{bmatrix} H \\ 0 \end{bmatrix},$$

where H is an $n \times n$ upper triangular matrix, and the zero block 0 is $(nt - n) \times n$. We shall show that $a = R_n H$. For if we write $U = (U_{ij})$, where the matrices U_{ij} are $n \times n$, then

$$\sum_{j=1}^{t} U_{1j} B_j = H,$$

so that $H \epsilon a$, implying that

$$R_n H \subset a.$$

If we then write $U^{-1} = V = (V_{ij})$, where the matrices V_{ij} are $n \times n$, then V belongs to R_{nt} (since U is a unit matrix of R_{nt}) and from $B = VT$ we find that

$$B_i = V_{i1} H, \qquad 1 \leqslant i \leqslant t,$$

implying that

$$a \subset R_n H.$$

This completes the proof of Theorem 1.

To prove Theorem 2, we first observe that for any left ideal M of R, the left ideal M_n of R_n can be principal only if M has a set of n or fewer generators. In particular, if R is non-Noetherian, M can be chosen to violate this condition.

We now assume that R is a Dedekind ring and that any ideal in R can be generated by at most n elements. Let S be a nonprincial ideal in R, and let $\mathcal{S}$ be the subset of R_n consisting of all matrices with first column entries in S and all other entries arbitrary members of R. Clearly, $\mathcal{S}$ is a left ideal in R_n. We shall show that $\mathcal{S}$ is not principal.

Suppose the contrary. Let $X = (x_{ij})$ generate $\mathcal{S}$, so that $\mathcal{S} = R_n X$. Clearly the x_{i1} generate S; $S = \{x_{11}, x_{21}, \ldots, x_{n1}\}$. We may assume that x_{11} is not zero, since we may interchange the rows of X by left multiplication by a permutation matrix. Let $d = \det X$. Since $\mathcal{S}$ contains nonsingular matrices (for example, diag $(x_{11}, 1, \ldots, 1)$) X must be nonsingular and thus d is a nonzero element of S. Let $Y = (y_{ij})$ be the adjoint of X, so that

$$XY = YX = dI.$$

Then $Y \epsilon R_n$, and if C is any matrix in $\mathscr{S}$, every element of CY must be divisible by d. First choose $C = x_{i1} E_{11}$, $1 \leq i \leq n$. We obtain

$$\{d\} | \{x_{i1} y_{1j}\}, \qquad 1 \leq i, j \leq n. \tag{1}$$

Next choose $C = E_{1j}$, $2 \leq i \leq n$. We obtain

$$\{d\} | \{y_{ij}\}, \qquad 2 \leq i \leq n, \qquad 1 \leq j \leq n. \tag{2}$$

Put $y = \{y_{11}, y_{12}, \ldots, y_{1n}\}$. Then (2) implies that $y\{d\}^{n-1} | \{\det Y\}$; and since $\det Y = d^{n-1}$, $y = \{1\} = R$. Hence $\{y_{11}, y_{12}, \ldots, y_{1n}\} = \{1\} = R$. But now (1) and (2) imply that $\{d\} | \{x_{i1}\}$, $1 \leq i \leq n$. Write

$$x_{i1} = \beta_i d, \qquad \beta_i \epsilon R, \qquad 1 \leq i \leq n. \tag{3}$$

Since $d \epsilon S$ and the x_{i1} are a basis for S, elements r_i of R exist such that

$$d = \sum_{i=1}^{n} r_i x_{i1}.$$

But now (3) implies that

$$\sum_{i=1}^{n} r_i \beta_i = 1,$$

and hence

$$S = \{x_{11}, x_{21}, \ldots, x_{n1}\} = \{\beta_1, \beta_2, \ldots, \beta_n\}\{d\} = \{d\}.$$

Thus S is principal, a contradiction. This completes the proof of Theorem 2.

3. References

[1] Curtis, C., and Reiner, I., Representation theory of finite groups and associated algebras, Interscience (1962).
[2] MacDuffee, C. C., The theory of matrices, Chelsea (1946).
[3] McCoy, N. H., The theory of rings, Macmillan (1964).

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