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A Digital Computer Technique for Calculating the Step Response of Lumped or Distributed Networks

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This paper discusses a technique to solve step response problems for lumped or distributed networks with the aid of a digital computer. The Rosenbrock graphical cursor technique for obtaining the step response from the frequency response through the inverse Laplace Transformation was adapted for computer use. In addition, it was modified to increase its accuracy when used with a digital computer.

The response of a series RLC lumped network is computed and the numerical solution is compared to the analytical solution. Also, a numerical solution is given for the step response of a transmission line possessing skin-effect metal loss and Debye dielectric loss.

Key words: Digital computer program; distributed networks; inverse Laplace Transformation; lumped networks; Rosenbrock cursor; step response; transmission line.

1. Introduction

The transient properties of linear physical systems may be studied from two viewpoints, namely, the frequency domain and the time domain. Usually the information obtained from the system response to a unit step input is more directly relevant than the frequency domain response. However, the mathematical analysis of such systems is usually easier to accomplish in the frequency domain. The time domain response is conveniently measured [Samulon, 1956; Oliver 1961].

The connection between the time and frequency domains is the Laplace transform for semi-infinite time functions ($f(t)=0,\ t<0$) or the Fourier transform for time functions that exist for $-\infty<t<+\infty$ [Gardner and Barnes, 1942; Hildebrand, 1962]. It is not usually too difficult to obtain the Laplace transform $F(s)$ of a time function $f(t)$. $F(s)$ is given by

$$F(s)=\int_0^\infty f(t)e^{-st}dt; \tag{1}$$

manipulations of the transform $F(s)$ are also straightforward. The real difficulty in this class of problems lies in obtaining the inverse Laplace transformation given by

$$f(t)=\frac{1}{2\pi j}\int_{Br_1}F(s)e^{st}ds. \tag{2}$$

Br_1 is the Bromwich contour [McLachlan, 1963] from $\beta-j\infty$ to $\beta+j\infty$. β is chosen so that all the singularities of the integrand are on its left.

The unit step response $h(t)$ of a stable linear system, initially at rest, is

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$$h(t)=\frac{1}{2\pi j}\int_{-j\infty}^{j\infty}\frac{A(s)}{s}e^{st}ds \tag{3}$$

in which $A(s)$ is the system transfer function [Rosenbrock, 1955].

2. Derivation of Rosenbrock Cursor

Rosenbrock [1955, 1959] developed a graphical technique for evaluation of (3). In an appendix [Rosenbrock, 1955], he showed that $h(t)$ may be expressed as

$$h(t)=\frac{2}{\pi}\int_{\omega=0}^{\infty} Re[A(j\omega)]\frac{\sin(\omega t)}{\omega}d\omega \tag{4}$$

or

$$h(t)=h(t=\infty)+\frac{2}{\pi}\int_{\omega=0}^{\infty}\text{Im}[A(j\omega)]\frac{\cos(\omega t)}{\omega}d\omega. \tag{5}$$

The transfer function, $A(s)$, of the linear system must satisfy the condition that there are no poles in the right-hand half-plane or on the imaginary axis, although a pole of $A(s)/s$ at the origin is permissible. The unique feature of the Rosenbrock method lies in the change of the variable of integration from ω to $\ln\omega$. Notice that

$$\frac{d\omega}{\omega}=d(\ln\omega). \tag{6}$$

Substituting (6) into (5) obtains

$$h(t)=h(\infty)+\frac{2}{\pi}\int_{\omega=0}^{\infty}\text{Im}[A(j\omega)]\ \cos(\omega t)\ d(\ln\omega) \tag{7}$$

Letting

$$\theta=\omega t, \tag{8}$$

(7) may be approximated by

$$h(t)\simeq h(\infty)+\frac{2}{\pi}\sum_{n=1}^{\infty}\text{Im}[A(j\omega_n)]\ \cos(\theta_n)[\Delta_n(\ln\theta)]. \tag{9}$$

The product $\cos(\theta_n)[\Delta_n(\ln\theta)]$ is the incremental area under the curve of $\cos(\theta)$ versus $\ln\theta$, in figure 1. With the assumption that $\text{Im}[A(j\omega)]$ remains relatively constant throughout the interval $\Delta_n(\ln\theta)$, it is approximately true that

$$h(t)\simeq h(\infty)+\frac{2}{\pi}\sum_{n=1}^{N}\text{Im}[A(j\omega_n)]WF_n, \tag{10}$$

where

$$WF_n=\int_{\theta_a}^{\theta_b}\cos(\theta)\ d(\ln\omega). \tag{11}$$

or

$$WF_n=\int_{\theta_a}^{\theta_b}\frac{\cos(\theta)}{\theta}\ d\theta. \tag{12}$$

Equations (10) and (11) form the basis for the Rosenbrock cosine cursor.

Rosenbrock's graphical technique consists of laying his transparent cursor over a plot of the imaginary part of the transfer function, $\text{Im}[A(j\omega)]$, versus $\log(\omega)$ and summing up the intercepts of the cursor and the $\text{Im}[A(j\omega)]$ plot. The cursor is shifted over the $\text{Im}[A(j\omega)]$ plot for various times, t. A particular position of the cursor over the imaginary part plot yields $f(t_1)$. Another position of the cursor yields another value of $f(t)$, i.e., $f(t_2)$; and so on.

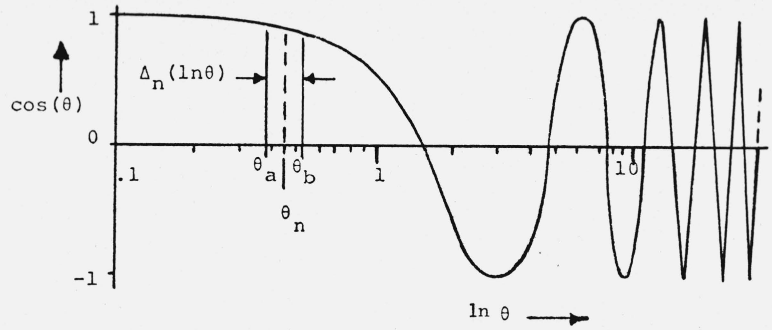

FIGURE 1. *Cos* (θ) *versus ln* (θ).

3. Computer Adaption of Rosenbrock Cursor

As a means of saving time and labor in step response problems, the authors adapted the Rosenbrock cursor so that modern-day digital computers may be used to solve this class of problems.

Using the FORTRAN IV language, the subprogram ROSEBK, Appendix 1, computes (10). It uses the tabulated values for the Rosenbrock cosine cursor [Rosenbrock, 1955; Gooch, 1960]. It is set up as REAL FUNCTION ROSEBK (*T*, *AIM*, *HINF*) where the parameters are: *T*, the value of time t in seconds; *AIM*, the name of another FUNCTION subprogram to calculate $\mathrm{Im}[A(j\omega_n)]$; and *HINF*, the value of $h(t=\infty)$. This program is adequate for well-behaved functions whose imaginary parts do not have rapid changes in them. In this program $N=35$.

To take advantage of the increased accuracy available with a digital computer, the cursor method was expanded to employ 168 data points ($N=168$). This modification consisted of breaking the cos (θ) versus log (θ) curve (fig. 1) into many segments as tabulated in table 1.

The small and large θ regions respectively, ($\theta<.01$) and ($\theta>\theta_{uL}$), need more discussion. If $\mathrm{Im}[A(j\omega)]$ behaves as $\simeq a\omega$ for $0<\theta<0.01$, then

$$\int_0^{\omega t=0.01} \mathrm{Im}[A(j\omega)]\ \frac{\cos(\omega t)\,d\omega}{\omega} \simeq \int_0^{\omega t=0.01} a\omega\frac{1}{\omega}\,d\omega \tag{13}$$

$$=1.0\times\mathrm{Im}\left[A\left(j\frac{0.01}{t}\right)\right]. \tag{14}$$

An approximation is also necessary for the region $\theta>31\frac{\pi}{2}$.

Table 1. *Scheme for subdividing cos* (θ) *versus log* (θ) *curve*

θ region	Division
$0 \to 0.01$	1 point (approximation)
$0.01 \to 0.1$	20 equal $\Delta(\ln\omega)$ segments
$0.1 \to 3\frac{\pi}{2}$	70 equal $\Delta(\ln\omega)$ segments
$3\frac{\pi}{2} \to 7\frac{\pi}{2}$	20 equal $\Delta(\ln\omega)$ segments
$7\frac{\pi}{2} \to 15\frac{\pi}{2}$	12 equal samples/period
$15\frac{\pi}{2} \to 31\frac{\pi}{2}$	8 equal samples/period
$31\frac{\pi}{2} \to \theta_{uL}$	1 point
$\theta_{uL} \to \infty$	θ_{uL} chosen such that $\int_{\theta_{uL}}^{\infty} \frac{\cos(\theta)}{\theta}d\theta \simeq 0$

The integral $$\int_{\theta_{uL}}^{\infty} \mathrm{Im}[A(j\omega)] \frac{\cos(\theta)}{\theta} d\theta$$

is approximately 0 if $\mathrm{Im}[A(j\omega)]$ is close to 0 or essentially constant, because $\cos(\theta)$ alternates very rapidly for large θ when plotted on a logarithmic scale (fig. 1). For $\theta > 31\frac{\pi}{2}$ the sum of the positive and negative contributions tend to cancel [Gooch, 1960]. The upper limit, θ_{uL}, can be found from the cosine-integral function $Ci(x)$ [Jahnke and Emde, 1945],

$$Ci(x) = -\int_{x}^{\infty} \frac{\cos(\theta)}{\theta} d\theta \tag{15}$$

$$\simeq \frac{\sin(x)}{x}, \; x \gg 1. \tag{16}$$

Thus θ_{uL} is chosen to be the next point after $\theta = 31\frac{\pi}{2}$ when $\sin(\theta) = 0$, i.e.,

$$\theta_{uL} = 32\frac{\pi}{2} \cdot \tag{17}$$

A computer program was written to calculate θ_n and WF_n, (11) or (12), using the subdivision scheme listed in table 1.

The low and high θ approximations impose limits upon the values of time, t, at which accurate solutions can be obtained. The limits can be visualized by considering the graphical technique of placing the transparent cursor over a plot of $\mathrm{Im}[A(j\omega)]$ versus $\log \omega$. For small t, the cursor will be shifted to the right. Two possible sources of error may arise. One, the function $\mathrm{Im}[A(j\omega)]$ to the left of the cursor may no longer be adequately approximated by a linear function of ω. Second, there may be rapid oscillations or changes in the $\mathrm{Im}[A(j\omega)]$ curve under the cursor for large values of ω (see fig. 4) such that the assumption, that $\mathrm{Im}[A(j\omega)]$ is essentially a constant for each sample segment $d(\ln \omega)$ is no longer correct.

The determination of the upper and lower limits of validity for t must be considered as a special case for each problem. For large values of time, t, the cursor will be shifted to the left. Errors encountered in this region will probably not be as significant as those for small t. In this case a significant portion of the $\mathrm{Im}[A(j\omega)]$ curve is to the right of the cursor. Thus the assumption that $\mathrm{Im}[A(j\omega)]$ for $\theta > \theta_{uL}$ is close to 0 or is essentially constant is no longer valid, but the $\cos\theta$ curve changes sign rapidly and the large θ summation may tend to cancel out.

Appendix 2 contains the listing for ROSBKM subprogram for computing (10) with $N = 168$. It is set up as REAL FUNCTION ROSBKM (*T*, *AIM*, *HINF*). The parameters have the same definition as used in ROSEBK.

4. Examples

To demonstrate this technique of step response calculation and to obtain a measure of the accuracy involved, two examples are presented below. The first example is the current step response of a lumped series RLC circuit driven by a unit step voltage source. The second example deals with a distributed circuit and presents the original solution for the voltage step response of a coaxial transmission line with simple skin effect and a dielectric with Debye molecular relaxation. To the authors' knowledge this problem has not been solved prior to the work reported here.

RLC Series Circuit. The initial conditions are assumed to be zero. The transfer function is the loop admittance

$$Y(s) = \frac{sC}{1 + sRC + s^2LC} \cdot \tag{18}$$

The analytic expression for $i(t)$ due to a unit voltage step applied at $t = 0$ is

$$i(t) = \frac{1}{\omega_n L} e^{-\alpha t} \sin(\omega_n t). \qquad t > 0; \tag{19}$$

where,

$$\alpha_1 = \frac{R}{2L}, \tag{20}$$

$$\omega_0 = \frac{1}{\sqrt{LC}}, \tag{21}$$

and

$$\omega_n = \sqrt{\omega_0^2 - \alpha_1^2}. \tag{22}$$

The imaginary part of $Y(j\omega)$ is

$$\mathrm{Im}[Y(j\omega)] = \frac{\omega C\left[1-\left(\frac{\omega}{\omega_0}\right)^2\right]}{\left[1-\left(\frac{\omega}{\omega_0}\right)^2\right]^2 + (\omega RC)^2}. \tag{23}$$

A program was written to calculate $i(t)$ from (19) and also from the Rosenbrock cursors, $N=35$ and $N=168$ using (23). The results are shown in figure 2. Only a few of the points calculated using the cursor are shown. The computations were made from $t=0.01$ μs to $t=5.00 \mu s$. The error was computed as a percentage of the absolute maximum of $i(t)$, (19). The average error using ROSEBK ($N=35$) was ± 5.32 percent, while the average error was ± 0.0177 percent using ROSBKM ($N=168$).

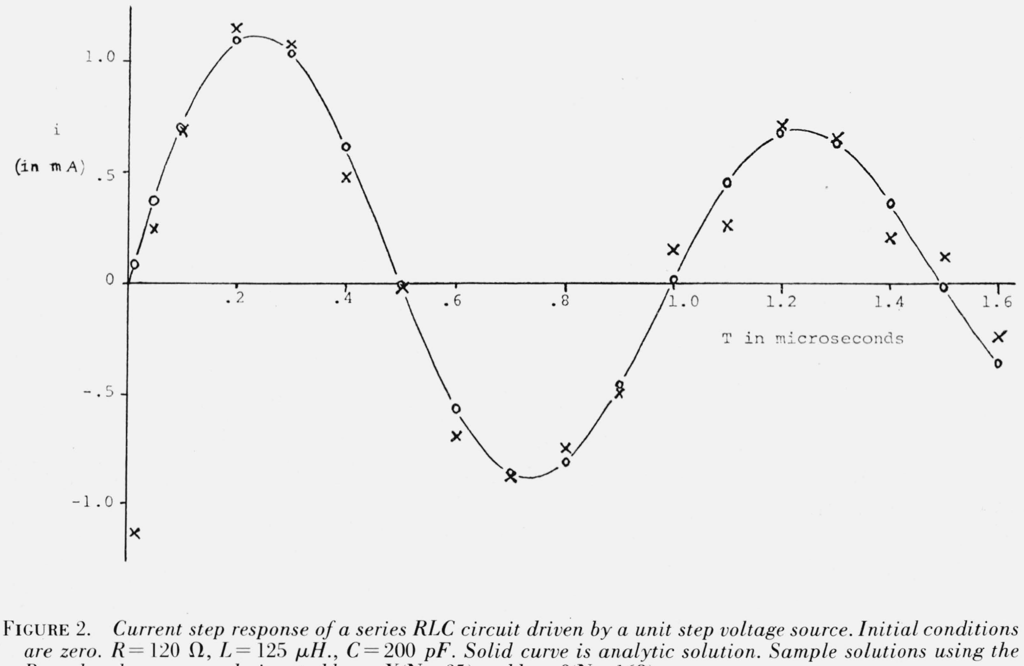

FIGURE 2. *Current step response of a series RLC circuit driven by a unit step voltage source. Initial conditions are zero. $R=120\ \Omega$, $L=125\ \mu H.$, $C=200\ pF$. Solid curve is analytic solution. Sample solutions using the Rosenbrock cursor are designated by an* X(N=35) *and by a* 0(N=168).

Lossy Transmission Line. This example demonstrates the application of the Rosenbrock cursor technique to more complicated problems. It was used to calculate the voltage step response of a coaxial transmission line with losses resulting from two causal material models: (1) simple skin effect and (2) Debye dielectric molecular relaxation. The equivalent circuit per unit length is shown in figure 3.

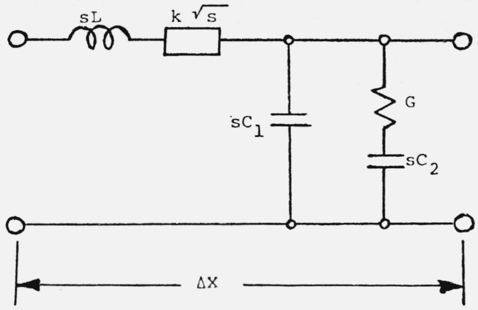

FIGURE 3. *Equivalent circuit for transmission line with simple skin effect and a dielectric with Debye molecular relaxation.*

The propagation function, $\gamma(s)$, is defined as the geometric mean of the series impedance $Z(s)$ and the shunt admittance $Y(s)$ per meter. Consequently $\gamma(s)$ is given by

$$\gamma(s)=\sqrt{Y(s)Z(s)}=\frac{s}{V_1}\left[\left(1+\frac{k}{L}\frac{1}{\sqrt{s}}\right)\frac{\left(\frac{C_0}{C_1\tau}+s\right)}{(1/\tau+s)}\right]^{1/2} \tag{24}$$

where

$$\tau=C_2/G, \qquad \text{s.} \tag{25}$$

$$C_0=C_1+C_2. \qquad \text{F/m} \tag{26}$$

and

$$V_1=\frac{1}{\sqrt{LC_1}} \qquad \text{m/s.} \tag{27}$$

The series inductance L, shunt capacitance C, and the skin effect constant k, are defined in Appendix 3. The transfer function of the line is

$$A(s)=e^{-\gamma(s)X} \tag{28}$$

which is an exponential function with an irrational argument. The time delay of the line is given by

$$TD_1=\lim_{\omega\to\infty}\mathrm{Im}\left[\frac{X\gamma(j\omega)}{\omega}\right] \qquad \text{s.} \tag{29}$$

$$=X/V_1 \tag{30}$$

To improve the accuracy of the computer calculations, it was desirable to remove the time delay of the line by a time shift,

$$T=t-TD_1. \tag{31}$$

Thus a new transfer function, $\hat{A}(s)$, must be considered.

$$\hat{A}(s)=e^{-s\frac{X}{V_1}(\zeta(s)-1)} \tag{32}$$

where

$$\zeta(s)=\left[\left(1+\frac{k}{L}\frac{1}{\sqrt{s}}\right)\frac{\left(\frac{C_0}{C_1\tau}+s\right)}{(1/\tau+s)}\right]^{1/2} \tag{33}$$

and

$$\zeta(\omega)=\eta(\omega)-j\varphi(\omega). \tag{34}$$

The minus sign is used with $\mathrm{Im}[\zeta(j\omega)]$ so that $\varphi(j\omega)$ and $\alpha(j\omega)$ are positive. Using (34). $\hat{A}(j\omega)$ may than be split into real and imaginary parts,

$$\hat{A}(j\omega)=e^{-\alpha_2 X}\left[\cos\left(\omega(\eta-1)\frac{X}{V_1}\right)-j\sin\left(\omega(\eta-1)\frac{X}{V_1}\right)\right], \tag{35}$$

where

$$\alpha_2=\frac{\omega}{V_1}\varphi(\omega). \tag{36}$$

A 100-ft. hypothetical coaxial line having a liquid dielectric was used in the numerical calculations. The data used in the calculations is listed in table 2. A plot of transmission versus frequency (fig. 4) shows a sharp cutoff, low-pass filter characteristic. The imaginary part of $\hat{A}(j\omega)$, (35) was used in the Rosenbrock cursor and is shown in figure 5. The unit step response of the hypothetical line was calculated using ROSBKM (fig. 6).

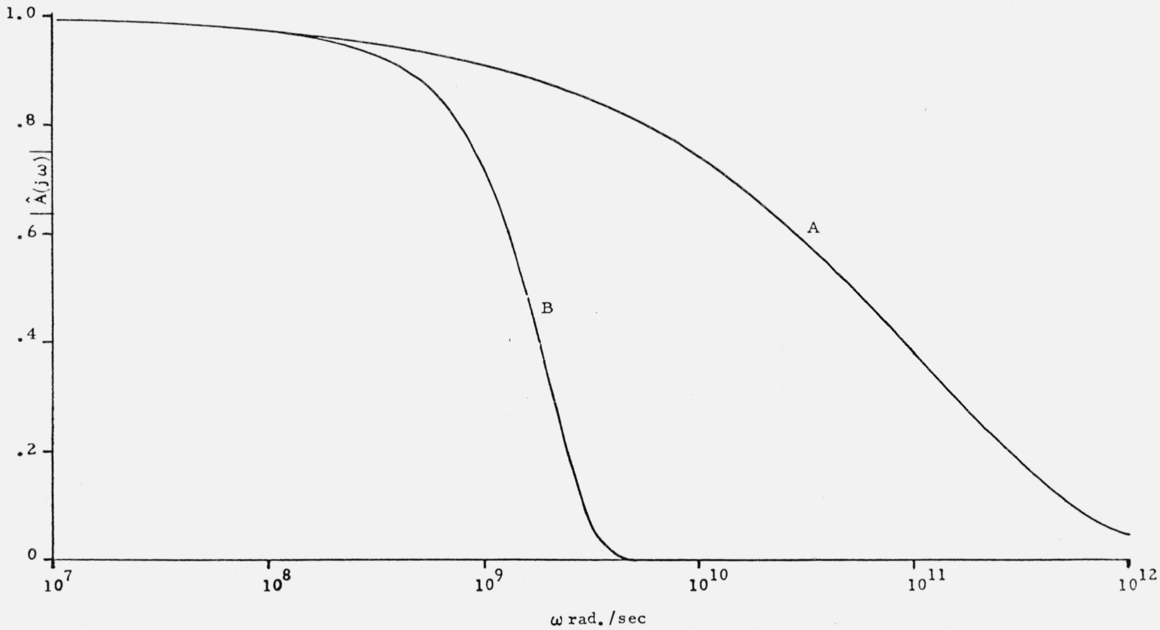

FIGURE 4. *Magnitude of the transfer function* [Â(jω)] *versus frequency. Curve A is a coaxial cable with only simple skin effect losses. Curve B is for a cable with simple skin effect and a dielectric with Debye molecular relaxation.*

The general shape of the calculated step response (fig. 6) can be verified from physical reasoning. For very high frequencies ($\omega \gg 1/\tau$), the shunt admittance $Y(s)$ reduces to simply a pure susceptance $j\omega C_1$. Likewise for low frequencies ($\omega \ll 1/\tau$), the shunt admittance is simply $j\omega(C_1+C_2)$. Thus, the very high frequencies propagate at a velocity V_1, (27), while the low frequencies propagate at the slower velocity,

$$V_2=\frac{1}{\sqrt{L(C_1+C_2)}} \tag{37}$$

These two different propagation velocities result in a longer delay time TD_2, (38), for the low frequencies than for the high frequencies, TD_1 (30).

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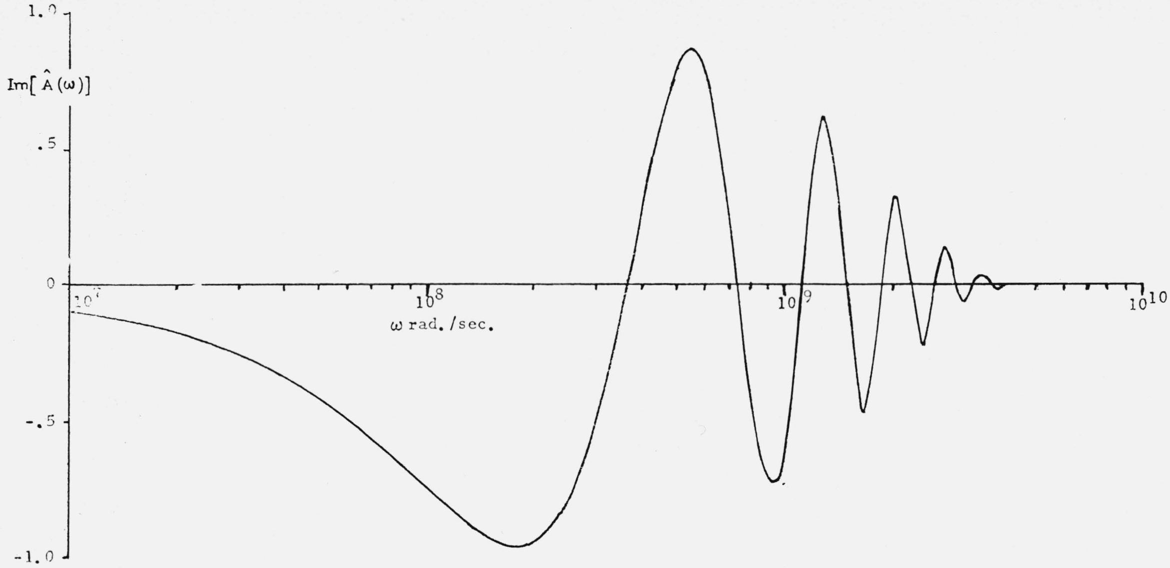

FIGURE 5. *Imaginary part of the transfer function Im[Â(jω)] versus frequency for coaxial cable with simple skin effect and a dielectric with Debye molecular relaxation.*

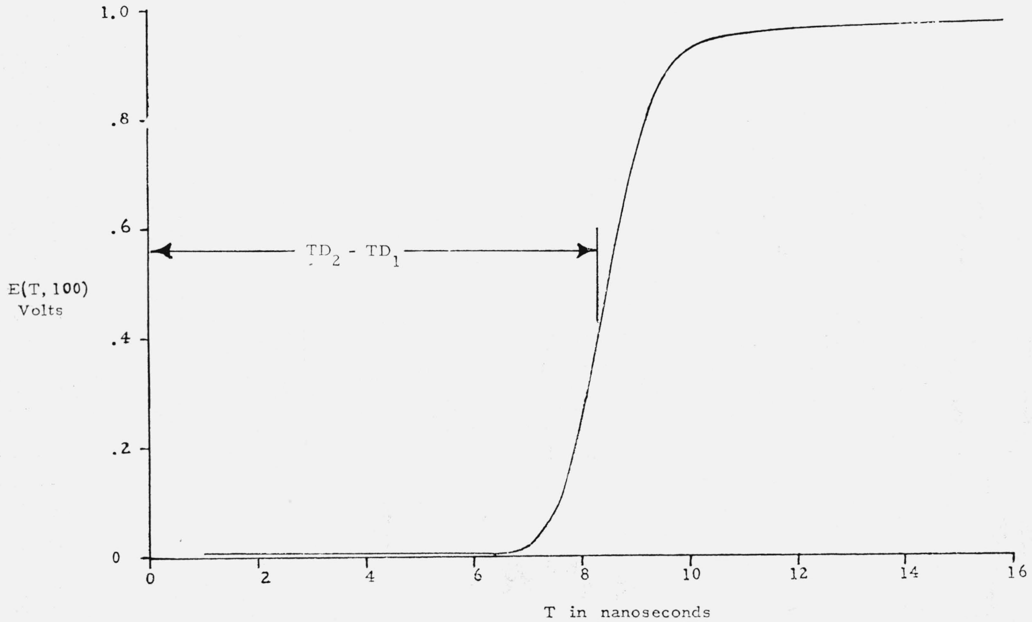

FIGURE 6. *Unit step response of 100 ft of theoretical coaxial transmission line with simple skin effect and a dielectric with Debye molecular relaxation.*

Table 2. Hypothetical coaxial transmission line data

Line length	100 ft.
Inner conductor radius	0.204 cm
Outer conductor radius	0.710 cm
Conductor conductivity	6.30 $\times 10^7$ mho/m
Low frequency relative dielectric constant	2.24
High frequency relative dielectric constant	2.00
Dielectric relaxation time	29.5 ps
Low frequency characteristic impedance	50.0 Ω

$$TD_2 = \frac{X}{V_2} \tag{38}$$

The difference between the two time delays is 8.38 ns. Thus the initial part of the output ($T < 8$ ns) is produced by the high frequency components of the input step, while the latter portion of the output ($T > 9$ ns) is produced predominantly by the low frequency components. Consequently, it follows that in comparison to a lossless line possessing a propagation velocity V_1, the output waveform's 50 percent delay time should be about 8 ns larger than that of the lossless line. That such is the case is shown in figure 6.

It should be pointed out that the desire to solve this transmission line problem was the prime motivation in developing the modified Rosenbrock cursor, ROSBKM. Because an analytic solution to this problem was not available, the cited results were compared to those obtained by a different numerical technique, namely an improved Fourier Series method [Manney, 1968]. For amplitude values greater than 0.01 the differences were less than 0.5 percent as compared to the waveform maximum value of unity. For amplitudes less than 0.01 and T less than 7 ns the ROSBKM results departed from the zero amplitude axis. The reasons for the departure are presently being determined [Ives, 1968].

The two examples shown here used analytic expressions for $\mathrm{Im}[A(j\omega)]$. It is also possible for minimum phase transmission line losses to obtain $h(t)$ from experimentally measured transmission line attenuation versus frequency data. Gooch [1960], has shown how this may be accomplished using the graphical Rosenbrock cursor technique. Another extension would be to read the measured data into a computer and to use interpolating polynomials for $\mathrm{Im}[A(j\omega)]$ and use ROSEBKM to find $h(t)$.

5. Conclusion

This paper has discussed the extension to digital computer computations of a tractable graphical technique for evaluating the step response of a linear physical system. Examples were included to demonstrate its use and accuracy.

One of the examples presented an original solution for the step response of a coaxial line possessing combined losses resulting from two causal material models: (1) simple skin effect and (2) Debye dielectric molecular relaxation.

Finally, further extensions of the method discussed here are possible. In particular, because it is assumed that $\mathrm{Im}[A(j\omega)]$ remains relatively constant throughout the interval $\Delta_n(\ln\theta)$ (fig. 1) Rosenbrock's procedure makes no use of the variation of $\mathrm{Im}[A(j\omega)]$ within the interval. If it is necessary to more effectively use the data within the interval, then the variation of $\mathrm{Im}[A(j\omega)]$ within the interval should be expressed in terms of some approximating function.

APPENDIX I

```
      REAL FUNCTION ROSEBK(T,AIM,HINF)

      THIS PROGRAM CALCULATES THE STEP RESPONSE OF A LINEAR SYSTEM.

      THE ROSENBROCK COSINE CURSOR TECHNIQUE IS USED WITH N=35.

      T=TIME IN SECONDS.

      AIM IS THE NAME OF A FUNCTION SUBPROGRAM THAT CALCULATES THE
      IMAGINARY PART OF THE SYSTEM TRANSFER FUNCTION, IM(A(JW)), AT A
      GIVEN FREQUENCY W.  W IS IN RADIANS/SECOND.

      HINF=VALUE OF TIME FUNCTION, H(T), AT T=INFINITY.

      DIMENSION THETA(35),WFM(35)
      DATA(THETA( 1)=  0.110),(WFM( 1)= 0.086)
      DATA(THETA( 2)=  0.133),(WFM( 2)= 0.087)
      DATA(THETA( 3)=  0.159),(WFM( 3)= 0.087)
      DATA(THETA( 4)=  0.191),(WFM( 4)= 0.087)
      DATA(THETA( 5)=  0.230),(WFM( 5)= 0.088)
      DATA(THETA( 6)=  0.276),(WFM( 6)= 0.089)
      DATA(THETA( 7)=  0.331),(WFM( 7)= 0.091)
      DATA(THETA( 8)=  0.398),(WFM( 8)= 0.093)
      DATA(THETA( 9)=  0.478),(WFM( 9)= 0.097)
      DATA(THETA(10)=  0.574),(WFM(10)= 0.102)
      DATA(THETA(11)=  0.689),(WFM(11)= 0.111)
      DATA(THETA(12)=  0.828),(WFM(12)= 0.127)
      DATA(THETA(13)=  0.994),(WFM(13)= 0.157)
      DATA(THETA(14)=  1.190),(WFM(14)= 0.233)
      DATA(THETA(15)=  1.430),(WFM(15)= 0.626)
      DATA(THETA(16)=  1.720),(WFM(16)=-0.560)
      DATA(THETA(17)=  2.070),(WFM(17)=-0.180)
      DATA(THETA(18)=  2.480),(WFM(18)=-0.109)
      DATA(THETA(19)=  2.980),(WFM(19)=-0.088)
      DATA(THETA(20)=  3.580),(WFM(20)=-0.096)
      DATA(THETA(21)=  4.300),(WFM(21)=-0.220)
      DATA(THETA(22)=  5.130),(WFM(22)= 0.200)
      DATA(THETA(23)=  6.080),(WFM(23)= 0.110)
      DATA(THETA(24)=  7.210),(WFM(24)= 0.155)
      DATA(THETA(25)=  8.540),(WFM(25)=-0.160)
      DATA(THETA(26)= 10.100),(WFM(26)=-0.130)
      DATA(THETA(27)= 12.500),(WFM(27)= 0.098)
      DATA(THETA(28)= 15.600),(WFM(28)=-0.123)
      DATA(THETA(29)= 18.800),(WFM(29)= 0.148)
      DATA(THETA(30)= 21.900),(WFM(30)=-0.173)
      DATA(THETA(31)= 25.100),(WFM(31)= 0.197)
      DATA(THETA(32)= 28.200),(WFM(32)=-0.222)
      DATA(THETA(33)= 31.400),(WFM(33)= 0.247)
      DATA(THETA(34)= 34.500),(WFM(34)=-0.271)
      DATA(THETA(35)= 37.700),(WFM(35)= 0.568)
      W=THETA(1)/T
      SUM=6.*(AIM(W)/WFM(1))
      DO 1 I=2,35
      W=THETA(I)/T
    1 SUM=SUM+AIM(W)/WFM(I)
      ROSEBK=HINF+SUM/100.
      RETURN
      END
```

APPENDIX II

```
REAL FUNCTION ROSBKM(T,AIM,HINF)
THIS PROGRAM CALCULATES THE STEP RESPONSE OF A LINEAR SYSTEM.
THE ROSENBROCK COSINE CURSOR TECHNIQUE IS USED WITH N=168.
T=TIME IN SECONDS.
AIM IS THE NAME OF A FUNCTION SUBPROGRAM THAT CALCULATES THE
IMAGINARY PART OF THE SYSTEM TRANSFER FUNCTION, IM(A(JW)), AT A
GIVEN FREQUENCY W.  W IS IN RADIANS/SECOND.
HINF=VALUE OF TIME FUNCTION, H(T), AT T=INFINITY.
DIMENSION THETA(168),WF(168)
DATA(THETA(  1)= 1.0592537252E  -2),(WF(  1)= 1.1512279585E  -1)
DATA(THETA(  2)= 1.1885022274E  -2),(WF(  2)= 1.1512112353E  -1)
DATA(THETA(  3)= 1.3335214322E  -2),(WF(  3)= 1.1511901820E  -1)
DATA(THETA(  4)= 1.4962356561E  -2),(WF(  4)= 1.1511636777E  -1)
DATA(THETA(  5)= 1.6788040181E  -2),(WF(  5)= 1.1511303112E  -1)
DATA(THETA(  6)= 1.8836490895E  -2),(WF(  6)= 1.1510883055E  -1)
DATA(THETA(  7)= 2.1134890398E  -2),(WF(  7)= 1.1510354243E  -1)
DATA(THETA(  8)= 2.3713737057E  -2),(WF(  8)= 1.1509688520E  -1)
DATA(THETA(  9)= 2.6607250598E  -2),(WF(  9)= 1.1508850442E  -1)
DATA(THETA( 10)= 2.9853826190E  -2),(WF( 10)= 1.1507795393E  -1)
DATA(THETA( 11)= 3.3496543916E  -2),(WF( 11)= 1.1506467212E  -1)
DATA(THETA( 12)= 3.7583740430E  -2),(WF( 12)= 1.1504795202E  -1)
DATA(THETA( 13)= 4.2169650344E  -2),(WF( 13)= 1.1502690383E  -1)
DATA(THETA( 14)= 4.7315125895E  -2),(WF( 14)= 1.1500040755E  -1)
DATA(THETA( 15)= 5.3088444424E  -2),(WF( 15)= 1.1496705359E  -1)
DATA(THETA( 16)= 5.9566214355E  -2),(WF( 16)= 1.1492506803E  -1)
DATA(THETA( 17)= 6.6834391758E  -2),(WF( 17)= 1.1487221860E  -1)
DATA(THETA( 18)= 7.4989420937E  -2),(WF( 18)= 1.1480569663E  -1)
DATA(THETA( 19)= 8.4139514165E  -2),(WF( 19)= 1.1472196866E  -1)
DATA(THETA( 20)= 9.4406087627E  -2),(WF( 20)= 1.1461659030E  -1)
DATA(THETA( 21)= 1.0279020267E  -1),(WF( 21)= 5.4749055107E  -2)
DATA(THETA( 22)= 1.0860633719E  -1),(WF( 22)= 5.4715265234E  -2)
DATA(THETA( 23)= 1.1475156358E  -1),(WF( 23)= 5.4677551496E  -2)
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DATA(THETA( 25)= 1.2810482907E  -1),(WF( 25)= 5.4588481809E  -2)
DATA(THETA( 26)= 1.3535333037E  -1),(WF( 26)= 5.4536053555E  -2)
DATA(THETA( 27)= 1.4301197055E  -1),(WF( 27)= 5.4477544072E  -2)
DATA(THETA( 28)= 1.5110395633E  -1),(WF( 28)= 5.4412250747E  -2)
DATA(THETA( 29)= 1.5965380751E  -1),(WF( 29)= 5.4339390066E  -2)
DATA(THETA( 30)= 1.6868743130E  -1),(WF( 30)= 5.4258089093E  -2)
DATA(THETA( 31)= 1.7823220080E  -1),(WF( 31)= 5.4167374985E  -2)
DATA(THETA( 32)= 1.8831703795E  -1),(WF( 32)= 5.4066164135E  -2)
DATA(THETA( 33)= 1.9897250117E  -1),(WF( 33)= 5.3953249636E  -2)
DATA(THETA( 34)= 2.1023087795E  -1),(WF( 34)= 5.3827287731E  -2)
DATA(THETA( 35)= 2.2212628270E  -1),(WF( 35)= 5.3686782763E  -2)
DATA(THETA( 36)= 2.3469476011E  -1),(WF( 36)= 5.3530070597E  -2)
DATA(THETA( 37)= 2.4797439434E  -1),(WF( 37)= 5.3355300284E  -2)
DATA(THETA( 38)= 2.6200542451E  -1),(WF( 38)= 5.3160414295E  -2)
DATA(THETA( 39)= 2.7683036649E  -1),(WF( 39)= 5.2943126033E  -2)
DATA(THETA( 40)= 2.9249414191E  -1),(WF( 40)= 5.2700896503E  -2)
DATA(THETA( 41)= 3.0904421409E  -1),(WF( 41)= 5.2430907494E  -2)
DATA(THETA( 42)= 3.2653073201E  -1),(WF( 42)= 5.2130033813E  -2)
DATA(THETA( 43)= 3.4500668215E  -1),(WF( 43)= 5.1794811748E  -2)
DATA(THETA( 44)= 3.6452804916E  -1),(WF( 44)= 5.1421406880E  -2)
DATA(THETA( 45)= 3.8515398542E  -1),(WF( 45)= 5.1005577865E  -2)
DATA(THETA( 46)= 4.0694699027E  -1),(WF( 46)= 5.0542638974E  -2)
DATA(THETA( 47)= 4.2997309952E  -1),(WF( 47)= 5.0027420543E  -2)
DATA(THETA( 48)= 4.5430208535E  -1),(WF( 48)= 4.9454226707E  -2)
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DATA(THETA( 49)= 4.8000766787E  -1),(WF( 49)= 4.8816793220E  -2)
DATA(THETA( 50)= 5.0716773851E  -1),(WF( 50)= 4.8108243315E  -2)
DATA(THETA( 51)= 5.3586459592E  -1),(WF( 51)= 4.7321044583E  -2)
DATA(THETA( 52)= 5.6618519546E  -1),(WF( 52)= 4.6446967071E  -2)
DATA(THETA( 53)= 5.9822141264E  -1),(WF( 53)= 4.5477044637E  -2)
DATA(THETA( 54)= 6.3207032152E  -1),(WF( 54)= 4.4401541782E  -2)
DATA(THETA( 55)= 6.6783448887E  -1),(WF( 55)= 4.3209928509E  -2)
DATA(THETA( 56)= 7.0562228502E  -1),(WF( 56)= 4.1890867072E  -2)
DATA(THETA( 57)= 7.4554821202E  -1),(WF( 57)= 4.0432214970E  -2)
DATA(THETA( 58)= 7.8773325076E  -1),(WF( 58)= 3.8821050574E  -2)
DATA(THETA( 59)= 8.3230522776E  -1),(WF( 59)= 3.7043727281E  -2)
DATA(THETA( 60)= 8.7939920209E  -1),(WF( 60)= 3.5085965708E  -2)
DATA(THETA( 61)= 9.2915787473E  -1),(WF( 61)= 3.2932994846E  -2)
DATA(THETA( 62)= 9.8173202134E  -1),(WF( 62)= 3.0569754488E  -2)
DATA(THETA( 63)= 1.0372809486E   0),(WF( 63)= 2.7981173717E  -2)
DATA(THETA( 64)= 1.0959729774E   0),(WF( 64)= 2.5152545146E  -2)
DATA(THETA( 65)= 1.1579859523E   0),(WF( 65)= 2.2070014132E  -2)
DATA(THETA( 66)= 1.2235077811E   0),(WF( 66)= 1.8721208209E  -2)
DATA(THETA( 67)= 1.2927370038E   0),(WF( 67)= 1.5096032569E  -2)
DATA(THETA( 68)= 1.3658833943E   0),(WF( 68)= 1.1187661710E  -2)
DATA(THETA( 69)= 1.4431685961E   0),(WF( 69)= 6.9937567920E  -3)
DATA(THETA( 70)= 1.5248267937E   0),(WF( 70)= 2.5179394964E  -3)
DATA(THETA( 71)= 1.6111054225E   0),(WF( 71)=-2.2284511938E  -3)
DATA(THETA( 72)= 1.7022659184E   0),(WF( 72)=-7.2242993780E  -3)
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DATA(THETA( 74)= 1.9003530559E   0),(WF( 74)=-1.7816514900E  -2)
DATA(THETA( 75)= 2.0078799283E   0),(WF( 75)=-2.3299028355E  -2)
DATA(THETA( 76)= 2.1214909480E   0),(WF( 76)=-2.8797326857E  -2)
DATA(THETA( 77)= 2.2415303720E   0),(WF( 77)=-3.4201037984E  -2)
DATA(THETA( 78)= 2.3683619359E   0),(WF( 78)=-3.9373196189E  -2)
DATA(THETA( 79)= 2.5023699565E   0),(WF( 79)=-4.4148100620E  -2)
DATA(THETA( 80)= 2.6439604962E   0),(WF( 80)=-4.8330319743E  -2)
DATA(THETA( 81)= 2.7935625935E   0),(WF( 81)=-5.1695527926E  -2)
DATA(THETA( 82)= 2.9516295629E   0),(WF( 82)=-5.3994048656E  -2)
DATA(THETA( 83)= 3.1186403686E   0),(WF( 83)=-5.4958170368E  -2)
DATA(THETA( 84)= 3.2951010761E   0),(WF( 84)=-5.4314461342E  -2)
DATA(THETA( 85)= 3.4815463849E   0),(WF( 85)=-5.1802379623E  -2)
DATA(THETA( 86)= 3.6785412501E   0),(WF( 86)=-4.7200378403E  -2)
DATA(THETA( 87)= 3.8866825921E   0),(WF( 87)=-4.0360310394E  -2)
DATA(THETA( 88)= 4.1066011072E   0),(WF( 88)=-3.1250108611E  -2)
DATA(THETA( 89)= 4.3389631792E   0),(WF( 89)=-2.0003247347E  -2)
DATA(THETA( 90)= 4.5844728953E   0),(WF( 90)=-6.9712238045E  -3)
DATA(THETA( 91)= 4.8132736282E   0),(WF( 91)= 4.2744563048E  -3)
DATA(THETA( 92)= 5.0215684925E   0),(WF( 92)= 1.2881474466E  -2)
DATA(THETA( 93)= 5.2388773367E   0),(WF( 93)= 2.1258997455E  -2)
DATA(THETA( 94)= 5.4655902411E   0),(WF( 94)= 2.8924519540E  -2)
DATA(THETA( 95)= 5.7021141679E   0),(WF( 95)= 3.5335580857E  -2)
DATA(THETA( 96)= 5.9488736892E   0),(WF( 96)= 3.9919731387E  -2)
DATA(THETA( 97)= 6.2063117521E   0),(WF( 97)= 4.2119692665E  -2)
DATA(THETA( 98)= 6.4748904710E   0),(WF( 98)= 4.1454627137E  -2)
DATA(THETA( 99)= 6.7550919587E   0),(WF( 99)= 3.7596220600E  -2)
DATA(THETA(100)= 7.0474191923E   0),(WF(100)= 3.0454922967E  -2)
DATA(THETA(101)= 7.3523969138E   0),(WF(101)= 2.0267168345E  -2)
DATA(THETA(102)= 7.6705725749E   0),(WF(102)= 7.6689519987E  -3)
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DATA(THETA(106)= 9.0870546603E   0),(WF(106)=-3.9734782958E  -2)
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DATA(THETA(111)= 1.1257373677E   1),(WF(111)= 1.1812596422E  -2)
DATA(THETA(112)= 1.1780972453E   1),(WF(112)= 3.1013723979E  -2)
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      DATA(THETA(113)= 1.2304571228E    1),(WF(113)= 4.0621066799E   -2)
      DATA(THETA(114)= 1.2828170003E    1),(WF(114)= 3.9000773613E   -2)
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      DATA(THETA(116)= 1.3875367555E    1),(WF(116)= 9.7163251075E   -3)
      DATA(THETA(117)= 1.4398966331E    1),(WF(117)=-9.2501146800E   -3)
      DATA(THETA(118)= 1.4922565107E    1),(WF(118)=-2.4493078391E   -2)
      DATA(THETA(119)= 1.5446163882E    1),(WF(119)=-3.2360678401E   -2)
      DATA(THETA(120)= 1.5969762658E    1),(WF(120)=-3.1324007350E   -2)
      DATA(THETA(121)= 1.6493361433E    1),(WF(121)=-2.2225019943E   -2)
      DATA(THETA(122)= 1.7016960209E    1),(WF(122)=-7.9132542564E   -3)
      DATA(THETA(123)= 1.7540558985E    1),(WF(123)= 7.6012475654E   -3)
      DATA(THETA(124)= 1.8064157760E    1),(WF(124)= 2.0238180653E   -2)
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      DATA(THETA(126)= 1.9111355311E    1),(WF(126)= 2.6172497792E   -2)
      DATA(THETA(127)= 1.9634954087E    1),(WF(127)= 1.8664407829E   -2)
      DATA(THETA(128)= 2.0158552863E    1),(WF(128)= 6.6746551113E   -3)
      DATA(THETA(129)= 2.0682151638E    1),(WF(129)=-6.4513005203E   -3)
      DATA(THETA(130)= 2.1205750414E    1),(WF(130)=-1.7242853145E   -2)
      DATA(THETA(131)= 2.1729349190E    1),(WF(131)=-2.3004943621E   -2)
      DATA(THETA(132)= 2.2252947966E    1),(WF(132)=-2.2476170369E   -2)
      DATA(THETA(133)= 2.2776546741E    1),(WF(133)=-1.6087172661E   -2)
      DATA(THETA(134)= 2.3300145518E    1),(WF(134)=-5.7713259545E   -3)
      DATA(THETA(135)= 2.3954643987E    1),(WF(135)= 1.2164049017E   -2)
      DATA(THETA(136)= 2.4740042150E    1),(WF(136)= 2.8558962636E   -2)
      DATA(THETA(137)= 2.5525440314E    1),(WF(137)= 2.7727533397E   -2)
      DATA(THETA(138)= 2.6310838477E    1),(WF(138)= 1.1185907716E   -2)
      DATA(THETA(139)= 2.7096236640E    1),(WF(139)=-1.0760080582E   -2)
      DATA(THETA(140)= 2.7881634804E    1),(WF(140)=-2.5343093759E   -2)
      DATA(THETA(141)= 2.8667032967E    1),(WF(141)=-2.4686225530E   -2)
      DATA(THETA(142)= 2.9452431131E    1),(WF(142)=-9.9875408397E   -3)
      DATA(THETA(143)= 3.0237829295E    1),(WF(143)= 9.6466781555E   -3)
      DATA(THETA(144)= 3.1023227458E    1),(WF(144)= 2.2778202330E   -2)
      DATA(THETA(145)= 3.1808625621E    1),(WF(145)= 2.2246175808E   -2)
      DATA(THETA(146)= 3.2594023786E    1),(WF(146)= 9.0211019511E   -3)
      DATA(THETA(147)= 3.3379421948E    1),(WF(147)=-8.7420945861E   -3)
      DATA(THETA(148)= 3.4164820112E    1),(WF(148)=-2.0684785217E   -2)
      DATA(THETA(149)= 3.4950218276E    1),(WF(149)=-2.0245116396E   -2)
      DATA(THETA(150)= 3.5735616439E    1),(WF(150)=-8.2252000384E   -3)
      DATA(THETA(151)= 3.6521014603E    1),(WF(151)= 7.9926190319E   -3)
      DATA(THETA(152)= 3.7306412766E    1),(WF(152)= 1.8943782059E   -2)
      DATA(THETA(153)= 3.8091810929E    1),(WF(153)= 1.8574352439E   -2)
      DATA(THETA(154)= 3.8877209093E    1),(WF(154)= 7.5583546917E   -3)
      DATA(THETA(155)= 3.9662607256E    1),(WF(155)=-7.3615066264E   -3)
      DATA(THETA(156)= 4.0448005419E    1),(WF(156)=-1.7473109114E   -2)
      DATA(THETA(157)= 4.1233403584E    1),(WF(157)=-1.7158338403E   -2)
      DATA(THETA(158)= 4.2018801747E    1),(WF(158)=-6.9915293813E   -3)
      DATA(THETA(159)= 4.2804199910E    1),(WF(159)= 6.8227696271E   -3)
      DATA(THETA(160)= 4.3589598074E    1),(WF(160)= 1.6214338577E   -2)
      DATA(THETA(161)= 4.4374996237E    1),(WF(161)= 1.5942935625E   -2)
      DATA(THETA(162)= 4.5160394401E    1),(WF(162)= 6.5037906556E   -3)
      DATA(THETA(163)= 4.5945792564E    1),(WF(163)=-6.3575094445E   -3)
      DATA(THETA(164)= 4.6731190728E    1),(WF(164)=-1.5124749052E   -2)
      DATA(THETA(165)= 4.7516588891E    1),(WF(165)=-1.4888331517E   -2)
      DATA(THETA(166)= 4.8301987054E    1),(WF(166)=-6.0796658102E   -3)
      DATA(THETA(167)= 1.0000000000E   -2),(WF(167)= 1.0000000000E    0)
      DATA(THETA(168)= 5.0265482465E    1),(WF(168)= 2.0124038192E   -2)
      PI=3.141592654
      SUM=0.
      DO 10 I=1,168
      W=THETA(I)/T
   10 SUM=SUM+AIM(W)*WF(I)
      ROSBKM=HINF+SUM*2./PI
      RETURN
      END
```

Appendix III. Coaxial Line Parameter Definitions

series inductance

$$L=\frac{\mu}{2\pi}\ln\left(\frac{r_0}{r_i}\right)$$

shunt capacitance

$$C=\frac{2\pi\epsilon}{\ln\left(\frac{r_o}{r_i}\right)}$$

skin effect constant

$$k=\sqrt{\frac{\mu}{\sigma}}\frac{1}{2\pi}\left(\frac{1}{r_i}+\frac{1}{r_0}\right)$$

In the above μ is the magnetic permeability, ϵ the dielectric constant, σ the conductor conductivity, and r_i and r_0 the radius of the inner and outer conductors.

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