

The Traffic Assignment Problem for a General Network*

Stella C. Dafermos¹ and Frederick T. Sparrow²

(February 18, 1969)

A transportation network is considered. The traffic demands associated with pairs of nodes and the (convex) traveling cost functions associated with the links are assumed given. The two problems of finding the traffic patterns which either minimize the total cost or equilibrate the users' costs are formulated, and algorithms are constructed for the solution of these problems.

Key words: Algorithm; least cost; traffic allocation; transportation.

Introduction

Many economic systems can be visualized as networks where nodes stand for commodities, and links and paths stand for simple and complex production processes. The type of system which can be thus described in the most natural way is probably a transportation network. In this case the nodes stand for "cities," the links stand for roads directly connecting two cities, and the paths stand for roads connecting two cities directly or indirectly.

A certain demand is associated with every pair of connected nodes of the network. This demand will be distributed among paths which join the pair of nodes. This gives rise to a traffic pattern, the determination of which is known as the *traffic assignment problem*. With every link of the network we associate a "traveling" cost which is assumed to be a function of the "traffic volume" on the link. We assume that the units traveling along this link uniformly share this cost.

In some cases the traffic pattern can be regulated by some central authority, as for example, a network used for the transportation of military supplies or for a railroad network. It is obvious that in this case, the problem which the central authority faces is to determine the traffic pattern which minimizes the total cost over the whole network.

On the other hand a broad class of transportation networks can be described as *user optimized*. Here travel patterns are set up by individual users each choosing the cheapest way (in the light of other users' decisions) to arrive at his respective destination, rather than having his travel pattern dictated by a choice consistent with some aggregate system optimum.

That the two above criteria lead generally to different traffic patterns was observed first by Pigou [1, p. 194]³ in an example of a simple two node, two link network. Interest in this problem has been revived by Wardrop [2], who calculates the traffic patterns according to the above two criteria for the case of a network consisting of two nodes connected by n independent paths and for a special cost function. Wardrop discusses briefly the case of a general network and sketches the equilibrium equations, but he does not discuss their solution.

Since 1952 several authors have reexamined the problem of flow patterns in a transportation network. For a complete bibliography we refer to a survey article by Beckmann [3]. We should observe here that two problems discussed by Wardrop, the problem of calculating the flow patterns according to the above two criteria, and the problem of planning an optimal investment allocation for improvement of the traffic network, still remain open.

*An invited paper. This work was undertaken under contract CST-1278 with the National Bureau of Standards. We wish to acknowledge the many constructive comments of Alan Goldman of the Applied Mathematics Division of the Bureau, and Dr. George Nemhauser of Johns Hopkins University.

¹Present address: Cornell University, Ithaca, N.Y. 14850.

²Present address: The Johns Hopkins University, Baltimore, Md. 21218.

³Figures in brackets indicate the literature references at the end of this paper.

Some progress towards the calculation of the flow patterns has been made. Almond [4] has constructed an algorithm for the solution of the user optimized network in the case of very simple networks. However, no extension of the algorithm for more complicated networks and no proof of convergence has been provided so far.

A different method of attack is based on the observation that the user optimization problem can itself be reformulated as a total cost minimization problem for an appropriately chosen objective function [5, 3].

When viewed in this manner the problem is of the "multicommodity network flow" class, which has been considered [13, 14] in the literature. Tomlin [14] has shown that for the case of linear cost (congestion) functions, the problem reduces to a linear programming problem that can be solved fairly efficiently by the Dantzig-Wolfe decomposition principle. Others [11] have suggested the use of convex programming techniques to get around the nonlinearity of the objective function. In fact, it was the enormous number of constraints associated with the convex programming formulation of the problem for the simplest of networks that led us to develop the special algorithms presented in the paper.

Returning to the Tomlin algorithm, it should be emphasized that his algorithm takes advantage of the linearity of the objective function; on the contrary, the success of ours hinges on the nonlinearity of the objective function, as will be demonstrated below. The algorithms should be viewed as a contribution to the theory of nonlinear multicommodity flow, as well as a contribution to the traffic flow literature.

In the present paper we mainly try to solve the open problem of the calculation of the traffic pattern in a general network, for the two criteria proposed by Wardrop. Some progress has been made [10] on Wardrop's resource allocation problem; these results will be reported in a later publication.

The paper is divided into two sections: the first concerns itself with problem formulation. Section 1.1 introduces the notation to be used, and the concept of a feasible flow pattern for a network. Section 1.2 describes two problems associated with transportation networks. The first, P_1 , is to find a feasible flow pattern that minimizes the total cost of traveling in the network; the second, P_2 , to find the feasible flow pattern that would be arrived at if users considered only their own interests in choosing these paths. Section 1.3 spells out the conditions that are assumed concerning the congestion functions for the links of the network, and then goes on to give the necessary and sufficient conditions for the existence, uniqueness, and stability of a solution for problem P_1 . Then the same conditions are derived for problem P_2 by showing that there is always a problem P_{12} associated with problem P_2 whose solution is that for P_2 , yet whose formulation is that of P_1 . This theorem is a translation into the present paper's perspective and notation of the result of Jorgensen [5] referred to before, and implies that every traffic assignment problem of a user optimized network can be solved by solving the associated problem of a total optimized network. In addition, we extend Jorgensen's work by examining the stability of user optimized networks, as well as giving a more general condition for the user optimized and total cost optimized travel patterns to coincide. The section concludes with the conditions on the congestion functions that will cause the solutions of P_1 and P_2 to coincide.

Section 2 constructs algorithms for the solution of problems P_1 and P_2 . Section 2.1 introduces the concept of an equilibration operator, and the conditions that must hold for such operators to obtain a solution to P_1 , referred to as the process of "inducing an algorithm for the solution of P_1 ." The section concludes with the introduction of the notion of disjoint paths. In section 2.2, we construct two equilibration operators, E_{dsj} for networks with disjoint paths, and E_{nds_j} , for any network. The operators are first applied to quadratic models, and we discuss under what conditions they induce algorithms for P_1 . In brief, we show that E_{dsj} induces an algorithm for *simple* (disjoint paths) and *almost simple* (see text) networks with quadratic cost functions, and give evidence that E_{dsj} converges rapidly to a solution. Next, we show that the operator E_{nds_j} induces an algorithm for problem P_1 for arbitrary networks with quadratic cost functions.

Section 2.3 extends the results of the previous section to cases where the cost function is required only to be twice continuously differentiable and convex, rather than quadratic. Section 2.4 briefly compares the two operators and presents the respective conditions that appear favorable for their use.

1. The Problem of the Traffic Distribution in a Transportation Network

1.1. Generalities

We start by introducing the concept of a transportation network. Let \mathcal{G} be a network in the sense of Ford and Fulkerson [7, ch. 1, sec. 1], i.e., \mathcal{G} is a pair $(\mathcal{N}, \mathcal{L})$ where \mathcal{N} is a collection of elements which will be called *nodes* and \mathcal{L} is a set of pairs of ordered elements of \mathcal{N} which will be called *links*.

By a path connecting the ordered pair $w = (x, y)$ of nodes we mean a sequence of links $(x_1, x_2), (x_2, x_3), \dots, (x_{n-1}, x_n)$ where x_1, x_2, \dots, x_n are distinct nodes, $x_1 = x$, and $x_n = y$. Thus a path here is a chain in the terminology of [7, ch. 1, sec. 1]. In particular, every link is a path. The set of all paths of \mathcal{G} will be denoted by \mathcal{P} . A pair w of nodes will be called connected if there exists at least one path connecting w . The set of all connected (ordered) pairs of nodes of \mathcal{G} will be denoted by \mathcal{W} . The set of all allowable travel paths which connect a w will be denoted by \mathcal{P}_w^A .

With every $w = (x, y) \in \mathcal{W}$ we associate a nonnegative demand d_w for travel with origin x and destination y . This demand will be distributed among all paths in \mathcal{P}_w . Suppose that $p \in \mathcal{P}_w$. By f_p we denote the part of d_w which travels through p . Thus we have the conservation equations

$$d_w = \sum_{p \in \mathcal{P}_w} f_p. \quad (1.1)$$

We define

$$\mathcal{F} \equiv \{f_p : p \in \mathcal{P}\}, \quad \mathcal{D} \equiv \{d_w : w \in \mathcal{W}\}. \quad (1.2)$$

A fixed value of \mathcal{F} will be called a *flow pattern* since it characterizes completely the flow. In the present paper we assume that the traffic flows are nonnegative real numbers and that the links of the network have infinite capacity.

We will assume that a cost c_a is associated with every $a \in \mathcal{L}$ of \mathcal{G} . The value of c_a is assumed to be a function of the total amount of traffic \bar{f}_a through a . That is,

$$c_a = c_a(\bar{f}_a) \quad (1.3)$$

where

$$\bar{f}_a = \sum_{p \in \mathcal{P}} \delta_{ap} f_p \quad (1.4)$$

with

$$\delta_{ap} = \begin{cases} 1 & \text{if } a \text{ is contained in } p, \\ 0 & \text{otherwise.} \end{cases} \quad (1.5)$$

We define

$$\bar{\mathcal{F}} \equiv \{\bar{f}_a : a \in \mathcal{L}\}, \quad \mathcal{C} \equiv \{c_a(\bar{f}_a) : a \in \mathcal{L}, \bar{f}_a \in [0, \infty)\}.$$

The triple $\mathcal{T} \equiv \{\mathcal{G}, \mathcal{D}, \mathcal{C}\}$ will be called a *transportation network*.

Throughout the paper we consider problems of the following type: A transportation network \mathcal{T} is given and the flow pattern \mathcal{F} is the basic unknown. So far \mathcal{F} has to conform to the conserva-

⁴ We assume that \mathcal{P}_w is a reasonably small set which can be enumerated in advance with little difficulty. This assumption, certainly a plausible one to make for traffic networks, avoids the problem of computing all paths in a network, an enormously time consuming task for large networks.

tion equation (1.1). An \mathcal{F} which satisfies (1.1) will be called a *feasible flow pattern*. The set of all feasible flow patterns (for fixed \mathcal{G} , \mathcal{D}) will be denoted by $\mathcal{L}[\mathcal{G}, \mathcal{D}]$. It is obvious that there exists a unique feasible flow pattern only in the case for which for any $w \in \mathcal{W}$ which is connected by more than one path, $d_w = 0$ holds. Leaving aside this trivial case, we observe that there is an infinity of feasible flow patterns.

We are now ready to formulate the two basic problems with which we will deal in the paper.

1.2. Formulation of the Problems P_1 and P_2

PROBLEM $P_1[\mathcal{T}]$: Given a transportation network $\mathcal{T} = (\mathcal{G}, \mathcal{D}, \mathcal{C})$, find a feasible solution $\mathcal{F}_1(\mathcal{T})$ which minimizes the total cost

$$C(\bar{\mathcal{F}}) \equiv \sum_{a \in \mathcal{L}} c_a(\bar{f}_a) \quad (1.6)$$

spent in the network.

A solution $\mathcal{F}_1(\mathcal{T})$ of problem $P_1[\mathcal{T}]$ will be called a “system optimizing” flow pattern.

As noted in the introduction, this is a reasonable problem but in many cases the network is in fact “user optimized.” Each user of a link a will be charged with a portion of the total cost c_a on this link. It is natural to assume that there is full interaction between all units traveling on link a ; that is, the cost is distributed uniformly among them. Thus, the share of the cost of each unit traveling on a will be given by

$$\bar{c}_a = \bar{c}_a(\bar{f}_a) \equiv \frac{c_a(\bar{f}_a)}{\bar{f}_a} \quad (1.7)$$

In consequence, the personal cost \bar{c}_p of a unit traveling on $p \in \mathcal{P}$ will be given by

$$\bar{c}_p = \sum_{a \in \mathcal{L}} \delta_{ap} \bar{c}_a \quad (1.8)$$

where the incidence symbols δ_{ap} have been introduced by (1.5).

In order to make clear the notion of a flow pattern which is “user optimized,” we introduce the following definition.

DEFINITION (1.1): For given $\mathcal{T} = (\mathcal{G}, \mathcal{D}, \mathcal{C})$, by an *equilibrium flow pattern* \mathcal{F}' we mean a feasible flow pattern with the following property. Let $w \in \mathcal{W}$ such that $d_w > 0$. Choose any $p \in \mathcal{P}_w$ for which $f'_p > 0$, and any number Δf , $0 < \Delta f < f'_p$. Consider another path $q \in \mathcal{P}_w$. Then the individual cost $\bar{c}_p(\mathcal{F}') \Delta f$ of Δf in the original flow pattern \mathcal{F}' is not greater than the individual cost $\bar{c}_q(\mathcal{F}'') \Delta f$ in the flow pattern \mathcal{F}'' defined by

$$\begin{aligned} f''_p &\equiv f'_p - \Delta f, \\ f''_q &\equiv f'_q + \Delta f, \\ f''_r &\equiv f'_r, \quad r \in \mathcal{P}, r \neq p, q.^5 \end{aligned} \quad (1.9)$$

In other words, an equilibrium flow pattern is an equilibrium point in the sense of Nash (e.g. [8, sec. 7.8]) of the noncooperative game among the various users of the network. Having given the definition of an equilibrium flow pattern, we now formulate problem P_2 .

PROBLEM $P_2[\mathcal{T}]$: Given a transportation network \mathcal{T} , find an equilibrium flow pattern $\mathcal{F}_2 = \mathcal{F}_2(\mathcal{T})$.

⁵ Under the assumption $\mathbf{4}_2$ to be imposed shortly, this condition implies that no distribution of Δf among several paths of $\mathcal{P}_w - \{p\}$ reduces the cost for Δf .

1.3. Study of the Solutions to Problems P_1 and P_2

It is not to be expected that the problems P_1, P_2 , formulated above are well posed unless some conditions are set on the form of the cost functions $c_a(\bar{f}_a)$. Whenever we consider the problem P_1 we will assume that the above functions satisfy the following assumptions for all $a \in \mathcal{L}$.

1. $c_a(\bar{f}_a)$ is continuous on $[0, \infty)$.
2. $c_a(0) = 0$.
3. $c_a(\bar{f}_a)$ is strictly increasing on $[0, \infty)$.
4. $c_a(\bar{f}_a)$ is strictly convex on $[0, \infty)$.

The interpretation of conditions 1–3 is obvious. Conditions 4₁, for differentiable $c_a(\bar{f}_a)$, means that the rate of increase of the cost, i.e., the marginal cost, is a strictly increasing function of the traffic flow \bar{f}_a (congestion effect).

Whenever we consider the problem P_2 we will assume that conditions 1–3 above are satisfied but, in the place of 4₁, we will impose the condition

- 4₂. $\bar{c}_a(\bar{f}_a)$ is strictly increasing on $[0, \infty)$

with the understanding that

$$\bar{c}_a(0) \equiv \lim_{\bar{f}_a \rightarrow 0^+} \frac{c_a(\bar{f}_a)}{\bar{f}_a}.$$

Assumption 4₂ provides a slightly different interpretation of the congestion effect with the emphasis placed on the individual rather than on the marginal cost. In fact conditions 1, 2, 3, 4₁ imply condition 4₂. (However, conditions 1, 2, 3, 4₂ do not imply, in general, condition 4₁.)

The simplest model which satisfies the above requirements corresponds to a cost function of the form

$$c_a(\bar{f}_a) = g_a \bar{f}_a^2 + h_a \bar{f}_a, \quad g_a > 0, \quad h_a \geq 0 \quad (1.10)$$

and will be called the *quadratic model*. In this model the congestion effect depends linearly on the traffic flow.

Having specified the admissible form of \mathcal{C} , let us consider the problem $P_1[\mathcal{T}]$, $\mathcal{T} = (\mathcal{G}, \mathcal{D}, \mathcal{C})$. Recall that this problem calls for the vector $(\mathcal{F}, \bar{\mathcal{F}})$ which solves the minimization problem:

$$\min C(\bar{\mathcal{F}}) = \sum_{a \in \mathcal{L}} c_a(\bar{f}_a) \quad (1.11)$$

subject to

$$\bar{f}_a - \sum_{p \in \mathcal{P}} \delta_{ap} f_p = 0, \quad a \in \mathcal{L},$$

$$f_p \geq 0, \quad p \in \mathcal{P}, \quad (1.12)$$

$$\sum_{p \in \mathcal{P}_w} f_p = d_w, \quad w \in \mathcal{W}.$$

Observe that a fixed \mathcal{F} induces a unique $\bar{\mathcal{F}}$ through (1.12)₁. But it is possible that more than one feasible \mathcal{F} induces the same $\bar{\mathcal{F}}$. The set of all feasible \mathcal{F} which induce a given fixed $\bar{\mathcal{F}}$ will be denoted by $R[\bar{\mathcal{F}}]$. On account of (1.12), $R[\bar{\mathcal{F}}]$ is a convex set. $\bar{\mathcal{F}}$ will be called *feasible* if $R[\bar{\mathcal{F}}]$ is non-empty. The set of all feasible $\bar{\mathcal{F}}$ will be denoted by $\mathcal{F}[\mathcal{G}, \mathcal{D}]$. In appendix I we give an example of a transportation network such that for some $\bar{\mathcal{F}} \in \mathcal{F}[\mathcal{G}, \mathcal{D}]$, $R[\bar{\mathcal{F}}]$ contains an infinite number of elements.

Note that if $(\mathcal{F}', \bar{\mathcal{F}}')$, $(\mathcal{F}'', \bar{\mathcal{F}}'')$ satisfy the constraints (1.12) so does any convex combination

$$(\mathcal{F}, \bar{\mathcal{F}}) \equiv \lambda'(\mathcal{F}', \bar{\mathcal{F}}') + \lambda''(\mathcal{F}'', \bar{\mathcal{F}}''), \quad \lambda', \lambda'' > 0, \quad \lambda' + \lambda'' = 1.$$

On the other hand and on account of 4₁,

$$C(\bar{\mathcal{F}}) \leq \lambda' C(\bar{\mathcal{F}}') + \lambda'' C(\bar{\mathcal{F}}'')$$

and equality may hold only if $\bar{\mathcal{F}}' = \bar{\mathcal{F}}''$. Consequently, P_1 is a convex minimization problem in $\mathcal{L} \oplus \bar{\mathcal{L}}$ and, in particular, it is strictly convex in $\bar{\mathcal{L}}$. Using the theory of convex programming and observing that the total cost function depends only on $\bar{\mathcal{F}}$, we arrive at the following theorem.

THEOREM (1.1): *Given $\mathcal{F} = (\mathcal{G}, \mathcal{D}, \mathcal{C})$, there exists a unique $\bar{\mathcal{F}}_1 \in \bar{\mathcal{L}}[\mathcal{G}, \mathcal{D}]$ such that $C(\bar{\mathcal{F}}_1)$ is the minimum of $C(\bar{\mathcal{F}})$ over $\bar{\mathcal{L}}[\mathcal{G}, \mathcal{D}]$. Every element $\mathcal{F}_1 \in R[\bar{\mathcal{F}}_1]$ is a solution of problem P_1 .*

Thus, problem P_1 always possesses solutions and in particular it possesses a unique solution if and only if $R[\bar{\mathcal{F}}_1]$ consists of a unique element.

In the special case where the $c_a(\bar{f}_a)$ are differentiable functions we can prove the following theorem.

THEOREM (1.2): *The flow pattern $\mathcal{F} \in \mathcal{L}$ is a solution of problem P_1 if and only if it has the following property. For any $w \in \mathcal{W}$ connected by precisely the paths p_1, \dots, p_m , these paths can be so numbered that*

$$\begin{aligned} c'_{p_1}(\bar{\mathcal{F}}) = \dots = c'_{p_s}(\bar{\mathcal{F}}) &\equiv M_w \leq c'_{p_{s+1}}(\bar{\mathcal{F}}) \leq \dots \leq c'_{p_m}(\bar{\mathcal{F}}), \\ f_{p_r} &> 0, \quad r = \bar{1}, \dots, s, \\ f_{p_r} &= 0, \quad r = s + 1, \dots, m, \end{aligned} \tag{1.13}$$

where we use the notation

$$c'_p(\bar{\mathcal{F}}) \equiv \sum_{a \in \mathcal{L}} \delta_{ap} c'_a(\bar{f}_a),$$

$$c'_a(\bar{f}_a) \equiv \frac{dc_a(\bar{f}_a)}{d\bar{f}_a}.$$

PROOF OF SUFFICIENCY: Assume that $\mathcal{F} \in \mathcal{L}$ satisfies (1.13). Let $\mathcal{F} + \Delta\mathcal{F} \in \mathcal{L}$ be a feasible reallocation. The change of the total cost is given by

$$\Delta C = \sum_{a \in \mathcal{L}} [c_a(\bar{f}_a + \Delta\bar{f}_a) - c_a(\bar{f}_a)].$$

Applying the mean value theorem and using the fact that the functions $c'_a(\bar{f}_a)$ are (strictly) increasing we obtain

$$\Delta C \geq \sum_{a \in \mathcal{L}} c'_a(\bar{f}_a) \Delta\bar{f}_a. \leftarrow \Delta\bar{f}_a \tag{1.14}$$

Recalling (1.4),

$$\Delta\bar{f}_a = \sum_{p \in \mathcal{P}} \delta_{ap} \Delta f_p. \tag{1.15}$$

Then,

$$\Delta C \geq \sum_{a \in \mathcal{L}} \sum_{p \in \mathcal{P}} \delta_{ap} c'_a(\bar{f}_a) \Delta f_p = \sum_{p \in \mathcal{P}} \Delta f_p c'_p(\bar{\mathcal{F}}). \tag{1.16}$$

Note that if $f_p = 0$, then $\Delta f_p \geq 0$. Thus, using (1.13),

$$\Delta f_p c'_p(\bar{\mathcal{F}}) \geq \Delta f_p M_w \tag{1.17}$$

where w is the pair of nodes which is connected by p . Recalling that $\sum_{p \in \mathcal{P}_w} \Delta f_p = 0$, we obtain from (1.16) and (1.17),

$$\Delta C \geq 0$$

which proves that \mathcal{F} is a solution of problem P_1 .

Q.E.D.

PROOF OF NECESSITY: Suppose that $\mathcal{F} \in \mathcal{L}$ is a solution of problem P_1 , but there exist paths $p, q \in \mathcal{P}_w$ such that $f_p > 0$ and

$$c'_p(\bar{\mathcal{F}}) - c'_q(\bar{\mathcal{F}}) = \epsilon > 0. \quad (1.18)$$

Assume now that a portion Δf of f_p is reallocated to the path q . The change of the total cost is given by

$$\Delta C = \sum_{a \in \mathcal{L}} \delta_{ap}^q [c_a(\bar{f}_a - \Delta f) - c_a(\bar{f}_a)] + \sum_{a \in \mathcal{L}} \delta_{aq}^p [c_a(\bar{f}_a + \Delta f) - c_a(\bar{f}_a)]$$

where

$$\delta_{ap}^q \equiv \begin{cases} \delta_{ap} & \text{if } a \text{ is not contained in } q, \\ 0 & \text{if } a \text{ is contained in } q, \end{cases} \quad (1.19)$$

and δ_{aq}^p is defined in an analogous fashion.

Applying the mean value theorem and recalling that $c'_a(\bar{f}_a)$ are (strictly) increasing functions we end up with

$$\Delta C < \left\{ - \sum_{a \in \mathcal{L}} \delta_{ap}^q c'_a(\bar{f}_a - \Delta f) + \sum_{a \in \mathcal{L}} \delta_{aq}^p c'_a(\bar{f}_a + \Delta f) \right\} \Delta f.$$

Now $\sum_{a \in \mathcal{L}} \delta_{ap}^q c'_a(\bar{f}_a - \Delta f)$, $\sum_{a \in \mathcal{L}} \delta_{aq}^p c'_a(\bar{f}_a + \Delta f)$ are continuous functions of Δf . Hence, we may choose a positive Δf (for feasibility it must be such that $f_p - \Delta f \geq 0$, whence $\Delta f \leq f_p$) such that

$$\sum_{a \in \mathcal{L}} \delta_{ap}^q c'_a(\bar{f}_a - \Delta f) > \sum_{a \in \mathcal{L}} \delta_{ap}^q c'_a(\bar{f}_a) - \frac{\epsilon}{3},$$

$$\sum_{a \in \mathcal{L}} \delta_{aq}^p c'_a(\bar{f}_a + \Delta f) < \sum_{a \in \mathcal{L}} \delta_{aq}^p c'_a(\bar{f}_a) + \frac{\epsilon}{3}$$

Hence,

$$\Delta C < - \sum_{a \in \mathcal{L}} \delta_{ap}^q c'_a(\bar{f}_a) \Delta f + \frac{\epsilon}{3} \Delta f + \sum_{a \in \mathcal{L}} \delta_{aq}^p c'_a(\bar{f}_a) \Delta f + \frac{\epsilon}{3} \Delta f.$$

It is easily seen that the above inequality may be written in the form

$$\Delta C < \left\{ - \sum_{a \in \mathcal{L}} \delta_{ap} c'_a(\bar{f}_a) + \sum_{a \in \mathcal{L}} \delta_{aq} c'_a(\bar{f}_a) + \frac{2\epsilon}{3} \right\} \Delta f$$

or, using (1.18),

$$\Delta C < -\frac{\epsilon}{3} \Delta f < 0$$

which is a contradiction to the assumption that \mathcal{F} is a solution of problem P_1 .

Q.E.D.

Actually it can be shown that (1.13) are simply the Kuhn-Tucker conditions (see [9, ch. 6]) for the minimization problem (1.11), (1.12). However, these conditions have been derived independently here in order to keep the paper self-contained.

In the case of the quadratic model, P_1 reduces to a quadratic programming problem and (1.13) become linear.

From the convexity of problem P_1 we can obtain additional information, namely that the solution is *stable*. To make this precise we introduce the following definition.

DEFINITION (1.2): Let $\epsilon > 0$. We say that $\bar{\mathcal{F}} \in \mathcal{L}$ satisfies (1.13) modulo ϵ if and only if for any p, q connecting any $w \in \mathcal{W}$ and such that $f_p \geq d\epsilon$, either

$$|c'_p(\bar{\mathcal{F}}) - c'_q(\bar{\mathcal{F}})| \leq \frac{c}{d}\epsilon \quad (1.20)$$

holds, or else both

$$c'_p(\bar{\mathcal{F}}) < c'_q(\bar{\mathcal{F}}) + \frac{c}{d}\epsilon \quad (1.21)$$

and

$$f_q < d\epsilon$$

where c, d are arbitrary but fixed magnitudes having dimensions of cost and traffic flow, respectively, and included in order to make ϵ dimensionless.

THEOREM (1.3): Let $\mathcal{T} = (\mathcal{G}, \mathcal{D}, \mathcal{C})$ be a transportation network with twice continuously differentiable cost functions, and \mathcal{F}' a solution of problem $P_1[\mathcal{T}]$. Then there exist numbers K and L which depend solely on \mathcal{T} , such that

$$C(\bar{\mathcal{F}}) - C(\bar{\mathcal{F}}') < cK\epsilon, \quad (1.22)$$

$$\|\bar{\mathcal{F}} - \bar{\mathcal{F}}'\|^2 \equiv \sum_{a \in \mathcal{L}} |\bar{f}_a - \bar{f}'_a|^2 < d^2L\epsilon, \quad (1.23)$$

for any $\bar{\mathcal{F}} \in \mathcal{L}[\mathcal{T}]$ which satisfies (1.13) modulo ϵ , $\epsilon > 0$.

PROOF: Assume that $\bar{\mathcal{F}} \in \mathcal{L}$ satisfies (1.13) modulo ϵ . We set $\Delta\bar{\mathcal{F}} \equiv \bar{\mathcal{F}}' - \bar{\mathcal{F}}$. As in the proof of Theorem (1.2) (see eq (1.16)), we have

$$C(\bar{\mathcal{F}}') - C(\bar{\mathcal{F}}) \geq \sum_{p \in \mathcal{P}} \Delta f_p c'_p(\bar{\mathcal{F}}). \quad (1.24)$$

We decompose $\mathcal{W}, \mathcal{W} = \mathcal{W}_1 \cup \mathcal{W}_2$, so that if $w \in \mathcal{W}_1$, then $f_p < d\epsilon$ for all $p \in \mathcal{P}_w$, while if $w \in \mathcal{W}_2$, then $f_p \geq d\epsilon$ for at least one $p \in \mathcal{P}_w$. In particular one of $\mathcal{W}_1, \mathcal{W}_2$ may be empty.

For $w \in \mathcal{W}_1$

$$\Delta f_p > -d\epsilon, \quad p \in \mathcal{P}_w.$$

Hence

$$\sum_{w \in \mathcal{W}_1} \sum_{p \in \mathcal{P}_w} \Delta f_p c'_p(\bar{\mathcal{F}}) \geq -c \frac{d}{c} \epsilon \sum_{w \in \mathcal{W}_1} \sum_{p \in \mathcal{P}_w} c'_p(\bar{\mathcal{F}}) \geq -cK_1\epsilon \quad (1.25)$$

where K_1 can be easily estimated in terms of elements of \mathcal{T} .

Fix now $w \in \mathcal{W}_2$, and let $\mathcal{P}_w = \{p_1, \dots, p_{m_w}\}$. Suppose that

$$f_{p_r} \geq d\epsilon, \quad r = 1, \dots, s, \quad \text{and} \quad f_{p_r} < d\epsilon, \quad r = s+1, \dots, m_w.$$

We write

$$\mu_{p_r} \equiv c'_{p_r}(\bar{\mathcal{F}}) - c'_{p_1}(\bar{\mathcal{F}}), \quad r = 1, \dots, m_w$$

and we observe that since $\bar{\mathcal{F}}$ satisfies (1.13) modulo ϵ ,

$$|\mu_{p_r}| < \frac{c}{d}\epsilon, \quad r = 1, \dots, s,$$

$$\mu_{p_r} > -\frac{c}{d}\epsilon, \quad r = s+1, \dots, m_w.$$

It is also easy to obtain an estimate of the form

$$\mu_{p_r} \leq \frac{c}{d}E_w, \quad r = s+1, \dots, m_w$$

where the number E_w depends at most on \mathcal{T} and w . For example

$$\frac{c}{d} E_w \equiv \max_r \sum_{a \in \mathcal{J}} \delta_{ap_r} c'_a \left(\sum_{w \in \mathcal{W}_a} d_w \right) \quad (1.26)$$

where \mathcal{W}_a is the set of all $w \in \mathcal{W}_2$ which are connected by at least one path containing a . Notice also the useful estimates

$$\begin{aligned} \sum_{p \in \mathcal{P}} \Delta f_p &= 0, \\ \Delta f_{p_r} &> -d\epsilon \quad , r = s+1, \dots, m_w, \\ |\Delta f_{p_r}| &\leq d_w \quad , r = 1, \dots, m_w. \end{aligned}$$

Hence

$$\begin{aligned} \sum_{p \in \mathcal{P}_w} \Delta f_p c'_p(\bar{\mathcal{F}}) &= c'_{p_1}(\bar{\mathcal{F}}) \sum_{p \in \mathcal{P}_w} \Delta f_p + \sum_{r=1}^{m_w} \mu_{p_r} \Delta f_{p_r} = \sum_{r=1}^s \mu_{p_r} \Delta f_{p_r} + \sum_{r=s+1}^{m_w} \mu_{p_r} \Delta f_{p_r} \\ &\geq \left\{ -sc \frac{d_w}{d} - (m_w - s)c \max \left(\frac{d_w}{d}, E_w \right) \right\} \epsilon \geq -m_w c \max \left(\frac{d_w}{d}, E_w \right) \epsilon. \end{aligned}$$

From this last inequality, (1.25), and (1.24) we deduce (1.22) for

$$K \equiv \sum_{w \in \mathcal{W}_2} m_w \max \left(\frac{d_w}{d}, E_w \right) + K_1. \quad (1.27)$$

We now proceed to the proof of (1.23). Note that

$$C(\bar{\mathcal{F}}) - C(\bar{\mathcal{F}}') = \sum_{a \in \mathcal{L}} \frac{\partial C}{\partial \bar{f}_a} \Delta \bar{f}_a + \frac{1}{2} \sum_{a, b \in \mathcal{L}} \frac{\partial^2 C}{\partial \bar{f}_a \partial \bar{f}_b} \Delta \bar{f}_a \Delta \bar{f}_b$$

with $\frac{\partial C}{\partial \bar{f}_a}$ calculated at the point $\bar{\mathcal{F}}'$ and $\frac{\partial^2 C}{\partial \bar{f}_a \partial \bar{f}_b}$ calculated at a fixed intermediate point $\bar{\mathcal{F}}' + \theta \Delta \bar{\mathcal{F}}$, $0 \leq \theta \leq 1$. Since $\bar{\mathcal{F}}'$ is a solution of problem P_1 and $\bar{\mathcal{F}}' + \Delta \bar{\mathcal{F}} = \bar{\mathcal{F}} \in \mathcal{L}$, then

$$\sum_{a \in \mathcal{L}} \frac{\partial C}{\partial \bar{f}_a} \Big|_{\bar{\mathcal{F}}'} \Delta \bar{f}_a = 0.$$

On the other hand

$$\frac{\partial^2 C}{\partial \bar{f}_a \partial \bar{f}_b} = \begin{cases} c''_a(\bar{f}_a), & \text{if } a = b, \\ 0, & \text{if } a \neq b. \end{cases}$$

Hence,

$$C(\bar{\mathcal{F}}) - C(\bar{\mathcal{F}}') = \frac{1}{2} \sum_{a \in \mathcal{L}} c''_a(\bar{f}_a) \Delta \bar{f}_a^2.$$

Since $c_a(\bar{f}_a)$ are strictly convex, the constant

$$\frac{c}{d^2} k \equiv \min_{a \in \mathcal{L}} \min_{\theta \in [0, 1]} c''_a(\bar{f}_a + \theta \Delta \bar{f}_a) \quad (1.28)$$

is positive and (1.23) follows with the help of (1.22) for

$$L \equiv \frac{2K}{k}. \quad (1.29)$$

Q.E.D.

The interpretation of the above theorem is that the solution of problem P_1 is stable. We emphasize here that, as follows from the proof, the constants K, L can be estimated explicitly in terms of known characteristics of the transportation network. Then, apart from its theoretical importance, the Theorem (1.3) is useful in practice since it provides a means of estimating the distance of a given feasible flow pattern from the solution of problem P_1 . An explicit application of the above observation will be presented in part 2 of the paper. With these comments we complete the study of problem P_1 .

We now proceed to a similar study for problem P_2 . We start by proving a theorem analogous to Theorem (1.2).

THEOREM (1.4): *The flow pattern $\mathcal{F} \in \mathcal{L}$ is a solution of problem P_2 if and only if it has the following property. For any $w \in \mathcal{W}$ connected by precisely the paths p_1, \dots, p_m , these paths can be so numbered that*

$$\begin{aligned} \bar{c}_{p_1}(\overline{\mathcal{F}}) = \dots = \bar{c}_{p_s}(\overline{\mathcal{F}}) &\equiv A_w \leq \bar{c}_{p_{s+1}}(\overline{\mathcal{F}}) \leq \dots \leq \bar{c}_{p_m}(\overline{\mathcal{F}}), \\ f_{p_r} &> 0, \quad r = 1, \dots, s, \\ f_{p_r} &= 0, \quad r = s+1 \dots, m. \end{aligned} \quad (1.30)$$

PROOF OF SUFFICIENCY: Assume that $\mathcal{F} \in \mathcal{L}$ satisfies (1.30). Let p, q be two paths connecting the same $w \in \mathcal{W}$ and such that $f_p > 0$. By (1.30)

$$\bar{c}_p(\overline{\mathcal{F}}) \leq \bar{c}_q(\overline{\mathcal{F}}). \quad (1.31)$$

Suppose that a portion Δf , of f_p , $0 < \Delta f \leq f_p$, selects the path q . By \mathcal{F}' we denote the resulting flow pattern. Recalling (1.19) we have

$$\bar{c}_q(\overline{\mathcal{F}'}) - \bar{c}_q(\overline{\mathcal{F}}) = \sum_{a \in \mathcal{L}} \frac{v_a}{v_q} [\bar{c}_a(\bar{f}_a + \Delta f) - \bar{c}_a(\bar{f}_a)] > 0$$

where use has been made of the fact that $\bar{c}_a(\bar{f}_a)$ is a strictly increasing function. In particular, recalling (1.31),

$$\bar{c}_q(\overline{\mathcal{F}'}) > \bar{c}_p(\overline{\mathcal{F}})$$

which shows that \mathcal{F} is an equilibrium point in the sense of Definition (1.1).

The proof of necessity is essentially a repetition of the proof of the necessity in Theorem (1.2) and will be omitted. Q.E.D.

The above conditions are nothing more than the standard average cost equality conditions an economist would expect to find in a system that is optimized by individuals acting independently of one another with no regard for total system optimization. Conditions (1.30) are known (but not in full generality) at least since the time of Pigou's treatise referred to in the introduction. Many authors consider the conditions themselves as a definition of problem P_2 .

Comparing (1.13) with (1.30), we observe that there exists a remarkable similarity between them. The role of the average cost \bar{c}_p in (1.30) is played in (1.13) by the marginal cost c'_p . Starting from the above observation, we will now show that there exists a close relationship between the set of problems P_1 and the set of problems P_2 .

DEFINITION (1.3): Given $P_2[\mathcal{F}] = P_2[\mathcal{G}, \mathcal{D}, {}_2\mathcal{C}]$, we construct a set of cost functions ${}_{21}\mathcal{C}$ in the following way. For any $a \in \mathcal{L}$ we set

$${}_{21}c_a(\bar{f}_a) \equiv \int_0^{\bar{f}_a} {}_2\bar{c}_a(\bar{f}) d\bar{f}. \quad (1.32)$$

Note that if ${}_2c_a$ satisfies conditions 1, 2, 3, 4₂, then ${}_{21}c_a$ satisfies conditions 1, 2, 3, 4₁. Furthermore, ${}_{21}c_a$ is continuously differentiable. The problem $P_1[\mathcal{G}, \mathcal{D}, {}_{21}\mathcal{C}]$ will be called problem P_1 associated with problem $P_2[\mathcal{G}, \mathcal{D}, {}_2\mathcal{C}]$ and will be denoted by $P_{12}[\mathcal{G}, \mathcal{D}, {}_2\mathcal{C}]$.

Similarly, given $P_1[\mathcal{G}, \mathcal{D}, {}_1\mathcal{C}]$ where ${}_1\mathcal{C}$ consists of continuously differentiable functions, we construct the set of cost functions ${}_{12}\mathcal{C}$ through the use of

$${}_{12}c_a(\bar{f}_a) \equiv {}_1c'_a(\bar{f}_a)\bar{f}_a. \quad (1.33)$$

Note that if ${}_1c_a(\bar{f}_a)$ satisfies conditions 1, 2, 3, 4₁, then ${}_{12}c_a(\bar{f}_a)$ satisfies conditions 1, 2, 3, 4₂. The problem $P_2[\mathcal{G}, \mathcal{D}, {}_{12}\mathcal{C}]$ will be called problem P_2 associated with problem $P_1[\mathcal{G}, \mathcal{D}, {}_1\mathcal{C}]$ and will be denoted by $P_{21}[\mathcal{G}, \mathcal{D}, {}_1\mathcal{C}]$.

The above given definition of the associated problem is justified by the following theorem.

THEOREM (1.5): Let \mathcal{F} be a solution of problem $P_2[\mathcal{G}, \mathcal{D}, \mathcal{C}]$. Then \mathcal{F} is also a solution of problem $P_{12}[\mathcal{G}, \mathcal{D}, \mathcal{C}]$. Similarly if \mathcal{F} is a solution of problem $P_1[\mathcal{G}, \mathcal{D}, \mathcal{C}]$ with \mathcal{C} consisting of continuously differentiable functions, then \mathcal{F} is also a solution of problem $P_{21}[\mathcal{G}, \mathcal{D}, \mathcal{C}]$.

PROOF: The proof follows from the construction of the associated cost functions. In fact we observe that (1.30) written for $P_2[\mathcal{G}, \mathcal{D}, {}_2\mathcal{C}]$ and (1.13) written for $P_1[\mathcal{G}, \mathcal{D}, {}_{21}\mathcal{C}]$ coincide. Similarly (1.13) written for $P_1[\mathcal{G}, \mathcal{D}, {}_{12}\mathcal{C}]$ coincides with (1.30) written for $P_2[\mathcal{G}, \mathcal{D}, {}_{12}\mathcal{C}]$. Q.E.D.

The notion of the associated problem is very simple but it will be of essential importance throughout the paper. For example note that Theorem (1.1) and Theorem (1.3) immediately imply the following corresponding theorem for problem P_2 .

THEOREM (1.6): Given $\mathcal{F} = (\mathcal{G}, \mathcal{D}, {}_2\mathcal{C})$ there exists a unique $\bar{\mathcal{F}} = \bar{\mathcal{F}}_2(\mathcal{G}, \mathcal{D}, {}_2\mathcal{C})$ such that every $\mathcal{F} \in R[\bar{\mathcal{F}}]$ is an equilibrium solution of problem $P_2[\mathcal{G}, \mathcal{D}, {}_2\mathcal{C}]$. Furthermore, this solution is stable (in a sense quite analogous to the notion of a stable solution of problem P_1 induced by Theorem (1.3)).

PROOF: Consider the problem $P_{12}[\mathcal{G}, \mathcal{D}, {}_2\mathcal{C}]$, i.e. the problem $P_1[\mathcal{G}, \mathcal{D}, {}_{12}\mathcal{C}]$, and let $\bar{\mathcal{F}} = \bar{\mathcal{F}}_1(\mathcal{G}, \mathcal{D}, {}_{12}\mathcal{C})$ be its solution. Obviously, $\bar{\mathcal{F}}$ is also the unique and stable solution to problem $P_2[\mathcal{G}, \mathcal{D}, {}_2\mathcal{C}]$.

Q.E.D.

We will close this section with certain simple observations about the associated problems. In the case of a quadratic model, the associated problem is also quadratic. More precisely, if ${}_1c_a(\bar{f}_a) = \frac{1}{2}g_a\bar{f}_a^2 + h_a\bar{f}_a$ then ${}_{12}c_a(\bar{f}_a) = g_a\bar{f}_a^2 + h_a\bar{f}_a$. Similarly if ${}_2c_a(\bar{f}_a) = g_a\bar{f}_a^2 + h_a\bar{f}_a$ then ${}_{21}c_a(\bar{f}_a) = \frac{1}{2}g_a\bar{f}_a^2 + h_a\bar{f}_a$. In general, it is obvious that ${}_{21}\mathcal{C} = {}_2\mathcal{C}$ and ${}_{12}\mathcal{C} = {}_1\mathcal{C}$.

A natural problem is the following: Suppose that a network \mathcal{G} is given. Characterize the type of \mathcal{C} for which the solutions of problem $P_1[\mathcal{G}, \mathcal{D}, \mathcal{C}]$ coincide with the solutions of problem $P_2[\mathcal{G}, \mathcal{D}, \mathcal{C}]$ for every \mathcal{D} . Such cases are extremely desirable because in them the pattern created by the individuals acting in their own self interests coincides with the pattern most economical for the total society. We have already seen that the solutions of the associated problem $P_2[\mathcal{G}, \mathcal{D}, \mathcal{C}]$ coincide with the solutions of the associated problem $P_{12}[\mathcal{G}, \mathcal{D}, \mathcal{C}]$. In consequence, the solutions of problem $P_1[\mathcal{G}, \mathcal{D}, \mathcal{C}]$ will coincide with the solutions of problem $P_2[\mathcal{G}, \mathcal{D}, \mathcal{C}]$ if and only if they coincide with the solutions of problem $P_{12}[\mathcal{G}, \mathcal{D}, \mathcal{C}]$.

Recalling (1.32), we conclude that the solutions of problem $P_1[\mathcal{G}, \mathcal{D}, \mathcal{C}]$ coincide with the solutions of problem $P_{12}[\mathcal{G}, \mathcal{D}, \mathcal{C}]$ if

$$c_a(\bar{f}_a) = \eta \int_0^{\bar{f}_a} \bar{c}_a(\bar{f}) d\bar{f}, \quad a \in \mathcal{L} \quad (1.34)$$

where η is an arbitrary positive constant, the same for all $a \in \mathcal{L}$. Integral equation (1.34) has the following solution:

$$c_a(\bar{f}_a) = c_a^0 \bar{f}_a^\eta \quad (1.35)$$

where c_a^0 is an arbitrary constant. In order for $c_a(\bar{f}_a)$, as given by (1.35), to satisfy conditions 1, 2, 3, 4₁ or 4₂, we restrict c_a^0, η so that $c_a^0 > 0, a \in \mathcal{L}, \eta > 1$. Actually (1.35) gives the most general type of cost functions which guarantee coincidence of the solutions of $P_1[\mathcal{G}, \mathcal{D}, \mathcal{C}]$ with those of $P_2[\mathcal{G}, \mathcal{D}, \mathcal{C}]$ for an arbitrary \mathcal{G} . It should be noted, though, that for special networks the class of such functions can be broadened.

2. Development of Algorithms for the Solution of the Problems P_1 and P_2

2.1. Generalities

In this section we develop algorithms for the solution of problem $P_1[\mathcal{T}]$. Obviously if such an algorithm is available, the solution of problem $P_2[\mathcal{T}]$ can be obtained as the solution of a P_1 problem, namely the associated problem $P_{21}[\mathcal{T}]$.

Roughly, the method of solution can be described as follows: Starting from an initial feasible flow pattern we construct a sequence of feasible flow patterns which converges to the optimal solution.

To be precise, we introduce the notion of an "equilibration operator."

A map

$$E_w : \mathcal{L} \rightarrow \mathcal{L}, \quad w \in \mathcal{W}$$

will be called an *equilibration operator associated with $w \in \mathcal{W}$* if it sends $\mathcal{F} \in \mathcal{L}$ into $\mathcal{F}' \in \mathcal{L}$ such that

$$f'_p = f_p$$

unless $p \in \mathcal{P}_w$.

A map

$$E : \mathcal{L} \rightarrow \mathcal{L}$$

will be called an *equilibration operator associated with a transportation network \mathcal{T}* , if E can be factored,

$$E = E_{w_n} \ 0 \ \dots \ 0 \ E_{w_1} \quad (2.1)$$

where $\{w_1, \dots, w_n\} = \mathcal{W}$ and E_{w_i} is an equilibration operator associated with w_i .

We now give our definition of an algorithm.

DEFINITION (2.1): Let \mathcal{T} be a transportation network and E an equilibration operator associated with \mathcal{T} . We will say that E induces an algorithm for the solution of problem $P_1[\mathcal{T}]$ if for any $\mathcal{F}^{(0)} \in \mathcal{L}$,

$$\bar{\mathcal{F}}^{(n)} \rightarrow \bar{\mathcal{F}}_1(\mathcal{T}), \quad n \rightarrow \infty \quad (2.2)$$

where

$$\mathcal{F}^{(n)} \equiv E^n \mathcal{F}^{(0)}, \quad n = 1, 2, \dots \quad (2.3)$$

and $\bar{\mathcal{F}}_1$ is a solution of problem P_1 .

The following theorem gives sufficient conditions for an equilibration operator to induce an algorithm for the solution of problem P_1 .

THEOREM⁶ (2.1): Let \mathcal{T} be a transportation network and E an equilibration operator associated with \mathcal{T} and having the following properties:

- (1) $E\mathcal{F} = \mathcal{F}$ for some $\mathcal{F} \in \mathcal{L}$ implies that \mathcal{F} satisfies (1.13) for all $w \in \mathcal{W}$, so that $\bar{\mathcal{F}} = \bar{\mathcal{F}}_1$.
- (2) E is a continuous mapping from \mathcal{L} to \mathcal{L} .
- (3) $C(\bar{E}\bar{\mathcal{F}}) \leq C(\bar{\mathcal{F}})$ for all $\mathcal{F} \in \mathcal{L}$.
- (4) $C(\bar{E}\bar{\mathcal{F}}) = C(\bar{\mathcal{F}})$ for some $\mathcal{F} \in \mathcal{L}$ implies that $E\mathcal{F} = \mathcal{F}$.

⁶ A proof of convergence along similar lines has been communicated independently to us by W. A. Horn of the National Bureau of Standards.

Then E induces an algorithm for the solution of problem $P_1[\mathcal{F}]$.

PROOF: Let $\mathcal{F}^{(0)} \in \mathcal{L}$ and $\mathcal{F}^{(n)} \equiv E^n \mathcal{F}^{(0)}$, $n = 1, 2, \dots$. We have to prove that

$$\overline{\mathcal{F}^{(n)}} \rightarrow \overline{\mathcal{F}_1(\mathcal{F})}, \quad n \rightarrow \infty. \quad (2.4)$$

We first prove that every convergent subsequence $\{\mathcal{F}^{(n_k)}\}$ of $\{\mathcal{F}^{(n)}\}$ converges to a solution \mathcal{F}_1 of $P_1[\mathcal{F}]$. In fact, let

$$\mathcal{F}^{(n_k)} \rightarrow \overline{\mathcal{F}}, \quad k \rightarrow \infty. \quad (2.5)$$

Since \mathcal{L} is closed, $\overline{\mathcal{F}} \in \mathcal{L}$. The sequence $\{C(\overline{\mathcal{F}^{(n_k)}})\}$ is decreasing and bounded from below by 0, hence is convergent. By Cauchy's theorem, given $\epsilon > 0$,

$$|C(\overline{E\mathcal{F}^{(n_k)}}) - C(\overline{\mathcal{F}^{(n_k)}})| = |C(\overline{\mathcal{F}^{(n_{k+1})}}) - C(\overline{\mathcal{F}^{(n_k)}})| < \frac{\epsilon}{3}$$

if $k \geq k_1(\epsilon)$.

Since $C(\overline{\mathcal{F}})$ is continuous, $\lim_{k \rightarrow \infty} C(\overline{\mathcal{F}^{(n_k)}}) = C(\overline{\mathcal{F}})$. Hence

$$|C(\overline{\mathcal{F}^{(n_k)}}) - C(\overline{\mathcal{F}})| < \frac{\epsilon}{3}$$

if $k \geq k_2(\epsilon)$. Furthermore, the continuity of $C(\overline{\mathcal{F}})$ implies the existence of δ_ϵ some such that

$$|C(\overline{\mathcal{F}}) - C(\overline{\mathcal{F}'})| < \frac{\epsilon}{3}, \quad \text{if } |\overline{\mathcal{F}} - \overline{\mathcal{F}'}| < \delta_\epsilon.$$

On the other hand, since E is a continuous mapping, given $\delta > 0$ there exists $\eta(\delta)$ such that

$$|E\overline{\mathcal{F}} - E\overline{\mathcal{F}'}| < \delta \quad \text{if } |\overline{\mathcal{F}} - \overline{\mathcal{F}'}| < \eta(\delta).$$

Finally, from (2.5) it follows that given $\eta > 0$, there exists $k_3(\eta)$ such that

$$|\overline{\mathcal{F}} - \overline{\mathcal{F}^{(n_k)}}| < \eta, \quad \text{if } k \geq k_3(\eta).$$

Suppose now that $k \geq \max\{k_1(\epsilon), k_2(\epsilon), k_3(\eta(\delta_\epsilon))\}$. Combining the above results we obtain

$$|C(\overline{E\mathcal{F}}) - C(\overline{\mathcal{F}})| \leq |C(\overline{E\mathcal{F}}) - C(\overline{E\mathcal{F}^{(n_k)}})| + |C(\overline{E\mathcal{F}^{(n_k)}}) - C(\overline{\mathcal{F}^{(n_k)}})| + |C(\overline{\mathcal{F}^{(n_k)}}) - C(\overline{\mathcal{F}})| < \epsilon.$$

But the left-hand side of the above inequality is independent of k and hence

$$C(\overline{E\mathcal{F}}) - C(\overline{\mathcal{F}}) = 0,$$

whence $\overline{\mathcal{F}}$ is a solution of problems P_1 by (4) and (1).

We now proceed to the proof of (2.4). Suppose that it is false. Then there exists a positive number δ and a subsequence $\{\mathcal{F}^{(n_k)}\}$ such that

$$\|\overline{\mathcal{F}^{(n_k)}} - \overline{\mathcal{F}_1}\| \equiv \sum_{a \in \mathcal{J}} |\bar{f}_a^{(n_k)} - \bar{f}_{1a}| > \delta. \quad (2.6)$$

The sequence $\{\mathcal{F}^{(n_k)}\}$ is bounded. By the theorem of Bolzano-Weierstrass there exists a converging subsequence $\{\mathcal{F}^{(n_{k_l})}\}$. As proved above

$$\mathcal{F}^{(n_{k_l})} \rightarrow \overline{\mathcal{F}_1}, \quad l \rightarrow \infty$$

where \mathcal{F}_1 is a solution of problem P_1 . In particular,

$$\overline{\mathcal{F}}^{(n_k l)} \rightarrow \overline{\mathcal{F}}_1, \quad l \rightarrow \infty$$

and this is a contradiction to (2.6).

Q.E.D.

The above theorem provides a criterion for establishing that a given equilibration operator induces an algorithm for the solution of problem P_1 . A limitation of the usefulness of the theorem may arise from the fact that it is not always easy to check whether assumptions 1–4 are satisfied. The following proposition simplifies this problem.

THEOREM (2.2): *Let $\{E_w : w \in \mathcal{W}\}$ be a collection of equilibration operators associated with the pairs of connected nodes of a transportation network \mathcal{T} . Suppose that for every $w \in \mathcal{W}$, E_w satisfies the following conditions.*

- (1') $E_w \mathcal{F} = \mathcal{F}$ for some $\mathcal{F} \in \mathcal{L}$ implies that \mathcal{F} satisfies (1.13) for this fixed w .
- (2') E_w is a continuous mapping from \mathcal{L} to \mathcal{L} .
- (3') $C(\overline{E_w \mathcal{F}}) \leq C(\overline{\mathcal{F}})$ for all $\mathcal{F} \in \mathcal{L}$.
- (4') $C(\overline{E_w \mathcal{F}}) = C(\overline{\mathcal{F}})$ for some $\mathcal{F} \in \mathcal{L}$ implies that $E_w \overline{\mathcal{F}} = \overline{\mathcal{F}}$.

Then any equilibration operator associated with \mathcal{T} and constructed by composition of the above collection $\{E_w : w \in \mathcal{W}\}$ satisfies conditions 1–4 of Theorem (2.1).

PROOF: Assumption 1 follows easily from 1' and the structure of an equilibration operator associated with a pair of nodes. Assumption 2 is an obvious consequence of 2'. Similarly 3 follows immediately from 3'. Finally 4 follows by a combination of 3' and 4'.

Q.E.D.

The above theorem reduces the problem of checking conditions 1–4 of Theorem (2.1) to the much simpler problem of checking conditions 1'–4' of Theorem (2.2).

Sometimes an equilibration operator E associated with a transportation network satisfies conditions 1 and 2 of Theorem (2.1) but it does not satisfy (or at least we cannot prove that it satisfies) conditions 3 and 4. Then of course we do not know whether E induces an algorithm for the solution of $P_1[\mathcal{T}]$. Nevertheless the sequence $\{E^n \mathcal{F}^{(0)}\}$ may lead to the solution of problem $P_1[\mathcal{T}]$ as shown by the following theorem, the proof of which is similar to the proof of Theorem (2.1).

THEOREM (2.3): *Suppose that an equilibration operator E satisfies conditions 1, 2 of Theorem (2.1). Suppose further that for some choice of $\mathcal{F}^{(0)}$ the sequence $\{\overline{E^n \mathcal{F}^{(0)}}\}$ converges as $n \rightarrow \infty$. Then $\{\overline{E^n \mathcal{F}^{(0)}}\}$ converges to the solution $\overline{\mathcal{F}}_1$, of the problem P_1 .*

REMARK (2.1): We have seen that an equilibration operator E which induces an algorithm for the solution of problem P_1 enables us to calculate through (2.2) the unique $\overline{\mathcal{F}}_1$ associated with a problem $P_1[\mathcal{T}]$. Then we know that $R[\overline{\mathcal{F}}_1]$ is the set of solutions of problem P_1 . The calculation of an element of $R[\overline{\mathcal{F}}_1]$, given $\overline{\mathcal{F}}_1$, amounts to finding a solution to the system (1.1), (1.4), which might be accomplished by phase 1 of the Simplex method. This requires a rather tedious calculation. However, as shown in the proof of Theorem (2.1), some elements of $R[\overline{\mathcal{F}}_1]$ can be obtained directly from the algorithm as limits of the convergent subsequences of $\{E^n \mathcal{F}^{(0)}\}$. In particular, if $R[\overline{\mathcal{F}}_1]$ consists of a unique element then

$$E^n \mathcal{F}^{(0)} \rightarrow \overline{\mathcal{F}}_1, \quad n \rightarrow \infty.$$

REMARK (2.2): The stability results of Theorem (1.3) can be employed here in order to estimate $\|\overline{\mathcal{F}}_1 - \overline{E^n \mathcal{F}^{(0)}}\|$, $n = 1, \dots$, and thus they provide a means for deciding whether the approximation is satisfactory, in which case the algorithm is terminated. In fact the estimation of the smallest ϵ modulo which $E^n \mathcal{F}^{(0)}$ satisfies (1.13) can be obtained for example by a method communicated to us by Alan Goldman and which is given in appendix 3. Then a use of estimates (1.22), (1.23) reveals the accuracy of the approximation.

Recall that the proof of Theorem (1.3) involved an estimate of the form

$$\|\overline{E^n \mathcal{F}^{(0)}} - \overline{\mathcal{F}}_1\|^2 \leq \frac{2d^2}{ck} [C(\overline{E^n \mathcal{F}^{(0)}}) - C(\overline{\mathcal{F}}_1)],$$

which enables us to estimate the convergence of $\{\overline{E^n \mathcal{F}^{(0)}}\}$ in terms of the convergence of $\{C(\overline{E^n \mathcal{F}^{(0)}})\}$. Thus when we apply the algorithm it is sufficient to inspect the sequence of the successive values of the total cost. From the rate of convergence of this sequence we can judge the rate of convergence of the sequence $\{\overline{E^n \mathcal{F}^{(0)}}\}$.

It is convenient to introduce here the concept of disjoint paths. Let $w \in \mathcal{W}$. The set \mathcal{P}_w will be called *disjoint* if there is no $a \in \mathcal{L}$ which is contained in more than one $p \in \mathcal{P}_w$.

A network \mathcal{G} will be called *simple* if \mathcal{P}_w is disjoint for every $w \in \mathcal{W}$.

In the following paragraphs we will construct two equilibration operators E_{dsj} , E_{nds} and we will discuss under what conditions they induce algorithms for the solution of problem P_1 . These operators will be introduced first for the quadratic model and then the definition will be extended to the case of general convex cost functions. In particular E_{dsj} can be applied more naturally to simple networks while E_{nds} has been designed for application to nonsimple networks, for which E_{dsj} need not induce an algorithm for the solution of problem P_1 .

2.2. The Quadratic Model

Let $\mathcal{F} = (\mathcal{G}, \mathcal{I}, \mathcal{C})$ be a transportation network with quadratic cost functions,

$$c_a(\bar{f}_a) = g_a \bar{f}_a^2 + h_a \bar{f}_a, \quad a \in \mathcal{L}.$$

According to (2.1), in order to define the equilibration operators E_{dsj} , E_{nds} associated with the given \mathcal{F} it is sufficient to define their factors E_w^{dsj} , E_w^{nds} , respectively, for every $w \in \mathcal{W}$.

a. The Equilibration Operator E_{dsj} for the Quadratic Model

We start by motivating the definition of E_w^{dsj} . Let $w \in \mathcal{W}$ such that $\mathcal{P}_w = \{p_1, \dots, p_m\}$ is disjoint, and consider any $\mathcal{F} \in \mathcal{L}$. By $\mathcal{L}_w^{\mathcal{F}}$ we denote the subset of \mathcal{L} such that

$$\mathcal{F}' \in \mathcal{L}_w^{\mathcal{F}} \iff \{f'_q = f_q : q \notin \mathcal{P}_w\}.$$

Let us seek the element $\mathcal{F}' \in \mathcal{L}_w^{\mathcal{F}}$ which

$$\text{minimizes } C(\overline{\mathcal{F}'}) \text{ over } \mathcal{L}_w^{\mathcal{F}}. \quad (2.7)$$

In order to solve this minimization problem it is convenient to introduce the following notation.

$$g_{p_r} \equiv \sum_{a \in \mathcal{L}} \delta_{ap_r} g_a, \quad (2.8)$$

$$\mu_{p_r}(\mathcal{F}) \equiv \sum_{a \in \mathcal{L}} \delta_{ap_r} [2g_a \bar{f}_a + h_a], \quad (2.9)$$

$$h_{p_r}(\mathcal{F}) \equiv \mu_{p_r} - 2g_{p_r} \bar{f}_{p_r}. \quad (2.10)$$

Without loss of generality assume that

$$h_{p_1}(\mathcal{F}) \leq \dots \leq h_{p_m}(\mathcal{F}).$$

A comparison with (1.13) leads to the conclusion that the solution \mathcal{F}' of (2.7) satisfies the following equilibrium condition:

$$\begin{aligned} \mu_{p_1} + 2g_{p_1}(f'_{p_1} - \bar{f}_{p_1}) &= \dots = \mu_{p_s} + 2g_{p_s}(f'_{p_s} - \bar{f}_{p_s}) \leq \mu_{p_{s+1}} + 2g_{p_{s+1}}(f'_{p_{s+1}} - \bar{f}_{p_{s+1}}) \\ &\leq \dots \leq \mu_{p_m} + 2g_{p_m}(f'_{p_m} - \bar{f}_{p_m}), \end{aligned}$$

$$\sum_{p_r \in \mathcal{P}_w} f'_{p_r} = d_w.$$

$$f'_{p_r} > 0, r = 1, \dots, s, f'_{p_r} = 0, r = s+1, \dots, m. \quad (2.11)$$

On account of (2.10), (2.11)₁ reads

$$M_w \equiv 2g_{p_1}f'_{p_1} + h_{p_1} = \dots = 2g_{p_s}f'_{p_s} + h_{p_s} \leq 2g_{p_{s+1}}f'_{p_{s+1}} + h_{p_{s+1}} \leq \dots \leq 2g_{p_m}f'_{p_m} + h_{p_m}$$

and the solution of (2.11) gives

$$f'_{p_r} = \frac{M_w - h_{p_r}}{2g_{p_r}}, \quad r = 1, \dots, s,$$

$$f'_{p_r} = 0, \quad r = s+1, \dots, m, \quad (2.12)$$

where

$$M_w = \frac{2d_w + \sum_{r=1}^s h_{p_r} / g_{p_r}}{\sum_{r=1}^s 1/g_{p_r}}. \quad (2.13)$$

Thus \mathcal{F}' may be calculated through (2.12), (2.13), provided that the critical index s is known.

We now give a procedure for the evaluation of s . From (2.11), (2.12) we obtain the condition

$$h_{p_m} \geq \dots \geq h_{p_{s+1}} \geq M_w > h_{p_s} \geq h_{p_{s-1}} \geq \dots \geq h_{p_1}. \quad (2.14)$$

Let

$$M_w^r = \frac{2d_w + \sum_{k=1}^r h_{p_k} / g_{p_k}}{\sum_{k=1}^r 1/g_{p_k}}, \quad r = 1, \dots, m. \quad (2.15)$$

The index s for which M_w^s satisfies (2.14) is the critical one. The existence of a unique s having this property follows from the existence of a unique solution to the minimization problem in question. Nevertheless, we demonstrate separately the existence and uniqueness of such an s , using the following identities which are of interest in themselves:

$$M_w^1 = \frac{2d_w}{1/g_{p_1}} + h_{p_1} > h_{p_1}, \quad (2.16)$$

$$(M_w^r - M_w^{r-1}) \sum_{k=1}^{r-1} 1/g_{p_k} = 1/g_{p_r} (h_{p_r} - M_w^r), \quad (2.17)$$

or

$$(M_w^r - M_w^{r-1}) \sum_{k=1}^r 1/g_{p_k} = 1/g_{p_r} (h_{p_r} - M_w^{r-1}). \quad (2.18)$$

Let S be the set of indices such that $r \in S$ if and only if $M_w^r > h_{p_r}$. From (2.16) it follows that $1 \in S$. Suppose that $r \in S$. Then $M_w^r > h_{p_r}$. From (2.17) $M_w^{r-1} > M_w^r$. Thus $h_{p_{r-1}} \leq h_{p_r} < M_w^r < M_w^{r-1}$, i.e., $(r-1) \in S$ and hence $1, 2, \dots, r \in S$. Let s be the maximum index in S . Then $h_{p_s} < M_w^s$. Furthermore, either $s = m$ or $s+1 \notin S$ which implies $h_{p_{s+1}} \geq M_w^{s+1}$. Using (2.17), (2.18), we conclude that $M_w^{s+1} \leq h_{p_{s+1}}$ implies $M_w^s \leq M_w^{s+1}$, which in turn implies $M_w^s \leq h_{p_{s+1}}$. Thus the existence of a unique s has been established and another method of construction of s (as the maximum index in S) has been given.

Summarizing, to calculate the solution \mathcal{F}' of the minimization problem (2.7) we apply the following procedure:

- (1) We calculate the quantities $g_{p_r}, h_{p_r}(\mathcal{F}), r = 1, \dots, m$.
- (2) We arrange $h_{p_r}(\mathcal{F})$ in nondescending order and we relabel them according to this order.

(3) We calculate $M_w^r, r=1, \dots, m$ from (2.15).

(4) If $M_w^m < h_{p_m}$ we set $s=m$. If $M_w^m \geq h_{p_m}$, we find the unique index s such that $h_{p_s} < M_w^s \leq h_{p_{s+1}}$ (or equivalently such that $h_{p_s} < M_w^s$ and $h_{p_{s+1}} > M_w^{s+1}$).

(5) We calculate $f_{p_r}^r, r=1, \dots, m$, using formula (2.12).

Suppose now that $w \in \mathcal{W}$ with \mathcal{P}_w disjoint. We define E_w^{dsj} by

$$E_w^{dsj} \bar{\mathcal{F}} \equiv \bar{\mathcal{F}}' \quad (2.19)$$

where $\bar{\mathcal{F}}'$ is the solution of the minimization problem (2.7) for the given $\bar{\mathcal{F}}$. This definition induces the definition of E_{dsj} for simple networks with quadratic cost functions.

Note that (2.7) simply states that $\bar{\mathcal{F}}' \equiv E_w^{dsj} \bar{\mathcal{F}}$ satisfies (1.13) for the pair w . Thus, if $E_w^{dsj} \bar{\mathcal{F}} = \bar{\mathcal{F}}$, $\bar{\mathcal{F}}$ satisfies (1.13) for the pair w , i.e., E_w^{dsj} satisfies condition 1' of Theorem (2.2). Condition 2' of the same theorem follows immediately from the continuity of the functions involved in (2.12). Finally, conditions 3', 4' are also satisfied since $\bar{\mathcal{F}}' \equiv E_w^{dsj} \bar{\mathcal{F}}$ minimizes $C(\bar{\mathcal{F}}')$ over the set $\mathcal{L}_w^{\bar{\mathcal{F}}}$.

By Theorems (2.1), (2.2) it follows that E_{dsj} induces an algorithm for a simple network with quadratic cost functions.

We should emphasize here that the effectiveness of the solution of the minimization problem (2.7) is due essentially to the assumption of the disjointness of \mathcal{P}_w and of the quadraticity of $\{c_a(\bar{f}_a) : a \in \mathcal{L}\}$.

We now proceed to extend the definition of E_w^{dsj} in the case for which \mathcal{P}_w is not disjoint. Note that the minimization problem (2.7) is meaningful even in this case. The solution of this problem would provide a natural extension of the definition of E_w^{dsj} to cases where \mathcal{P}_w is not disjoint. Unfortunately an effective solution of this problem does not seem possible. Thus we devise the following kind of extension:

We choose to calculate

$$\bar{\mathcal{F}}' \equiv E_w^{dsj} \bar{\mathcal{F}}$$

by following the steps (1), (2), (3), (4), and (5) described above.

This is clearly possible since this procedure, though motivated for disjoint \mathcal{P}_w , does not depend in its definition on the assumption of disjointness. In this way we retain the simplicity of the calculations. Let us show that the condition 1' of Theorem (2.2) is satisfied. Suppose that $\bar{\mathcal{F}}' \equiv E_w \bar{\mathcal{F}} = \bar{\mathcal{F}}$ for some $\bar{\mathcal{F}} \in \mathcal{L}$. From (2.12), $f_{p_r} = f_{p_r}' \equiv \frac{M_w - h_{p_r}}{2g_{p_r}}, r=1, \dots, s$, and $f_{p_r} = f_{p_r}' = 0, r=s+1, \dots, m$. Then, recalling (2.9) and (2.10), $\mu_{p_r} = M_w, r=1, \dots, s, \mu_{p_r} \geq M_w, r=s+1, \dots, m$. Now observe that μ_{p_r} is the (real) marginal cost along the path p_r . Hence the equilibrium equations (1.13) are satisfied for w and the proof of condition 1' is complete. Furthermore, we can prove, as before, that the condition 2' of Theorem (2.2) remains valid. On the other hand, the motivation which was present in the case of disjoint \mathcal{P}_w and which was justified by the proof of the validity of conditions 3', 4' of Theorem (2.2) is not present any more. Thus, it is not obvious that conditions 3', 4' are still satisfied, and we have to go through the following lengthy calculation in order to check whether they are valid.

Let us set $\Delta \bar{\mathcal{F}} = \bar{\mathcal{F}}' - \bar{\mathcal{F}}$. The change of the total cost is given by

$$\begin{aligned} C(\bar{\mathcal{F}}') - C(\bar{\mathcal{F}}) &= \sum_{a \in \mathcal{L}} [g_a(\bar{f}_a + \Delta \bar{f}_a)^2 + h_a(\bar{f}_a + \Delta \bar{f}_a) - g_a \bar{f}_a^2 - h_a \bar{f}_a] = \sum_{a \in \mathcal{L}} g_a \Delta \bar{f}_a^2 \\ &+ \sum_{a \in \mathcal{L}} [2g_a \bar{f}_a + h_a] \Delta \bar{f}_a = \sum_{p \in \mathcal{P}_w} \Delta f_p \sum_{a \in \mathcal{L}} [2g_a \bar{f}_a + h_a] \delta_{ap} + \sum_{a \in \mathcal{L}} g_a \Delta \bar{f}_a^2 \\ &= \sum_{p \in \mathcal{P}_w} \Delta f_p [2g_p f_p + h_p] + \sum_{a \in \mathcal{L}} g_a \Delta \bar{f}_a^2 \end{aligned}$$

where use has been made of (2.9). Thus

$$C(\overline{\mathcal{F}'}) - C(\overline{\mathcal{F}}) = -2 \sum_{p \in \mathcal{P}} g_p \Delta f_p^2 + \sum_{p \in \mathcal{P}} \Delta f_p [2g_p f'_p + h_p] + \sum_{a \in \mathcal{L}} g_a \overline{\Delta f}_a^2.$$

By (2.11)

$$\sum_{p \in \mathcal{P}} \Delta f_{p_r} [2g_p f'_p + h_p] = M_w \sum_{r=1}^s \Delta f_{p_r} - \sum_{r=s+1}^m (-\Delta f_{p_r}) h_{p_r} \leq M_w \sum_{r=1}^m \Delta f_{p_r} = 0. \quad (2.20)$$

Hence,

$$C(\overline{\mathcal{F}'}) - C(\overline{\mathcal{F}}) \leq \sum_{a \in \mathcal{L}} g_a \overline{\Delta f}_a^2 - 2 \sum_{p \in \mathcal{P}} g_p \Delta f_p^2.$$

Recalling the Definition (2.8) of g_p , and (1.4),

$$C(\overline{\mathcal{F}'}) - C(\overline{\mathcal{F}}) \leq \sum_{a \in \mathcal{L}} g_a \left\{ \left[\sum_{p \in \mathcal{P}_w} \delta_{ap} \Delta f_p \right]^2 - 2 \sum_{p \in \mathcal{P}_w} \delta_{ap} \Delta f_p^2 \right\}. \quad (2.21)$$

We want to study now the sign of $C(\overline{\mathcal{F}'}) - C(\overline{\mathcal{F}})$. It is convenient to consider links of a special type. A link a will be called *simple*, *double*, or *total* with respect to w depending upon whether a is contained in precisely one, two or all of the paths of \mathcal{P}_w . Note that if \mathcal{P}_w is disjoint, then all links contained in \mathcal{P}_w are simple.

Suppose now that a is a simple link contained in a path $p \in \mathcal{P}_w$. Then,

$$g_a \left\{ \left[\sum_{p \in \mathcal{P}_w} \delta_{ap} \Delta f_p \right]^2 - 2 \sum_{p \in \mathcal{P}_w} \delta_{ap} \Delta f_p^2 \right\} = -g_a \Delta f_p^2 \leq 0 \quad (2.22)$$

with equality holding if and only if $\Delta f_p = 0$.

Let a be a double link contained in two paths $p, q \in \mathcal{P}_w$. Then,

$$g_a \left\{ \left[\sum_{p \in \mathcal{P}_w} \delta_{ap} \Delta f_p \right]^2 - 2 \sum_{p \in \mathcal{P}_w} \delta_{ap} \Delta f_p^2 \right\} = g_a \{ [\Delta f_p + \Delta f_q]^2 - 2[\Delta f_p^2 + \Delta f_q^2] \} = -g_a (\Delta f_p - \Delta f_q)^2 \leq 0 \quad (2.23)$$

with equality holding if and only if $\Delta f_p = \Delta f_q$.

Suppose finally, that a is a total link with respect to w . Then,

$$g_a \left\{ \left[\sum_{p \in \mathcal{P}_w} \delta_{ap} \Delta f_p \right]^2 - 2 \sum_{p \in \mathcal{P}_w} \delta_{ap} \Delta f_p^2 \right\} = -2g_a \sum_{p \in \mathcal{P}_w} \Delta f_p^2 \leq 0 \quad (2.24)$$

with equality holding if and only if $\Delta f_p = 0, p \in \mathcal{P}_w$.

Thus (2.21) implies that, if all links contained in \mathcal{P}_w are simple and/or double and/or total links, then

$$C(\overline{\mathcal{F}'}) - C(\overline{\mathcal{F}}) \leq 0. \quad (2.25)$$

Two paths of \mathcal{P}_w will be called *directly connected* if they share at least one double link (with respect to w); will be called *connected* if they can be joined by a finite sequence of direct connections. We now introduce the following definition.

DEFINITION (2.2): The set \mathcal{P}_w will be called *almost disjoint* if the following conditions are satisfied.

- (1) All links contained in \mathcal{P}_w are simple, double, or total with respect to w .
- (2) Any two paths of \mathcal{P}_w , which consist exclusively of double links and are connected only to paths consisting exclusively of double links, must be connected to each other.

If \mathcal{P}_w is almost disjoint for all $w \in \mathcal{W}$, the transportation network will be called *almost simple*. In particular a simple network is also almost simple.

Assume now that \mathcal{P}_w is almost disjoint. Then (2.25) is satisfied. In addition, we claim that equality can hold only if $\Delta \mathcal{F} = 0$. In fact, suppose that equality holds in (2.25). From (2.22), (2.24) it follows that

$$\Delta f_p = 0$$

in the case where $p \in \mathcal{P}_w$ contains at least one simple and/or total link. Let $p, q \in \mathcal{P}_w$ be two paths which are directly connected. On account of (2.23),

$$\Delta f_p = \Delta f_q. \quad (2.26)$$

It is clear that then the validity of (2.26) extends to the case where p and q are connected. Thus $\Delta f_p = 0$ even for all $p \in \mathcal{P}_w$ which are connected to a path containing a simple and/or total link. In order to complete the proof it remains to consider the set \mathcal{P}'_w of paths in \mathcal{P}_w consisting exclusively of double links and not connected to any path containing simple and/or total links. On account of the definition of almost disjointness, if $p, q \in \mathcal{P}'_w$, then p and q are connected in which case

$$\Delta f_p = \Delta f_q \equiv \Delta f. \quad (2.27)$$

Recall the conservation equation

$$\sum_{p \in \mathcal{P}_w} \Delta f_p = \sum_{p \in \mathcal{P}'_w} \Delta f_p = 0. \quad (2.28)$$

Using (2.27), (2.28) we deduce that $\Delta f = 0$, i.e., (2.25) holds as equality only if $\Delta \mathcal{F} = 0$.

Combining the above results with the Theorems (2.1), (2.2) we reach the following conclusion.

THEOREM (2.4): *The equilibration operator E_{dsj} induces an algorithm which solves problem P_1 for almost simple networks, with quadratic cost functions.*

Actually, this result is the best possible in the sense that there exist networks with triple links for which the operator E_{dsj} does not induce an algorithm which solves problem P_1 for arbitrary initial $\mathcal{F}^{(0)}$. An example of such a network is given in appendix 2. Nevertheless, recall that E_{dsj}^{qs} satisfies conditions 1', 2' of Theorem (2.2) for arbitrary \mathcal{P}_w . From Theorem (2.3) it follows that if $\{E_{dsj}^h \mathcal{F}^{(0)}\}$ converges then it will converge to the solution $\bar{\mathcal{F}}_1$ of problem P_1 . In consequence, it is worthwhile to try an application of E_{dsj} even for networks which are not almost simple.

In order for an algorithm to be appropriate for application in practice, it is not enough that it converges; it must converge *rapidly*. We have the following evidence about rapid convergence of the algorithm induced by E_{dsj} in the case of almost simple networks. We consider the test network of figure 1 (with 60 paths) which is almost simple, but is not simple. We have developed a computer program which solves problem P_1 for this network by application of E_{dsj} . We have calculated the solution for a very wide range of demands and choice of the initial distribution. We have observed extremely rapid convergence. More specifically if \bar{g}, \bar{h} , were the average values of $ga, ha, a \in \mathcal{L}$, and \bar{d} is the average demand in the network, we chose $\frac{c}{d}$ entering in Definition (1.2) equal to $2\bar{g}\bar{d} + \bar{h}$, and after 5 iterations the flow pattern satisfied eqs. (1.13) modulo 10^{-5} . Furthermore, we have treated the same network algebraically and have shown that

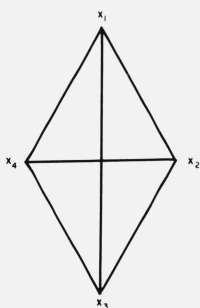


FIGURE 1.

$$\|E_{dsj}^n \mathcal{F}^{(0)} - E_{dsj}^{n-1} \mathcal{F}^{(0)}\|$$

decreases with the speed of a geometric progression with a ratio less than 1.

With these observations we conclude the discussion of the equilibration operator E_{dsj} for transportation networks with quadratic cost functions.

b. The Equilibration Operator E_{ndsj} for the Quadratic Model

In this section we introduce an equilibration operator E_{ndsj} which does induce an algorithm for the solution of problem P_1 for arbitrary transportation networks with quadratic cost functions.

Fix some $w \in \mathcal{W}$. The motivation of the introduction of E_w^{ndsj} is similar to that of E_w^{qsj} . Namely, E_w^{ndsj} will be selected to be a minimization operator of $C(\mathcal{F})$ but over a set less broad than the set $\mathcal{L}_w^{\mathcal{F}}$ defined earlier.

Let $\mathcal{F} \in \mathcal{L}$. We define $\mathcal{F}' \equiv E_w^{ndsj} \mathcal{F}$ by the minimization problem
 minimize $C(\mathcal{F}')$, over $\mathcal{F}' \in \hat{\mathcal{L}}_w^{\mathcal{F}}$ where $\hat{\mathcal{L}}_w^{\mathcal{F}}$, to be defined, is a subset of $\mathcal{L}_w^{\mathcal{F}}$ containing \mathcal{F} .
(2.29)

Note that the minimization problem (2.29) is very similar to (2.7). Such a definition of E_w^{ndsj} guarantees automatically that condition 3' of Theorem (2.2) is satisfied. The main difficulty is that we must select the set $\hat{\mathcal{L}}_w^{\mathcal{F}}$ in such a way so that conditions 1', 2', 4' of Theorem (2.2) are also satisfied while, at the same time, the solution of (2.29) can be obtained in an elegant way.⁷

The marginal cost on a path $t \in \mathcal{P}_w$ corresponding to the flow pattern \mathcal{F} is given by $\mu_{p_r}(\mathcal{F})$ of (2.9) with $p_r = t$.

Let $p, q \in \mathcal{P}_w$ be defined by

$$\begin{aligned} \mu_p(\mathcal{F}) &= \min \{ \mu_t(\mathcal{F}) : t \in \mathcal{P}_w \}^8, \\ \mu_q(\mathcal{F}) &= \max \{ \mu_t(\mathcal{F}) : t \in \mathcal{P}_w, f_t > 0 \}^9. \end{aligned} \tag{2.30}$$

We define the set $\hat{\mathcal{L}}_w^{\mathcal{F}}$ by

$$\hat{\mathcal{L}}_w^{\mathcal{F}} \equiv \{ \mathcal{F}' \in \mathcal{L} : f'_t = f_t, \text{ unless } t = p, q \}. \tag{2.31}$$

We now define $E_w^{ndsj} \mathcal{F} \equiv \mathcal{F}'$ as the solution of the minimization problem (2.29) over $\hat{\mathcal{L}}_w^{\mathcal{F}}$ as selected above.

The calculation of $\mathcal{F}' = E_w^{ndsj} \mathcal{F}$ amounts to the calculation of the two new flows f'_p, f'_q .

Let us define

$$g_p^q \equiv \sum_{a \in \mathcal{J}} \delta_{ap}^q g_a, \quad g_q^p \equiv \sum_{a \in \mathcal{J}} \delta_{aq}^p g_a \tag{2.32}$$

where the incidence symbols $\delta_{ap}^q, \delta_{aq}^p$ have been defined by (1.19).

The minimization problem (2.29) leads to the following equilibrium conditions analogous to (2.11):

$$\begin{aligned} \mu_p(\mathcal{F}) + 2g_p^q(f'_p - f_p) &\leq \mu_q(\mathcal{F}) + 2g_q^p(f'_q - f_q), \\ f'_p + f'_q &= f_p + f_q, \\ f'_p &\geq 0, f'_q &\geq 0, \end{aligned} \tag{2.33}$$

with the understanding that, if (2.33)₁ holds as a strict inequality, then the second one of (2.33)₃ must hold as an equality, and conversely, if both (2.33)₃ hold as strict inequalities, then (2.33)₁ must hold as an equality. The solution of (2.33) is given by

$$f'_p = f_p + \frac{\mu_q(\mathcal{F}) - \mu_p(\mathcal{F})}{2(g_p^q + g_q^p)},$$

⁷ If we selected $\hat{\mathcal{L}}_w^{\mathcal{F}} = \mathcal{L}_w^{\mathcal{F}}$, we would be led to the minimization problem (2.5). In this case we know that (2.29) does not have an elegant solution unless \mathcal{P}_w is disjoint. This observation emphasizes the fact that the selection of an appropriate $\hat{\mathcal{L}}_w^{\mathcal{F}}$ is not easy.

⁸ If μ attains its minimum for more than one path, then select p as any of those paths.

⁹ If μ attains its maximum for more than one path, then select q as any of those paths.

$$f'_q = f_q + \frac{\mu_p(\bar{\mathcal{F}}) - \mu_q(\bar{\mathcal{F}})}{2(g_p^q + g_q^p)}, \quad (2.34)$$

$$\text{if } f_q \geq \frac{\mu_q(\bar{\mathcal{F}}) - \mu_p(\bar{\mathcal{F}})}{2(g_p^q + g_q^p)},$$

or by

$$f'_p = f_p + f_q,$$

$$f'_q = 0, \quad (2.35)$$

$$\text{if } f_q < \frac{\mu_q(\bar{\mathcal{F}}) - \mu_p(\bar{\mathcal{F}})}{2(g_p^q + g_q^p)}.$$

Thus, the selection of $\hat{\mathcal{I}}_w^{\bar{\mathcal{F}}}$ has led to a minimization problem (2.29) whose solution can be calculated efficiently through (2.34) or (2.35).

We now proceed to prove that E_w^{ndsj} satisfies the conditions 1'-4' of Theorem (2.2). From (2.33) it is clear that $E_w^{ndsj}\bar{\mathcal{F}} = \bar{\mathcal{F}}$ if and only if $\mu_p(\bar{\mathcal{F}}) = \mu_q(\bar{\mathcal{F}})$ (recall that $f_q > 0$).

The paths p, q have been selected so that for any $t \in \mathcal{P}_w$ with $f_t > 0$,

$$\mu_p(\bar{\mathcal{F}}) \leq \mu_t(\bar{\mathcal{F}}) \leq \mu_q(\bar{\mathcal{F}}).$$

Thus, if $E_w^{ndsj}\bar{\mathcal{F}} = \bar{\mathcal{F}}$, then all paths $t \in \mathcal{P}_w$ with $f_t > 0$ have the same marginal cost and the equilibrium conditions are satisfied for w . Hence E_w^{ndsj} satisfies condition 1' of Theorem (2.2). Condition 2' is obviously satisfied.

Let us now calculate $\Delta C \equiv C(\bar{\mathcal{F}}) - C(E_w^{ndsj}\bar{\mathcal{F}})$. Obviously ΔC is given by

$$\begin{aligned} \Delta C &= \sum_{a \in \mathcal{L}} \delta q_p [g_a \bar{f}_a^2 + h_a \bar{f}_a - g_a \bar{f}_a'^2 - h_a \bar{f}_a'] + \sum_{a \in \mathcal{L}} \delta q_q [g_a \bar{f}_a^2 + h_a \bar{f}_a - g_a \bar{f}_a'^2 - h_a \bar{f}_a'] \\ &= \sum_{a \in \mathcal{L}} \delta q_p [2g_a \bar{f}_a + h_a - g_a (\bar{f}_a - \bar{f}_a')] (\bar{f}_a - \bar{f}_a') + \sum_{a \in \mathcal{L}} \delta q_q [2g_a \bar{f}_a + h_a - g_a (\bar{f}_a - \bar{f}_a')] (\bar{f}_a - \bar{f}_a'). \end{aligned}$$

Note that in the first sum $\bar{f}_a - \bar{f}_a' = f_p - f_p'$, while in the second sum $\bar{f}_a - \bar{f}_a' = f_q - f_q'$. Hence, recalling also the definition (2.32) of g_p^q, g_q^p , we get

$$\Delta C = (f_p - f_p') \mu_p^q - g_p^q (f_p - f_p')^2 + (f_q - f_q') \mu_q^p - g_q^p (f_q - f_q')^2 \quad (2.36)$$

where

$$\mu_p^q \equiv \sum_{a \in \mathcal{L}} \delta q_p [2g_a \bar{f}_a + h_a], \quad (2.37)$$

$$\mu_q^p \equiv \sum_{a \in \mathcal{L}} \delta q_q [2g_a \bar{f}_a + h_a]. \quad (2.38)$$

By (2.33)₂, $f_p - f_p' = -(f_q - f_q')$. Hence

$$\begin{aligned} \Delta C &= [\mu_p^q + g_p^q (f_p' - f_p) - \mu_q^p - g_q^p (f_q' - f_q)] (f_p - f_p') \\ &= [\mu_p^q + 2g_p^q (f_p' - f_p) - \mu_q^p - 2g_q^p (f_q' - f_q)] (f_p - f_p') + (g_p^q + g_q^p) (f_p - f_p')^2. \end{aligned} \quad (2.39)$$

But $\mu_p^q - \mu_q^p = \mu_p - \mu_q$. Then, using (2.33)₁ and the fact that $f_p - f_p' \leq 0$, we obtain

$$\Delta C \geq (g_p^q + g_q^p) (f_p - f_p')^2 \geq 0 \quad (2.40)$$

with equality holding only if $f'_p = f_p$, i.e., if $E_w^{dsj} \mathcal{F} = \mathcal{F}$. Hence conditions 3', 4' of Theorem (2.2) are satisfied. Combining the above results we reach the following conclusion:

THEOREM (2.5): *The equilibration operator E^{nds_j} induces an algorithm which solves problem P_1 for an arbitrary transportation network with quadratic cost functions.*

We wish to compare the algorithms induced by the equilibration operators E_{ds_j} , E_{nds_j} and to point out their corresponding advantages. We prefer to postpone this comparison until E_{ds_j} , E_{nds_j} have been extended to cover the case of nonquadratic cost functions. This extension is the subject of the next section.

2.3. The General Model

In this section, we will extend the definitions of the equilibration operators E_{ds_j} , E_{nds_j} to the case of a general transportation network $\mathcal{T} = (\mathcal{G}, \mathcal{D}, \mathcal{C})$. We will assume that c_a is twice continuously differentiable for all $a \in \mathcal{L}$.

Note that the minimization problems (2.7), (2.29), through which the operators $E_w^{ds_j}$, $E_w^{nds_j}$ have been introduced for the quadratic model, are well set also for the general model. Thus it appears that the proper extension of $E_w^{ds_j}$, $E_w^{nds_j}$ would be obtained through the same minimization problems set for general cost functions. However, a review of the theory described in section 2.2 indicates that the simplification induced by the assumption of quadratic cost functions lies in the fact that for such cost functions the equilibrium conditions (2.11), (2.33) corresponding to (2.7), (2.29), respectively, are linear. This fact permits a very effective and elegant solution for both minimization problems (2.7), (2.29).

In the case of general convex cost functions, the equilibrium conditions are in general non-linear and hence a simple solution of problems (2.7), (2.29) is no longer possible. In order to devise a working extension of the definition of E_{ds_j} , E_{nds_j} for the general model we use the following considerations:

By Taylor's theorem, if \bar{f}'_a is close to \bar{f}_a ,

$$c_a(\bar{f}'_a) \cong c_a(\bar{f}_a) + c'_a(\bar{f}_a)(\bar{f}'_a - \bar{f}_a) + \frac{1}{2}c''_a(\bar{f}_a)(\bar{f}'_a - \bar{f}_a)^2. \quad (2.41)$$

In particular, for quadratic $c_a(\bar{f}_a) = g_a\bar{f}_a^2 + h_a\bar{f}_a$,

$$c_a(\bar{f}'_a) = c_a(\bar{f}_a) + (2g_a\bar{f}_a + h_a)(\bar{f}'_a - \bar{f}_a) + g_a(\bar{f}'_a - \bar{f}_a)^2. \quad (2.42)$$

Comparing (2.41) with (2.42) we observe that the quadratic cost function $g_a(\bar{f}_a)\bar{f}'_a^2 + h_a(\bar{f}_a)\bar{f}'_a$ which approximates the general cost function $c_a(\bar{f}'_a)$ in the neighborhood of \bar{f}_a has coefficients

$$g_a(\bar{f}_a) = \frac{1}{2}c''_a(\bar{f}_a), \quad (2.43)$$

$$h_a(\bar{f}_a) = c'_a(\bar{f}_a) - c''_a(\bar{f}_a)\bar{f}_a. \quad (2.44)$$

Using as a motivation this observation, given $\mathcal{F} \in \mathcal{L}$ we construct the collection of quadratic cost functions

$$\mathcal{C}_{\mathcal{F}} \equiv \left\{ \frac{1}{2} c''_a(\bar{f}_a) \bar{f}'_a{}^2 + [c'_a(\bar{f}_a) - c''_a(\bar{f}_a) \bar{f}_a] \bar{f}'_a : a \in \mathcal{L} \right\}. \quad (2.45)$$

We now define $E_w^{ds_j}$ (resp. $E_w^{nds_j}$) by identifying $E_w^{ds_j} \mathcal{F}$ (resp. $E_w^{nds_j} \mathcal{F}$) for the transportation network $\mathcal{T} = (\mathcal{G}, \mathcal{D}, \mathcal{C})$ with $E_w^{ds_j} \mathcal{F}$ (resp. $E_w^{nds_j} \mathcal{F}$) for the transportation network $\mathcal{T}_{\mathcal{F}} = (\mathcal{G}, \mathcal{D}, \mathcal{C}_{\mathcal{F}})$, the latter being well defined since $\mathcal{C}_{\mathcal{F}}$ consists of quadratic cost functions. In other words, $\mathcal{F}' \equiv E_w^{ds_j} \mathcal{F}$ (resp. $\mathcal{F}' \equiv E_w^{nds_j} \mathcal{F}$) will be calculated through (2.12), (2.13) (resp. (2.34) or (2.35)) where

$$g_p(\mathcal{F}) \equiv \frac{1}{2} \sum_{a \in \mathcal{L}} \delta_{ap} c''_a(\bar{f}_a), \quad (2.46)$$

$$\mu_p(\mathcal{F}) \equiv \sum_{a \in \mathcal{L}} \delta_{ap} c'_a(\bar{f}_a), \quad (2.47)$$

$$h_p(\mathcal{F}) \equiv \mu_p(\mathcal{F}) - 2g_p(\mathcal{F})f_p. \quad (2.48)$$

$$g_p^q(\mathcal{F}) \equiv \frac{1}{2} \sum_{a \in \mathcal{L}} \delta_{ap} c''_a(\bar{f}_a). \quad (2.49)$$

Hence the definition of E_{dsj} , E_{ndsjs} has been extended to transportation networks with general convex cost functions.

The numerical application of these operators to a given $\mathcal{F} \in \mathcal{L}$ follows precisely the same pattern as in the case of the quadratic model, the only difference being that the coefficients g_p , h_p , μ_p , g_p^q are not constants any more but should be calculated at every step.

It remains to examine under what conditions E_{dsj} , E_{ndsjs} do induce algorithms for the solution of problem P_1 . As in the case of the quadratic model, conditions 1', 2' of Theorem (2.2) are satisfied. It is intuitively expected that 3', 4' of the same theorem are more apt to be valid if the approximation of \mathcal{C} by $\mathcal{C}_{\mathcal{F}}$ is faithful, that is if $c_a(\bar{f}_a)$ is sufficiently close to a quadratic function, $a \in \mathcal{L}$. This idea has been verified for *simple* networks in [10], where we give sufficient conditions so that E_w^{ndsjs} satisfies 3', 4'. We will go here through a similar analysis for the operator E_w^{ndsjs} in the case of a general network.

Fix $w \in \mathcal{W}$ and let $\mathcal{F} \in \mathcal{L}$. We set $\mathcal{F}' \equiv E_w^{ndsjs} \mathcal{F}$. The change of the total cost

$$\Delta C \equiv C(\bar{\mathcal{F}}) - C(\bar{\mathcal{F}}') \quad (2.50)$$

is given by

$$\Delta C = \sum_{a \in \mathcal{L}} \delta_{hp} [c_a(\bar{f}_a) - c_a(\bar{f}'_a)] + \sum_{a \in \mathcal{L}} \delta_{hq} [c_a(\bar{f}_a) - c_a(\bar{f}'_a)] \quad (2.51)$$

where $p, q \in \mathcal{P}_w$ are the paths which are "equilibrated." Applying Taylor's theorem,

$$\begin{aligned} \Delta C = & - \sum_{a \in \mathcal{L}} \delta_{hp} [c'_a(\bar{f}_a) (\bar{f}'_a - \bar{f}_a) + \frac{1}{2} c''_a(\hat{f}_a) (\bar{f}'_a - \bar{f}_a)^2] \\ & - \sum_{a \in \mathcal{L}} \delta_{hq} [c'_a(\bar{f}_a) (\bar{f}'_a - \bar{f}_a) + \frac{1}{2} c''_a(\hat{f}_a) (\bar{f}'_a - \bar{f}_a)^2] \end{aligned}$$

with \hat{f}_a between \bar{f}_a and \bar{f}'_a . Note that in the first sum $\bar{f}'_a - \bar{f}_a = f_p - f'_p$ while in the second sum $\bar{f}'_a - \bar{f}_a = f_q - f'_q$.

Let us set

$$\mu_q^p = \mu_q^p(\mathcal{F}) \equiv \sum_{a \in \mathcal{L}} \delta_{aq} c'_a(\bar{f}_a), \quad (2.52)$$

$$\mu_p^q = \mu_p^q(\mathcal{F}) \equiv \sum_{a \in \mathcal{L}} \delta_{ap} c'_a(\bar{f}_a), \quad (2.53)$$

$$\hat{g}_q^p \equiv \hat{g}_q^p(\hat{\mathcal{F}}) = \frac{1}{2} \sum_{a \in \mathcal{L}} \delta_{aq} c''_a(\hat{f}_a), \quad (2.54)$$

$$\hat{g}_p^q \equiv \hat{g}_p^q(\hat{\mathcal{F}}) = \frac{1}{2} \sum_{a \in \mathcal{L}} \delta_{ap} c''_a(\hat{f}_a), \quad (2.55)$$

and recall that $f_p - f'_p = - (f_q - f'_q)$. Then

$$\begin{aligned}
\Delta C &= [\mu_p^q + \hat{g}_p^q(f'_p - f_p) - \mu_q^p - \hat{g}_q^p(f'_q - f_q)](f_p - f'_p) \\
&= [\mu_p^q + 2g_p^q(f'_p - f_p) - \mu_q^p - 2g_q^p(f'_q - f_q)](f_p - f'_p) \\
&\quad + [2g_p^q - \hat{g}_p^q + 2g_q^p - \hat{g}_q^p](f_p - f'_p)^2
\end{aligned} \tag{2.56}$$

where g_p^q, g_q^p stand for $g_p^q(\mathcal{F}), g_q^p(\mathcal{F})$ as defined by (2.49). On account of the definition of E_w^{ndsj} , we have the equilibration condition (2.33)₁ written in the form

$$\mu_p^q + 2g_p^q(f_p - f'_p) - \mu_q^p - 2g_q^p(f_q - f'_q) \leq 0, \tag{2.57}$$

where use has been made of the obvious equation

$$\mu_p - \mu_q = \mu_p^q - \mu_q^p.$$

Combining (2.56) with (2.57) and the fact that $f_p - f'_p \leq 0$ we obtain

$$\Delta C \geq [2g_p^q - \hat{g}_p^q + 2g_q^p - \hat{g}_q^p](f'_p - f_p)^2. \tag{2.58}$$

From this last result we deduce that conditions 3', 4' of Theorem (2.2) are met if

$$2g_p^q(\mathcal{F}) - g_p^q(\hat{\mathcal{F}}) + 2g_q^p(\mathcal{F}) - g_q^p(\hat{\mathcal{F}}) > 0 \tag{2.59}$$

for all possible $\mathcal{F}, \hat{\mathcal{F}}$.

A sufficient condition for the validity of (2.59) is that

$$c_a''(\bar{f}_a) > \frac{1}{2} c_a''(\hat{f}_a) \tag{2.60}$$

for all possible $\bar{f}_a, \hat{f}_a, a \in \mathcal{L}$. Suppose that we know that \bar{f}_a, \hat{f}_a are limited in some interval I_a . Then \hat{f}_a will also be limited within I_a . Condition (2.60) is obviously satisfied if

$$\min_{I_a} c_a''(\bar{f}) > \frac{1}{2} \max_{I_a} c_a''(\bar{f}). \tag{2.61}$$

Thus, if (2.61) is valid for all $a \in \mathcal{L}$, then E_w^{ndsj} satisfies conditions 1', 2', 3', 4' of Theorem (2.2) for any $w \in \mathcal{W}$. Using Theorems (2.2) and (2.1) we reach the following conclusion:

THEOREM (2.6): *Let $\mathcal{T} = (\mathcal{G}, \mathcal{D}, \mathcal{C})$ be a (general) transportation network. If condition (2.61) is satisfied for all $a \in \mathcal{L}$, then the equilibration operator E_{ndsj} induces an algorithm for the solution of problem $P_2[\mathcal{T}]$.*

Actually condition (2.61) states that the oscillation of the function $c_a''(\bar{f}_a)$ on I_a is not very large, or in other words, that $c_a(\bar{f}_a)$ is sufficiently close to some quadratic function on I_a . Thus Theorem (2.6) is in accordance with the intuitive idea cited before.

In order to put the assertion of Theorem (2.6) into practical use we have to find intervals I_a with the property stated above.

Using the feasibility condition (1.1) we conclude that we may take

$$I_a = [\phi_a, \Phi_a] \tag{2.62}$$

where

$$\phi_a \equiv \sum_{w \in \mathcal{W}_a} d_w, \quad \Phi_a \equiv \sum_{w \in \mathcal{W}_a} d_w. \tag{2.63}$$

Here \mathcal{W}_a stands for the set of all $w \in \mathcal{W}$ which are connected by at least one path p containing the link a , and $\hat{\mathcal{W}}_a$ stands for the set of all $w \in \mathcal{W}_a$ with respect to which a is total (if $\hat{\mathcal{W}}_a = \emptyset$, then $\phi_a \equiv 0$).

Note that (2.61) is more apt to be satisfied if I_a is "small." Actually the I_a , as given by (2.63), are the best possible (i.e., the smallest possible) if we are to expect a convergent sequence $\{E_{nds_j}^n \mathcal{F}^{(0)}\}$ for every $\mathcal{F}^{(0)} \in \mathcal{L}$. In practice, though, we are merely interested in knowing whether $\{E_{nds_j}^n \mathcal{F}^{(0)}\}$ converges for a specific $\mathcal{F}^{(0)}$, namely the one selected as the starting point. If we restrict ourselves to this problem, then it is possible, at least for special types of networks, to obtain I_a which are *proper* subsets of the I_a given by (2.63) and hence are preferable.

Concluding this section, we want to emphasize that a very large subset of the set of cost functions which satisfy conditions 1, 2, 3, 4, of section 1.3 do satisfy also (2.61), and hence the operator E_{nds_j} induces an algorithm for the solution of problem P_1 for a very wide class of transportation networks. We also want to emphasize that it is worthwhile to try to solve problem P_1 by means of one of the operators E_{ds_j} or E_{nds_j} even if conditions 3', 4' of Theorem (2.2) are not met. In fact we have shown that the above operators always satisfy conditions 1, 2 of Theorem (2.1) and hence, according to Theorem (2.3), if $\{E^n \mathcal{F}^{(0)}\}$ converges, then it will converge to a solution of problem P_1^{10} .

2.4. Comparison Between E_{ds_j} and E_{nds_j}

From the theoretical point of view E_{nds_j} is superior to E_{ds_j} since it can be used for the solution of problem P_1 even in the case of not (almost) simple networks. From the practical point of view, though, E_{ds_j} also has some advantages. In fact $E_w^{ds_j}$ equilibrates the whole set of paths \mathcal{P}_w but does so perfectly only if \mathcal{P}_w is disjoint. On the other hand $E_w^{nds_j}$ equilibrates only the two "most unbalanced" paths but does so perfectly in all cases. It is clear that in the case of a network which is not almost simple we must apply E_{nds_j} .

In the case of an almost simple network we advise the application of E_{nds_j} if most of the links are simple and \mathcal{P}_w contains many (more than two) paths for at least one $w \in \mathcal{W}$. On the other hand, we advise the application of E_{ds_j} in the case of a network in which \mathcal{P}_w contains few (of the order of two) paths for all $w \in \mathcal{W}$ and there are relatively numerous double and/or total links in the network.

3. Appendix I. A Network With a Nontrivial $R[\overline{\mathcal{F}}]$

We consider the network of figure 2 with links

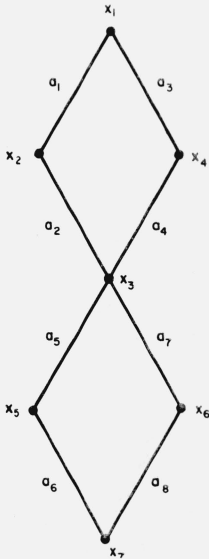


FIGURE 2.

$$a_1 = (x_1, x_2), a_2 = (x_2, x_3), a_3 = (x_1, x_4), a_4 = (x_4, x_3),$$

$$a_5 = (x_3, x_5), a_6 = (x_5, x_7), a_7 = (x_3, x_6), a_8 = (x_6, x_7),$$

and paths

$$p_1 = (a_1, a_2, a_5, a_6), p_2 = (a_1, a_2, a_7, a_8),$$

$$p_3 = (a_3, a_4, a_7, a_8), p_4 = (a_3, a_4, a_5, a_6).$$

Let d be the demand associated with the pair (x_1, x_7) . For this network we have

$$f_{p_1} + f_{p_2} + f_{p_3} + f_{p_4} = d,$$

$$f_{p_1} + f_{p_2} = \bar{f}_{a_1}, \quad f_{p_1} + f_{p_4} = \bar{f}_{a_5},$$

$$f_{p_1} + f_{p_2} = \bar{f}_{a_2}, \quad f_{p_1} + f_{p_4} = \bar{f}_{a_6},$$

$$f_{p_3} + f_{p_4} = \bar{f}_{a_3}, \quad f_{p_2} + f_{p_3} = \bar{f}_{a_7},$$

¹⁰ Such examples have been constructed.

$$f_{p_3} + f_{p_4} = \bar{f}_{a_4}, \quad f_{p_2} + f_{p_3} = \bar{f}_{a_8},$$

$$f_{p_s} \geq 0, \quad s = 1, \dots, 4. \quad (I.1)$$

The feasibility conditions on $\bar{\mathcal{F}}$ read:

$$\bar{f}_{a_s} \geq 0, \quad s = 1, \dots, 8,$$

$$\bar{f}_{a_1} = \bar{f}_{a_2}, \bar{f}_{a_3} = \bar{f}_{a_4}, \bar{f}_{a_5} = \bar{f}_{a_6}, \bar{f}_{a_7} = \bar{f}_{a_8}, \quad (I.2)$$

$$\bar{f}_{a_1} + \bar{f}_{a_3} = \bar{f}_{a_5} + \bar{f}_{a_7} = d.$$

Suppose now that $\bar{\mathcal{F}}$ is given such that (I.2) are satisfied.

Assume first that at least one of the \bar{f}_a is 0. Then it can be shown that there exists a unique solution of (I.1), i.e., in this case, $R[\bar{\mathcal{F}}]$ contains a unique element.

On the other hand, if $\bar{\mathcal{F}}$ is such that $\bar{f}_{a_s} > 0, s = 1, \dots, 8$, then it can be shown that the solutions of (I.1) form a one parameter family, i.e. $R[\bar{\mathcal{F}}]$ is a convex subset of a one dimensional vector space.

4. Appendix II. An Example of a Network Which is not Almost Simple and for Which E_{dsj} Fails

In this appendix we present an example which shows that the assertion of Theorem (2.4) is the best possible. To be precise, we consider the network of figure 3 with links

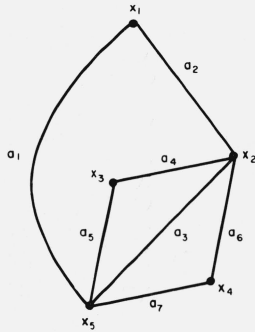


FIGURE 3.

$$a_1 = (x_1, x_5), a_2 = (x_1, x_2), a_3 = (x_2, x_3), a_4 = (x_2, x_3),$$

$$a_5 = (x_3, x_5), a_6 = (x_2, x_4), a_7 = (x_4, x_5),$$

and paths

$$p_1 = a_1, p_2 = (a_2, a_4, a_5), p_3 = (a_2, a_3),$$

$$p_4 = (a_2, a_6, a_7).$$

Let d be the demand associated with the pair (x_1, x_5) large enough so that all paths operate at a nonzero level. We assume that the cost functions are quadratic of the form

$$c_a(f_a) = g_a f_a^2 + h_a f_a,$$

where

$$g_{a_1} = g_{a_3} = 2g, g_{a_2} = 8g, g_{a_4} = g_{a_5} = g_{a_6} = g_{a_7} = g,$$

$$h_{a_1} = 3h, h_{a_2} = h_{a_4} = h_{a_5} = h_{a_6} = h_{a_7} = h, h_{a_3} = 2h,$$

with arbitrary $g > 0, h \geq 0$.

Note that the link a_2 is triple and hence this network is not almost simple. Thus Theorem (2.4) does not guarantee that $\{E_{dsj}^n \mathcal{F}^{(0)}\}$ converges. In fact we will prove the following interesting result. The sequence $\{E_{dsj}^n \mathcal{F}^{(0)}\}$ converges only if the initially chosen $\mathcal{F}^{(0)}$ satisfies $f_{p_1}^{(0)} = f_{1p_1}$, where \mathcal{F}_1 is a solution of problem P_1 .

To prove this result select $\mathcal{F} \in \mathcal{L}$. We want to calculate $\Delta f_{p_1} = f'_{p_1} - f_{p_1}$. It will turn out below that h_{p_1} is the smallest of the h_{p_r} . From (2.10), (2.12),

$$\Delta f_{p_1} = \frac{M - \mu_{p_1}}{2g_{p_1}}. \quad (\text{II.1})$$

Proceeding to the calculation and using (2.8), (2.9), (2.10) and the obvious relations

$$\bar{f}_{a_1} = f_{p_1}, \bar{f}_{a_2} = f_{p_2} + f_{p_3} + f_{p_4}, \bar{f}_{a_3} = f_{p_3}, \bar{f}_{a_4} = \bar{f}_{a_5} = f_{p_2}, \bar{f}_{a_6} = \bar{f}_{a_7} = f_{p_4},$$

we end up with

$$g_{p_1} = 2g, g_{p_2} = g_{p_3} = g_{p_4} = 10g,$$

$$\mu_{p_1} = 4gf_{p_1} + 3h,$$

$$h_{p_1} = 3h,$$

$$h_{p_2} = 16g(f_{p_4} + f_{p_3}) + 3h,$$

$$h_{p_3} = 16g(f_{p_2} + f_{p_4}) + 3h,$$

$$h_{p_4} = 16g(f_{p_3} + f_{p_2}) + 3h.$$

Whence, from (2.13),

$$M = \frac{13gd + 6h - 8gf_{p_1}}{2}$$

where use has been made of

$$f_{p_1} + f_{p_2} + f_{p_3} + f_{p_4} = d.$$

Substituting into (II.1) we obtain

$$\Delta f_{p_1} = \frac{13d - 16f_{p_1}}{8}. \quad (\text{II.2})$$

Suppose now that we apply E_{dsj} on \mathcal{F}' . Let $\Delta \mathcal{F}' \equiv E_{dsj} \mathcal{F}' - \mathcal{F}'$. From (II.2),

$$\Delta f'_{p_1} = \frac{13d - 16f'_{p_1}}{8} = \frac{13d - 16(f_{p_1} + \Delta f_{p_1})}{8} = \frac{13d - 16f_{p_1}}{8} - 2\Delta f_{p_1} = \Delta f_{p_1} - 2\Delta f_{p_1} = -\Delta f_{p_1}.$$

Thus, $\{E_{dsj}^n \mathcal{F}^{(0)}\}$ will cycle unless $\mathcal{F}^{(0)}$ has been chosen in such a way that $\Delta f_{p_1}^{(0)} = 0$.¹¹

5. Appendix III. Calculation of the Smallest Number Modulo Which a Flow Pattern Satisfies the Equations of Equilibrium

In this appendix we present a procedure which enables us to calculate the smallest number ϵ modulo which a given flow pattern \mathcal{F} satisfies the equilibrium equations (1.13).¹²

Fix $w \in \mathcal{W}$, and let f_1, \dots, f_m be the flows on the paths of \mathcal{P}_w and c'_1, \dots, c'_m the corresponding marginal costs. We want to find the minimum of the values of ϵ_w such that for any p, q in $\{1, \dots, m\}$, if $f_p \geq d\epsilon_w$, then either

$$|c'_p - c'_q| < c\epsilon_w/d \text{ holds, or both of } c'_p < c'_q + c\epsilon_w/d \text{ and } f_q < d\epsilon_w \text{ hold.}$$

¹¹ Note that if $\Delta f_{p_1}^{(0)} = 0$ we can apply Theorem (2.4), since a_2 becomes then *total* for the network emerging by the omission of the path p_1 .

¹² This method has been kindly communicated to us by Alan Goldman.

This last disjunction is equivalent to an exclusive disjunction: either $|c'_p - c'_q| < c\epsilon_w/d$ holds, or $f_q < d\epsilon_w$ and $c'_p - c'_q < c\epsilon_w/d \leq |c'_p - c'_q|$ hold, with the last condition equivalent to the conjunction of $c'_p < c'_q$ and $c\epsilon_w/d \leq c'_q - c'_p$.

Number so that $0 = f_0 \leq f_1 \leq \dots \leq f_m$. We ask whether it is possible to choose ϵ_w in the interval $(f_t/d, f_{t+1}/d]$. For such an ϵ_w , the requirement is that if $p > t$, then for each q either $\epsilon_w > (d/c)|c'_p - c'_q|$, or $q \leq t$ and $c'_p < c'_q$ and $\epsilon_w \leq (d/c)(c'_q - c'_p)$. That is, if $p > t$, then (a) $q > t$ implies $\epsilon_w > (d/c)|c'_p - c'_q|$, while (b) $q \leq t$ and $c'_p \geq c'_q$ imply the same conclusion. Since this is to hold for all $p > t$, we see that (a) is equivalent to

$$\epsilon_w > (d/c) \max\{|c'_p - c'_q| : p > t, q > t\} = (d/c) [\max_{p>t} c'_p - \min_{q>t} c'_q], \quad (\text{III.1})$$

while (b) is equivalent to

$$\epsilon_w > (d/c) \max\{c'_p - c'_q : q \leq t < p, c'_p \geq c'_q\} = (d/c) \max\{\max_{p>t} c'_p - \min_{q \leq t} c'_q, 0\}. \quad (\text{III.2})$$

Combining (III.1) and (III.2), and noting that the lower bound for ϵ_w in (III.1) is nonnegative, we get

$$\epsilon_w > (d/c) [\max_{p>t} c'_p - \min_{q>t} c'_q, \max_{p>t} c'_p - \min_{q \leq t} c'_q] = (d/c) [\max_{p>t} c'_p - c'_{\min}] \quad (\text{III.3})$$

where $c'_{\min} = \min\{c'_1, \dots, c'_m\}$. But (III.3) can hold, for an ϵ_w in interval $(f_t/d, f_{t+1}/d]$, if and only if

$$f_{t+1}/d > (d/c) [\max_{p>t} c'_p - c'_{\min}]. \quad (\text{III.4})$$

Thus the infimum of the allowable ϵ_w -values is f_t/d for the smallest t such that (III.4) holds.

We set

$$\epsilon \equiv \min_{w \in \mathcal{W}} \epsilon_w$$

and obviously this ϵ is the smallest number modulo which (1.13) are satisfied by \mathcal{F} .

6. References

- [1] Pigou, A. C., The Economics of Welfare, First edition (Macmillian, London, 1920).
- [2] Wardrop, J. G., Some theoretical aspects of road traffic research, Proc. Inst. Civ Eng. Part II, **1**, 325-378 (1952).
- [3] Beckmann, M. J., On the theory of traffic flow in networks, Traffic Quart. **21**, 109-116 (1967).
- [4] Almond, J., Traffic Assignment With Flow Dependent Journey Times, presented at the 3rd Int. Symp. on the Theory of Traffic Flow, New York, June 1965. Published in "Vehicular Traffic Science" pp. 222-234 (American Elsevier, New York, 1967).
- [5] Jorgensen, N. O., Some Aspects of the Urban Traffic Assignment Problem, Graduate Report, Institute of Transportation and Traffic Engineering, University of California, Berkeley, July 1963.
- [6] Beckmann, M. J., McGuire, C. B., Winsten, C. B., Studies in the Economics of Transportation (Yale University Press, New Haven, Conn. 1965).
- [7] Ford, L. R., and Fulkerson, D. R., Flows in Networks (Princeton, Princeton, N. J., 1962).
- [8] Luce, R. D., and Raiffa, H., Games and Decisions (John Wiley & Sons, New York, N. Y., 1965).
- [9] Hadley, G., Nonlinear and Dynamic Programming (Addison Wesley, Reading, Mass., 1964).
- [10] Dafermos, S. C., Traffic Assignment and Resource Allocation in Transportation Networks, PhD Thesis, Johns Hopkins University (1968).
- [11] Charnes, A., and Cooper, W. W., Multicopy Traffic Network Models, Proceedings of the Symposium on the Theory of Traffic Flow (R. Herman, Ed.) (1961).
- [12] Charnes, A., and Cooper, W. W., External Principles for simulating traffic flow in a network, Proc. Natl. Acad. Sci. U.S. **44**, pp. 201-204 (1958).
- [13] Ford, L. R., and Fulkerson, D. R., A suggested computation for maximal and multicommodity network flows, Management Sci. **51**, 97-101 (1958).
- [14] Tomlin, J. A., Minimum-cost multicommodity network flows, JORSA pp. 45-54 (Jan.-Feb., 1966).

(Paper 73B2-290)