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# Automorphic Integrals With Preassigned Periods

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Let $\Gamma$ be a discrete group of real $2\times 2$ matrices of determinant 1. Generalizing the usual notion of abelian integral, Eichler has defined $\Phi(\tau)$ to be an automorphic integral of degree $2n-2$ on $\Gamma$ if (1) $\Phi|A=\Phi+\omega_A$ for all $A\epsilon\Gamma$. Here $n$ is a positive integer, $\omega_A$ is a polynomial in $\tau$ of degree $2n-2$ or less, and $\Phi|A=(c\tau+d)^{2n-2}\Phi(A\tau)$, where $\tau$ is confined to the upper half-plane. A consequence of (1) is that (2) $\omega_{AB}=\omega_A|B+\omega_B$. If $\Phi$ has at most poles but no logarithmic singularities, $\Phi$ is said to be of the second kind and this requires (3) $\omega_A=Q|(A-1)$ for all elements $A$ that fix a real cusp of a fundamental region of $\Gamma$, where $Q$ is a polynomial of degree $\leqslant 2n-2$. Eichler proved that the necessary conditions (2) and (3′) are also sufficient for the existence of a $\Phi$ on $\Gamma$ with the preassigned "periods" $\omega_A$, but only when $\Gamma$ is a subgroup of finite index in the modular group. Here (3′) is a stronger version of (3). In the present paper this is generalized to all groups $\Gamma$ that are finitely-generated and have translations, and we use the correct conditions (2), (3) rather than (2), (3′).

Key words: Automorphic form; automorphic integral; fundamental region; group; Poincaré series.

1. Let $\Gamma$ be an $H$-group (defined in section 2). Following Eichler [1][1] and Petersson [5] we introduce a certain generalization of the abelian integral. For a function $f(\tau)$ defined on the upper half-plane $H$ and an integer $r$, set

$$f(\tau)\,|_r A=(c\tau+d)^r f(A\tau),\; A=\begin{pmatrix} a & b\\ c & d\end{pmatrix}$$

where $A$ is real and has positive determinant; also set

$$f\,|_r(\alpha_1A_1+\alpha_2A_2)=\alpha_1 f\,|_r A_1+\alpha_2 f\,|_r A_2$$

for constants $\alpha_1$, $\alpha_2$. We say $\Phi(\tau)$ is an automorphic integral of degree $2n-2\geqslant 0$ on $\Gamma$ if $\Phi$ satisfies three conditions:

$$\Phi \text{ is meromorphic in } H \tag{1}$$

$$\Phi\,|_{2n-2}(A-1)=\omega_A,\; A\epsilon\Gamma \tag{2}$$

where $1=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ and $\omega_A$ is a polynomial of degree $\leqslant 2n-2$. Before we can describe the third condition, certain preliminaries are required.

Let $P_p=P$ be a generator of the (cyclic) stabilizer of $\Gamma$ at the parabolic cusp $p$. If $V_p\infty=p$, $V_p\epsilon SL(2,R)$, then

$$V_p^{-1}PV_p=S_p=\begin{pmatrix}1 & \lambda_p\\ 0 & 1\end{pmatrix},$$

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[1] Figures in brackets indicate the literature references at the end of this paper.

and $\lambda_p > 0$ for the right choice of $P$ (either $P$ or $P^{-1}$); moreover $\lambda_p$ is independent of the choice of $V_p$. Since $(f|A)|B = f|AB$, it follows from (2) that, with $| = |_{2n-2}$,

$$\Phi|V_p|S_p = \Phi|PV_p = (\Phi + \omega_p)|V_p = \Phi|V_p + \omega_p|V_p,$$

and here $\omega_p|V_p$ is also a polynomial of degree $\leq 2n-2$. Let $q(\tau) = q_p(\tau)$ be a polynomial of degree $\leq 2n-1$ that satisfies the above equation, i.e., $q|S_p = q + \omega_P|V_p$. Then $(\Phi|V_p - q)|S_p = \Phi|V_p - q$ and there is an expansion

$$\Phi|V_p = \sum_h a_{ph} e^{2\pi i h\tau/\lambda p} + q_p(\tau).$$

The series in the right member is called the Fourier series of $\Phi$ at $p$. We say $\Phi$ is meromorphic at $p$ if its Fourier series at $p$ converges for $\operatorname{Im}\tau \geq \tau_0(p)$ and is left-finite. In that case

$$\Phi(\tau)|V_p = \sum_{h=h_0(p)} a_{hp} e^{2\pi i h\tau/\lambda}{}_p + q_p(\tau); \tag{3}$$

$h_0$ is called the order of $\Phi$ at $p$. The third condition in the definition of an automorphic integral is now:

$\Phi$ is meromorphic at each cusp of $\Gamma$.

The function $\Phi$ is called an integral of the third kind. If the degree of $q_p$ is $\leq 2n-2$ for all $p$, $\Phi$ is said to be of the second kind. When $\Phi$ is of the second kind and $\Phi$ is holomorphic in $H$ and at the cusps (order of $\Phi$ nonnegative at each cusp), we say $\Phi$ is of the first kind. The usual abelian integral is the case $n=1$.

The polynomial $\omega_A$ is called the period polynomial associated with $A$. From (2) we derive

$$\omega_{AB} = \Phi|(AB-1) = \Phi|(A-1)|B + \Phi|B-1),$$

that is,

$$\omega_{AB} = \omega_A|B + \omega_B;\ A, B \in \Gamma. \tag{4}$$

The relation (4) is therefore a necessary condition in order that a set of polynomials $\{\omega_A(\tau), A \in \Gamma\}$ of degree $\leq 2n-2$ be the period polynomials of an automorphic integral of degree $2n-2$.

Let $N$ be a normal fundamental region (Dirichlet region) for $\Gamma$ and let $p$ be a cusp of $N$. Since the Fourier series of $\Phi|V_p$ is invariant under $S_p$, we calculate from (3):

$$\omega_P = \Phi|(P-1) = \Phi|V_p|(S_p V_p^{-1} - V_p^{-1}) = q_p|V_p^{-1}P - q_p|V_p^{-1} = (q_p|V_p^{-1})|(P-1) = \Theta_p|(P-1).$$

Now suppose $\Phi$ is of the second kind. Then $q_p$, and therefore

$$\Theta_p = q_p|V_p^{-1},$$

are a polynomials of degree $\leq 2_n - 2$. That is,

$$\omega_p = \Theta_p|(P-1) \tag{5}$$

for each cusp $p$ lying in $N$. This condition, along with (4), is a necessary condition. A stronger requirement than (5) is

$$\omega_p \equiv 0 \tag{5'}$$

at each cusp of $N$, where, as always, $P$ is the normalized generator of $\Gamma_p$.

In [1] Eichler shows, by applying the Riemann-Roch theorem, that the conditions (4) and (5) are not only necessary but also sufficient in order that $\{\omega_A\}$ be the period polynomials of an integral of the second kind. In [2] Eichler shows by an elementary method that (4) and (5′) are sufficient conditions, but only for $\Gamma$ a subgroup of finite index in the modular group.

Eichler's argument consists in setting up two Poincaré series, the quotient of which is the desired atuomorphic integral $\Phi$. To establish convergence of the numerator series, he develops an estimate of $\omega_A$, namely.

$$|\omega_A(\tau)| \leq C(\tau) f(c^2+d^2),$$

where $C$ is a function of $\tau$ alone, $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, and $f$ is a function defined in theorem 1, below. This estimate depends in turn on some lemmas, which are established for general $H$-groups $\Gamma$ (see definition in Sec. 2), but the estimate for $\omega_A$ is proved only for subgroups of the modular group.

*It is the purpose of this paper to carry through Eichler's argument for* $\Gamma$, *an arbitrary $H$-group,* using conditions (4) and (5), the correct conditions. This requires in particular the use of a different type of Poincaré series than in [2]. I am indebted to M. I. Knopp for some helpful correspondence on the role of conditions (5) and (5′).

The map $\{A \to \omega_A, A \epsilon \Gamma\}$ is the basis of Eichler's construction of a cohomology of $\Gamma$. Such a map is called a cocycle if (4) is satisfied; it is a coboundary if $\omega_A = \Theta|(A-1)$ for all $A \epsilon \Gamma$, where $\Theta$ is a fixed polynomial of degree $\leq 2n-2$. See [1] for details.

The connection of automorphic integrals with automorphic forms is two-fold. First, an automorphic integral is an automorphic form if its periods are all zero. Secondly, the $(2n-1)-st$ derivative of $\Phi$ is an automorphic form of degree $-2n$. (This is easily proved by expressing the derivative as a Cauchy integral.) Let $\Phi(\tau) = d^{(2n-1)}\Phi | d^{2n-1}\tau$. Then $\Phi$ is of the second kind if and only if $\Phi$ has no constant term in its Fourier series at $p$ for all $p$. $\Phi$ is of the first kind if and only if $\Phi$ is a cusp form. Here $\Phi$ may have poles at interior points of $H$, but the principal parts must be such that $\Phi$ remains single-valued.

2. Let $LF(2, R)$ be the group of all linear-fractional transformations $\bar{A}\tau = (a\tau + b)/(c\tau + d)$ with real $a, b, c, d$ and $ad-bc=1$. Let $SL(2, R)$ be the group of all real $2 \times 2$ matrices $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of determinant 1. Then $LF(2, R) \simeq SL(2, R)/\{I, -I\}$, the correspondence being $\pm A \to \bar{A}$, and we may write $A\tau$ in place of $\bar{A}\tau$ without confusion. We regard $LF(2, R)$ as acting on the upper half-plane $H$.

Next we introduce $H$-groups; these are horocyclic, possess translations, and have normal fundamental regions of finite hyperbolic area. An equivalent to the last condition is that the fundamental region should have a finite number of sides, which implies that the group is finitely generated. For the facts about $H$-groups used in the following, see for example [3], [4], or [6]. In general we do not distinguish between the matrix group $\Gamma$ and the transformation group $\Gamma/\{I, -I\}$, i.e., we identify $A$ and $-A$.

Let $\Gamma$ be an $H$-group. The stabilizer of $\infty$ is denoted by $\Gamma_\infty$; it is cyclic and we let

$$S=\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}, \lambda > 0 \tag{6}$$

be a generator. Write

$$\Gamma = \Gamma_\infty \cdot M \tag{7}$$

to denote that $M$ is a system of coset representatives of $\Gamma$ with respect to $\Gamma_\infty$.

Let $N$ be a normal polygon (Dirichlet region) for $\Gamma$ with center $w_0 = x_0 + iy_0$ and having $\infty$ as a cusp. We may assume

$$|x_0| \leq \lambda/2, \; y_0 > 0. \tag{8}$$



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To see this, consider $S^m(N)=N_1$; then $N_1$ is a normal polygon with $\infty$ as a cusp and the center of $N_1$ satisfies (8) if $m$ is chosen suitably. Let $p$ be a cusp of $N$. If $P$ fixes $p$, then $S^mPS^{-m}=P_1$ fixes the cusp $p_1=S^mp$ of $N$. If condition (5) is satisfied for the fundamental region $N$, we have

$$\omega_P=q_p|(P-1), \qquad \omega_s=q_s|(S-1)$$

and the latter implies

$$\omega_{S^m}=q_s|(S^m-1)$$

for all integers $m$. Hence

$$\omega_{P_1}=q_s|(S^m-1)|PS^{-m}+q_p|(P-1)|S^{-m}+q_s|(S^{-m}-1).$$

Setting

$$Q=(q_s|S^m-1)+q_p)|S^{-m},$$

we see after a little calculation that this is equivalent to

$$\omega_{P_1}=Q|(p_1-1).$$

Thus condition (5) is fulfilled for the fundamental region $N_1$. We shall therefore fix a fundamental region $N$, which is a normal polygon having $\infty$ as a cusp and center $w_0$ satisfying (8). $N$ itself lies in the strip $|x|\leq\lambda$.

We can now select a system of representatives $M$ so that $Aw_0$ lies in the strip $|x|\leq\lambda/2$ for $A\epsilon M$. If this is not true for $A$, it is true for $S^mA$ with a suitable $m$, and $A$ and $S^mA$ lie in the same coset.

For the moment we denote by $m_1, m_2, \ldots$ positive constants that depend only on $\Gamma$ and on $y_0$. Later we shall permit these constants to depend on other parameters also.

The sides of $N$ are arranged in conjugate pairs, and the elements of $\Gamma$ mapping one side of a pair on the other form a system of generators for $\Gamma$. Call this system

$$\mathscr{B}=\{B_1, B_2, \ldots, B_s\}; \tag{9}$$

it is finite since $N$ has a finite number of sides. In [1] Eichler gives an algorithm for representing $A\epsilon\Gamma$ as a word in the generators $\mathscr{B}$. We proceed by induction: if we have already chosen $D_1, \ldots, D_i$ in the representation

$$A=D_1D_2\ldots D_i\ldots, D_j\epsilon\mathscr{B}$$

connect the point $D_1D_2\ldots D_i(w_0)$ with $Aw_0$ by a hyperbolic straight line. This line leaves the polygon $D_1\ldots D_i(N)$ and enters a neighboring polygon, and it is seen that the latter can be written as $D_1\ldots D_iD_{i+1}(N)$ with a $D_{i+1}\epsilon\mathscr{B}$. To start the process join $w_0$ to $Aw_0$. The process terminates (see [2]) and we get the representation:

$$A=D_1D_2\ldots D_h, D_j\epsilon\mathscr{B}. \tag{10}$$

Note that a power of an element occurring in (10) is written as a product, i.e., $B_1^m$ would be written as $B_1\ldots B_1=D_{j+1}D_{j+2}\ldots D_{j+m}$.

The following two lemmas are proved in [2] (theorems 1 and 2) on the assumption that $\Gamma$ is an $H$-group and are therefore available to us. Define

$$\mu(B)=\alpha^2+\beta^2+\gamma^2+\delta^2, \tag{11}$$

where $B=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\epsilon SL(2, R)$. Note that

$$\mu(B)\geq 2.$$

LEMMA 1. *Let (10) be given. A factor of (10) that is not parabolic is called a section. If $D_{j+1} \ldots D_k = P^{k-j}$ with $P$ parabolic, then $D_{j+1} \ldots D_k$ is called a section. Let $l$ be the number of sections in (10); then*

$$l \leqslant m_1 \log \mu(A) + m_2.$$

LEMMA 2. *Let (10) be given and let*

$$A = C_1 C_2 \ldots C_k, \tag{12}$$

where $C_i$ is a section. Then

$$\mu(C_t C_{t+1} \ldots C_k) \leqslant m_3 \mu(A),\ t = 1, 2, \ldots, k.$$

We suppose now that a system of polynomials $\{\omega_A(\tau) | A \epsilon \Gamma\}$ is given, of degree not exceeding $2n-2$, and that this system satisfies (4) and (5), the latter condition being satisfied for the fundamental region $N$ specified above. We are going to estimate $\omega_A$. We need some preliminary lemmas first.

LEMMA 3. *Let* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \epsilon \Gamma$, $c \neq 0$. *There exists a constant $m_4$ such that*

$$|c| \geqslant m_4 > 0.$$

This follows from the fact that $\Gamma$ is discrete and has translations; the proof is in [3], p. 45.

In the following we shall need a set of positive constants depending at most on $\Gamma$, $w_0$, a positive constant $\alpha$ to be defined later, and on the coefficients of the polynomials $\{\omega_B, B \epsilon \mathscr{B}\}$. From now on we denote such constants by $m_1, m_2, \ldots$.

We introduce the regions $E_\alpha$, $\alpha > 0$:

$$E_\alpha : |x| \leqslant 1/\alpha,\ y \geqslant \alpha. \tag{13}$$

Every compact subset of $H$ is contained in some $E_\alpha$.

LEMMA 4. *Let $c$, $d$ be real. For $\tau \epsilon E_\alpha$ we have*

$$m_7(c^2 + d^2) \leqslant |c\tau + d|^2 \leqslant m_6 |\tau|^2 (c^2 + d^2).$$

PROOF. When $c = 0$ the result holds provided $m_6 \geqslant \alpha^{-2}$. When $c \neq 0$,

$$\left|\frac{c\tau + d}{ci + d}\right|^2 = \left|\frac{\tau + d/c}{i + d/c}\right|^2 \leqslant \sup_{-\infty < u < \infty} \left|\frac{\tau + u}{i + u}\right|^2$$

$$= \sup \frac{(x+u)^2 + y^2}{1 + u^2} \leqslant 2 \sup \frac{|\tau|^2 + u^2}{1 + u^2}$$

$$\leqslant 2|\tau|^2 + \frac{u^2}{1+u^2}(1 - |\tau|^2)$$

$$\leqslant 2|\tau|^2 + 1 \leqslant (2 + \alpha^{-2})|\tau|^2 = m_6 |\tau|^2.$$

Also

$$\left|\frac{c\tau + d}{ci + d}\right|^2 \geqslant \inf_u \frac{(x+u)^2 + y^2}{1 + u^2} = \inf_u \varphi(u).$$

For $|u| \leqslant 2/\alpha$, $\varphi(u) \geqslant y^2/(1+u^2) \geqslant \alpha^2/(1+4\alpha^{-2})$, while for $|u| > 2/\alpha$, $\varphi(u) > u^2/4(1+u^2) \geqslant \alpha^{-2}/(1+4\alpha^{-2})$. Hence $\inf_u \varphi(u) \geqslant (1+4\alpha^{-2})^{-1} \min(\alpha^2, \alpha^{-2}) = m_7$.

**Theorem** 1. *For $A \epsilon M$, $\tau \epsilon E_\alpha$, we have*

$$|\omega_A(\tau)| \leqslant m_8 |\tau|^{2n-2} f(c^2+d^2),$$

where

$$f(u) = u^{n-1}(\log u + m_9). \tag{14}$$

Assume $A$ is factored into sections $C_i$ as in (12). Let $\tau \epsilon E_\alpha$. Writing $|$ for $|_{2n-2}$, we get from (4):

$$\omega_A = \omega_{C_1 \ldots C_{k-1}} | C_k + \omega_{C_k} = \ldots$$

$$= \omega_{C_1} | C_2 C_3 \ldots C_k + \omega_{C_2} | C_3 C_4 \ldots C_k + \ldots + \omega_{C_k}. \tag{15}$$

First, we shall assume $C_j$ is not parabolic, hence $C_j \epsilon \mathscr{B}$. We have

$$\omega_{C_j}(\tau) = \sum_{i=0}^{2n-2} b_{ij} \tau^i,$$

where the $b_{ij}$ depend on $j$. Since all $C_j$ belong to the fixed set $\mathscr{B}$, we may assume

$$\sum_{i=0}^{2n-2} |b_{ij}| \leqslant m_{10}$$

for all $j$. Writing

$$C_{j+1} \ldots C_k = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

we get

$$|(\omega_{Cj} | C_{j+1} \ldots C_k)| = |\gamma\tau+\delta|^{2n-2} |\omega_{Cj}(C_{j+1} \ldots C_k \tau)|$$
$$\leqslant \sum_{i=0}^{2n-2} |b_{ij}| \, |\alpha\tau+\beta|^i |\gamma\tau+\delta|^{2n-2-i},$$

and by lemma 4 this is not more than

$$m_6^{n-1} |\tau|^{2n-2} \sum_i |b_{ij}| (\alpha^2+\beta^2)^{i/2} (\gamma^2+\delta^2)^{n-1-i/2}.$$

But lemma 2 shows that

$$\alpha^2+\beta^2 \leqslant m_3 \mu(A), \qquad \gamma^2+\delta^2 \leqslant m_3 \mu(A),$$

so that

$$|(\omega_{Cj} | C_{j+1} \ldots C_k)| \leqslant m_{11} |\tau|^{2n-2} \mu^{n-1}(A) \sum_i |b_{ij}|,$$

or

$$|(\omega_{Cj} | C_{j+1} \ldots C_k)| \leqslant m_{12} |\tau|^{2n-2} \mu^{n-1}(A). \tag{16}$$

According to lemma 1 there are no more than $m_1 \log \mu(A) + m_2$ factors $C_j$, so we get

$$|\omega_A(\tau)| \leqslant m_{13} |\tau|^{2n-2} \mu^{n-1}(A) \{m_1 \log \mu(A) + m_2\}. \tag{17}$$

Next, suppose $C_j = P^m$ with $P$ parabolic, $P \epsilon \mathscr{B}$. By our assumption (5)

$$\omega_P = q_p | (P-1),$$

from which follows

$$\omega_{C_j}=q_p|(P^m-1)=q_p|(C_j-1).$$

Writing

$$q_p=\sum_{i=0}^{2n-2} a_{ij}\tau^i,\ j=j(p)$$

we can say that

$$\sum_i |a_{ij}|<m_{12}$$

for each $p$, since there are only a finite number of $P$ in $\mathscr{B}$.

Now

$$\omega_{C_j}|C_{j+i}\ .\ .\ .\ C_k=q_p|(C_j-1)|C_{j+i}\ .\ .\ .\ C_k$$

$$=q_p|C_jC_{j+i}\ .\ .\ .\ C_k-q_p|C_{j+i}\ .\ .\ .\ C_k.$$

Let $C_j\ .\ .\ .\ C_k=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}$; then

$$|(q_p|C_j\ .\ .\ .\ C_k)|\leq\sum_{i=0}^{2n-2}|a_{ij}|\,|\alpha\tau+\beta|^i|\gamma\tau+\delta|^{2n-2-i}.$$

Since $\mu(C_j\ .\ .\ .\ C_k)\leq m_3\mu(A)$ by lemma 2, the previous argument yields

$$|(q_p|C_j\ .\ .\ .\ C_k)|\leq m_{13}|\tau|^{2n-2}\mu^{n-1}(A),$$

with a similar estimate for $q_p|C_{j+1}\ .\ .\ .\ C_k$. It follows that $\omega_{C_j}|C_{j+1}\ .\ .\ .\ C_k$ is subject to the estimate (16), and therefore to (17). Each term of (15), then, is estimated by (17), which thus holds for arbitrary $A\epsilon\Gamma$.

Now we make use of the hypothesis that $A\epsilon M$. Recall that $|\mathrm{Re}\ Aw_0|\leq\lambda/2$ for $A\epsilon M$. Also we may assume $c\neq 0$, otherwise $A=\pm I$, and $\omega_I=\omega_{-I}=0$, the latter because $2n-2$ is even. Therefore

$$0<\mathrm{Im}\ Aw_0\leq 1/c^2y_0\leq 1/m_4^2y_0.$$

Hence $|Aw_0|^2\leq m_{14}$, which with lemma 4 gives

$$m_{15}(a^2+b^2)\leq|aw_0+b|^2\leq m_{14}|cw_0+d|^2\leq m'_{14}|w_0|^2(c^2+d^2)\leq m_{16}(c^2+d^2).$$

It follows that

$$\mu(A)\leq m_{17}(c^2+d^2), \tag{18}$$

and this inequality, inserted in (17), completes the proof.

3. The Poincaré series used by Eichler in [2] was of the type

$$\sum_{A\epsilon M}\frac{\omega_A(\tau)}{(c\tau+d)^{2n+2}},\ A=\begin{pmatrix}\cdot & \cdot\\ c & d\end{pmatrix}.$$

In order that this series should be independent of the particular system of representatives $M$, which is essential for later developments, we must have $\omega_{S^mA}(\tau)=\omega_A(\tau)$. By (3) this necessitates $\omega_{S^m}\equiv 0$ for all $m$, that is, $\omega_S\equiv 0$. This explains Eichler's condition (5′). Since we are using (5) rather than (5′), the above series is not independent of the choice of $M$ and we must proceed differently.

For this purpose we go back to the series proposed by Poincaré himself. Define

$$\Psi(\tau) = -\sum_{L\in\Gamma} \frac{\omega_L(\tau)}{((L\tau-i)(c\tau+d))^{2n+2}}, \tag{19}$$

$$\psi(\tau) = \sum_{L\in\Gamma} \frac{1}{((L\tau-i)(c\tau+d))^{2n+2}}, \quad L=\begin{pmatrix} \cdot & \cdot \\ c & d\end{pmatrix}. \tag{20}$$

The first step is to prove convergence; here we make use of the system $M$. We have $L=S^mA$, $A\in M$, so

$$-\Psi(\tau) = \sum_{A\in M} \frac{1}{(c\tau+d)^{2n+2}} \sum_{m=-\infty}^{\infty} \frac{\omega_{S^mA}(\tau)}{(A\tau-i+m\lambda)^{2n+2}}, \quad A=\begin{pmatrix} \cdot & \cdot \\ c & d\end{pmatrix}.$$

By (17) we can write, for $\tau\in E_\alpha$,

$$|\omega_{S^mA}(\tau)| \leq m_{13}|\tau|^{2n-2}\mu^{n-1/2}(S^mA).$$

But

$$\mu(S^mA) \leq \mu(S^m)\mu(A) \leq (m^2\lambda^2+2)m_{17}(c^2+d^2),$$

where we have used (18). Secondly, we get from lemma 4,

$$|\Psi(\tau)| \leq m_{18}|\tau|^{2n-2} \sum_{A\in M} \frac{1}{(c^2+d^2)^{3/2}} \sum_{m=\infty}^{\infty} \frac{3^n m^{2n-1}}{|m\lambda+A\tau-i|^{2n+2}}.$$

Let $E'_\alpha$ be the region obtained by deleting from $E_\alpha$ a small disk about each point of the finite set $\{L^{-1}i, L\in\Gamma\} \cap E_\alpha$. Then $L\tau\neq i$ for $\tau\in E'_\alpha$, $L\in\Gamma$. Suppose $|c|>c_0=\max(2, 2\alpha^{-1})$; then $E'_\alpha$ lies outside the isometric circle $|c\tau+d|=1$ of radius $1/|c|<\alpha/2$. Hence $L(E'_\alpha)$ is inside the isometric circle $|-c\tau+a|=1$. Since the radius of the latter circle is $1/|c|<1/2$, it follows that, for all such $c$, $|L\tau-i|>1/2$. For the finitely many $c$ in $|c|\leq c_0$ we have $|L\tau-i|>0$, as stated above. There is thus a positive constant $m_{19}$ such that

(*) $$|L\tau-i| > m_{19}, \quad \tau\in E'_\alpha, \ L\in\Gamma.$$

Also, since $A\in M$, we have when $c\neq 0$,

$$|A\tau| = \left|\frac{a}{c} + \frac{1}{c^2(\tau+d/c)}\right| \leq \left|\frac{a}{c}\right| + \frac{1}{m_4^2\,\alpha} < m_{21},$$

so

$$|A\tau-i| < m_{20}, \quad \tau\in E'_\alpha, \ c\neq 0.$$

Hence $|m\lambda+A\tau-i| \geq |m|\lambda-m_{20} \geq |m|\lambda/2$ for $|m|\lambda \geq 2m_{20}$, the inequality being established directly when $c=0$, i.e., $A\tau=\tau$. This gives for the inner sum of the double series above,

$$\sum_{\lambda|m|\geq 2m_{20}} < m'_{21} \sum_{m=1}^{\infty} m^{-3}.$$

Next, $\sum_m < m''_{21}$, the sum being extended over $\lambda|m|<2m_{20}$, because of the estimate (*). Since $\Sigma(c^2+d^2)^{-3/2}$ is known to converge ([3], p. 71), it follows that the double series converges absolutely uniformly in $E'_\alpha$ and its sum is holomorphic there, i.e., holomorphic in $E_\alpha$ except at the points of $\{L^{-1}i\}$, where it has a pole. But $\{L^{-1}i\}$ lies below some horizontal line, so $\Psi$ is holomorphic in $\mathrm{Im}\,\tau \geq m_{22}$. In this region we have the order of growth

$$|\Psi(\tau)| \leq m_{23}|\tau|^{2n-2}, \quad \mathrm{Im}\,\tau > m_{22}. \tag{21}$$

The same proof will show that $\psi$ is holomorphic in $\operatorname{Im}\tau > m_{22}$ and is meromorphic in $H$. Now all terms of $\psi$ tend to 0 as $\tau \to i\infty$, for $c=0$ implies $L\tau = \tau + k\lambda$ for some integer $k$. Hence $\psi \to 0$ as $\tau \to i\infty$. But $\psi$ is an automorphic form of degree $-2n-2$ (i.e., $\psi|_{-2n-2}B = \psi$ for $B \in \Gamma$) and so the last sentence implies that $\psi$ has a Fourier series

$$\psi(\tau) = \sum_{h=h_1}^{\infty} b_h e^{2\pi i h\tau/\lambda}, \ h_1 \geq 1. \tag{22}$$

Moreover, $\psi$ is not identically zero, since it has a pole at the points $\{L^{-1}i\}$.

Consider now

$$\Phi(\tau) = \frac{\Psi(\tau)}{\psi(\tau)}.$$

It is readily checked that

$$\Psi(\tau)|_{-4}B = \Psi(\tau) + \omega_B \psi(\tau), \ B \in \Gamma;$$

consequently

$$\Phi(\tau)|_{2n-2}B = \Phi(\tau) + \omega_B, \ B \in \Gamma. \tag{23}$$

That is, $\Phi$ satisfies the transformation formula of an automorphic integral with periods $\omega_B$. It is clear also that $\Phi$ is meromorphic in $H$.

To complete the proof that $\Phi$ is of the second kind, we must show, first, that $\Phi$ is meromorphic at the cusps and, secondly, that the additive polynomial $q_p(\tau)$ of (3) is of degree $\leq 2n-2$. The second condition is, however, immediately verified. Indeed, from (5) we have

$$\omega_P = \Theta_p | (P-1),$$

where $\Theta_p$ is of degree $\leq 2n-2$. Hence the equation (preceding (3))

$$q | (S_p - 1) = \Theta_P | (P-1) | V_p = \Theta_p | V_p | (S_p - 1)$$

is satisfied by $q = \Theta_p | V_p$, a polynomial of degree $\leq 2n-2$.

As to the first condition, since

$$\Phi(\tau)|_{2n-2}V_p = (\Psi(\tau)|_{-4}V_p)/(\psi(\tau)|_{-2n-2}V_p), \tag{24}$$

it is sufficient if numerator and denominator are each meromorphic at $\tau = i\infty (V_p\tau = p)$ and the denominator is not identically zero. Now $\psi|_{-2n-2}V_p$ is not identically zero, since $\psi$ is not. We must therefore investigate these functions at $i\infty$. Here $p$ runs over all cusps of $\Gamma$, not just the ones in $N$.

For $p=\infty$ choose $V_p = I$. By (22) we have $\psi = O(\exp(-2\pi h_1\tau/\lambda))$ while (21) shows that $\Psi = O(\tau^{2n-2})$. Hence $\Phi = O(\exp 2\pi h_1 \tau/\lambda)$ and so is $\Phi - q$, and the Fourier series of $\Phi$ is therefore left-finite.

When $p$ is finite there are two cases. Suppose $p$ is equivalent to $\infty$, i.e., there is a $B \in \Gamma$ such that $B\infty = p$. Then $\Phi|_{2n-2}B = \Phi + \omega_B = O(\exp 2\pi h_1\tau/\lambda)$ as $\tau \to i\infty$, for we have just proved $\Phi$ is of this order. In general, if $p_1$ is equivalent to $p_2$, the meromorphicity of $\Phi$ at one point implies meromorphicity at the other. Hence we can assume $p$ lies in $N$ and in particular $p$ is not equivalent to $\infty$.

Now choose

$$V_p = V = \begin{pmatrix} p & -1 \\ 1 & 0 \end{pmatrix}, \ p \in N, \ p \text{ finite}.$$

We have

$$\Psi|_{-4}V=\tau^{-4}\sum_{L\epsilon\Gamma}\frac{\omega_L(V\tau)}{((cV\tau+d)(LV\tau-i))^{2n+2}}$$

$$=\tau^{-4}\sum_{A\epsilon M}\frac{1}{(cV\tau+d)^{2n+2}}\sum_{m=-\infty}^{\infty}\frac{\omega_{S^mA}(V\tau)}{(AV\tau-i+m\lambda)^{2n+2}}.$$

Set

$$A_1=AV=\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1\end{pmatrix}=\begin{pmatrix} ap+b & -a \\ cp+d & -c\end{pmatrix}.$$

Certainly $c_1\neq 0$, otherwise $A_p=AV(\infty)=\infty$, but $p$ is not $\Gamma-$ equivalent to $\infty$. Moreover,

$$cV\tau+d=(c_1\tau+d_1)\tau^{-1}.$$

Rewriting, we get

$$\Psi|_{-4}V=\tau^{2n-2}\sum_{A_1\epsilon MV}\frac{1}{(c_1\tau+d_1)^{2n+2}}\sum_{-\infty}^{\infty}\frac{\omega_{S^mA}(V\tau)}{(A_1\tau-i+m\lambda)^{2n+2}}. \tag{25}$$

LEMMA 5. *For $\tau\epsilon E_\alpha$ we have*

$$|\omega_{S^mA}(V\tau)|\leq m_{28}((m^2\lambda^2+2)(c_1^2+d_1^2))^{n-1/2}.$$

We follow the proof of theorem 1, replacing $\tau$ by $V\tau$ and confining $\tau$ to $E_\alpha$. Then

$$|V\tau|=|p-1/\tau|\leq p+1/|\tau|\leq\tau+1/\alpha=m_{24}.$$

If we insert this estimate at each point in the proof of theorem 1, we obtain from (17) and (18):

$$|\omega_{S^mA}(V\tau)|\leq m_{25}\mu^{n-1/2}(S^mA)$$

$$\leq m_{26}((m^2\lambda^2+2)(c^2+d^2))^{n-1/2}.$$

But $c_1=cp+d$, $d_1=-c$, hence

$$c^2+d^2=d_1^2+(c_1+d_1p)^2\leq m_{27}(c_1^2+d_1^2).$$

This completes the proof of lemma 5.

Returning to (25) we can now establish, exactly as in the lines preceding (21), the absolute uniform convergence of $\Psi|_{-4}V$ in a region $\tau\epsilon E''_\alpha$ obtained by deleting small disks about the points $\{V^{-1}L^{-1}i, L\epsilon\Gamma\}$ from $E_\alpha$. If we let $\tau\rightarrow i\infty$ in $E''_\alpha$, each term $\rightarrow 0$, so that $\Psi|_{-4}V\rightarrow 0$.

By a similar but simpler analysis $\psi|_{-2n-2}V\rightarrow 0$ with $\tau\rightarrow i\infty$. Hence $\Psi$ and $\psi$ are meromorphic at each cusp $p$ and by (24) so is $\Phi$.

We have proved:

THEOREM 2. *Let $\Gamma$ be an H-group. Let $n$ be a positive integer, and let $\{\omega_A(\tau)|A\epsilon\Gamma\}$ be a system of polynomials of degree $2n-2$ or less satisfying conditions (4) and (5), the latter being fulfilled with respect to any normal fundamental region having $\infty$ as a cusp. Then there exists an automorphic integral of the second kind and of degree $2n-2$ whose system of period polynomials is exactly $\{\omega_A(\tau)\}$.*

4. The following result is known: Given a finite set of points in a fixed fundamental region $N$ and a principal part associated with each point, there exists an automorphic form $\Omega$ of degree $2n-2$ that has poles with the given principal parts and is holomorphic elsewhere in $\bar{N}$ with the possible exception of the cusps. See Petersson, Konstruktion von Modulformen . . ., Sitzungsber. Akad. Wiss. Heidelberg 1950, 417–494. The proof of this theorem depends on the Riemann-Roch theorem Since $\Phi+\Omega$ is obviously an integral of the second kind with the periods $\omega_A$, we have:
THEOREM 3. *Under the hypotheses of Theorem 2 there exists an automorphic integral of the second kind and of degree $2n-2$ that fulfills the conclusion of theorem 2 and in addition is holomorphic in $H$.*

Theorem 3 is similar to one to be published by S. Husseni and M. I. Knopp, Eichler cohomology and automorphic forms, which deals with integrals of the third kind. Their proof also involves the Riemann-Roch theorem.

*Note added in proof:*

The material of this paper is closely connected with Eichler cohomology. Theorem 3 and similar theorems that can be obtained by this method, can be used to prove the theorems of Eichler [1], Gunning (The Eichler cohomology groups and automorphic forms, Trans. Amer. Math. Soc. **100,** 44–62 (1961)), and Husseni-Knopp, quoted above. For an exposition see my paper in Proc. of the Atlas Symposium No. 2, Computers in Number Theory, Academic Press (to be published).

## References


[1] Eichler, M., Eine Verallgemeinerung der Abelsche Integrale, Math. Zeitschrift **67,** 267–298 (1957).

[2] Eichler, M., Grenzkreisgruppen und kettenbruchartige Algorithmen, Acta Arithmetica **XI,** 169–180 (1965).

[3] Lehner, J., A short course in automorphic functions, Holt, Rinehart, and Winston, New York, 1966.

[4] Macbeath, A. M., Discontinuous groups and birational transformations, Lecture Notes, Queen's College, Dundee, 1961.

[5] Petersson, H., Die linearen Relationen zwischen den ganzen Poincaréschen Reihen von reeller Dimension zur Modulgruppe, Abhandlungen aus dem math. Sem. Hamburgischen Universität, **12,** 415–472 (1938).

[6] Siegel, C. L. Ausgewählte Fragen der Funktionentheorie II, Göttingen, 1954.


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