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Cuttable and Cut-Reducible Matrices*

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An n square matrix having an $(n-1)$ principal minor which is block diagonal (or reducible) is called cuttable (or cut-reducible). The connectivity matrix of a graph having a cutpoint is cuttable. While neither block diagonal nor reducible, cuttable and cut-reducible matrices share with these matrices some of the theoretical and computational simplicity derived from a natural division into principal submatrices which are relatively independent of each other. Solution of linear compartmental systems are shown to be simplified by the presence of a cutpoint in the system.

Key words: Compartmental systems; connectivity; graph; matrix.

1. Introduction

Block diagonal and reducible matrices are characterized by the presence of certain submatrices all elements of which vanish. The digraphs associated with such matrices have the property that their points are divided into subsets, such that the flow between the subsets is restricted in some fashion. For block diagonal matrices, there is no flow between subsets, that is, the digraph is disconnected. For reducible matrices, flow between subsets is unidirectional. The computational advantage of block diagonal and reducible matrices arises from the fact that some of their properties may be expressed in terms of the properties of the smaller submatrices corresponding to the subsets.

This paper will discuss two other kinds of digraphs with restricted flow between subsets, and their matrices. For clarity and conciseness of the proofs, we shall assume that the system possesses only two subsets. The extention to more complicated systems will be made at the end of the paper. If all flow between subsets passes through a single point, the point is known as a *cutpoint.* The matrix of such a graph, which we shall call a *cuttable* matrix, possesses a diagonal element such that the cofactor of this element is block diagonal. If all flow in one direction passes through one point, we shall call the point a *cut-reduction* point. The matrix of such a digraph, which we shall call a *cut-reducible* matrix, has a diagonal element such that its cofactor is reducible. It will be shown that cuttable and cut-reducible matrices enjoy simplifying properties analogous to those of block diagonal and reducible matrices.

2. Properties of Cuttable Matrices

We will begin with some definitions and notation.

Definition. A square matrix M is *cuttable* if it contains a diagonal element m_{cc}, called the *central cut element*, such that the cofactor of m_{cc} is block diagonal. The row and column containing m_{cc} will be called the *cut row* and *cut column*, and their elements *cut elements.*

A cuttable matrix is shown in figure 1. It is characterized by the zero submatrix V_1 of dimensions $(n-1-r)xr$ in the upper right, and its transpose, V_2, in the lower left corner. The blocks of the cofactor of the central cut element will be denoted by A, B, etc. The $(n-r)\times(n-r)$ matrix consisting of the block A bordered by the appropriate elements of the cut row and column will be denoted by $\hat{A}$, $\hat{B}$, etc. The determinant of M will be denoted by $|M|$ as usual. The matrix formed

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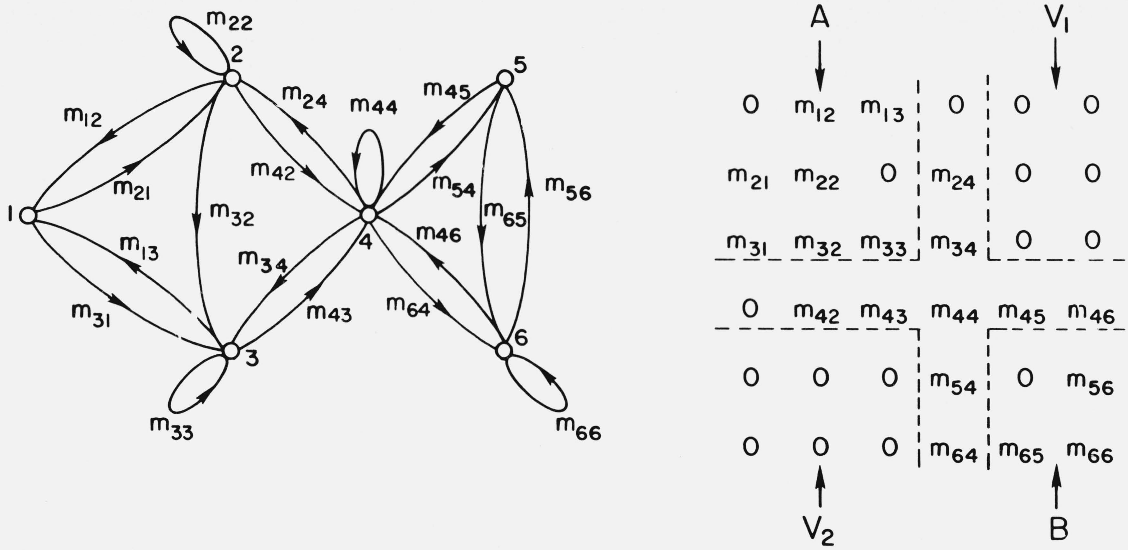


FIGURE 1. *A system with a cutpoint and its cuttable matrix. All connections between block* A, *which consists of points 1, 2, and 3, and block* B, *which consists of points 5 and 6, pass through point 4, so that deletion of point 4 disconnects* A *from* B. *The matrix expression of this connectivity is that the cofactor of* m_{44} *is block diagonal, with blocks* A *and* B.

by deleting the ith row and the jth column of M will be denoted by M_{ij}. Thus the cofactor of the element m_{ij} in the matrix M is given by $(-1)^{i+j}|M_{ij}|$. With the help of these definitions, we can state:

THEOREM 1. *If* M *is a cuttable matrix, then*

$$|\mathrm{M}| = |\mathrm{B}||\hat{\mathrm{A}}| + |\mathrm{A}||\hat{\mathrm{B}}| - \mathrm{m_{cc}}|\mathrm{A}||\mathrm{B}|.$$

PROOF: Note that the presence of an all zero submatrix of order $(n-1-r)xr$ is a necessary and sufficient condition for reducibility of an $(n-1)\times(n-1)$ matrix, so that if i and j are chosen so that at least one of the two all zero submatrices remains intact, the cofactor of m_{ij} is reducible. It is seen from inspection of figure 1 that the cofactor of any element in the cut row or cut column is reducible. Consider elements of the cut row. It can be seen that

$$
\begin{aligned}
(-1)^{c+j}|M_{cj}| &= |B|(-1)^{c+j}|\hat{A}_{cj}| && j<c \\
|M_{cc}| &= |B||A| && j=c \qquad (1)\\
(-1)^{c+j}|M_{cj}| &= |A|(-1)^{c+j}|\hat{B}_{cj}| && j>c
\end{aligned}
$$

That is, the cofactor of an element bordering a block is given by the product of its cofactor in the bordered block and the determinant of the other block. We can now write $|M|$ as an expansion of cofactors of elements of the cut row.

$$|M| = |B|\sum_{j=1}^{c-1} m_{cj}(-1)^{c+j}|\hat{A}_{cj}| + m_{cc}|A||B| + |A|\sum_{j=c+1}^{n} m_{cj}(-1)^{c+j}|\hat{B}_{cj}|. \qquad (2)$$

The sum of the first two terms is obviously $|B||\hat{A}|$, since $|A|$ is the cofactor of m_{cc} in $\hat{A}$. The sum of the second and third terms is similarly $|A||\hat{B}|$. (The determinant of $\hat{B}$ expanded by elements of the cut row is given by

$$\sum_{j'+1}^{r+1} m_{ij'}(-1)^{1+j'}\hat{B}_{ij'}$$

where j', the column index in $\hat{B}$ is given by $j'=j-c+1$. Since $(-1)^{1+j'}=(-1)^{2+j-c}=(-1)^{j-c}=(-1)^{j+c}$, this is equal to the sum of the second and third terms of equation (2)). Since the middle term has been counted twice it must be subtracted, so that

$$|M|=|B||\hat{A}|+|A||\hat{B}|-m_{cc}|A||B|. \tag{3}$$

THEOREM 2. *In a cuttable matrix the cofactor of any cut element bordering* A(B) *is given by the product of its cofactor in* Â(B̂) *and the determinant of* B(A). *The ratio of cofactors of any pair of cut elements bordering* A(B) *is therefore independent of* B(A).

PROOF: The argument of equation (1) can be applied to elements of the cut column yielding the equations

$$\begin{aligned} (-1)^{i+c}|M_{ic}| &= |B|(-1)^{i+c}|\hat{A}_{ic}| && i<c \\ |M_{cc}| &= |B||A| && i=c \\ (-1)^{i+c}|M_{ic}| &= |A|(-1)^{i+c}|\hat{B}_{ic}| && i>c \end{aligned} \tag{4}$$

Note that the cofactor of every element bordering the block A, i.e. m_{cj} with $j\leqslant c$ and m_{ic}, with $i\leqslant c$ contains $|B|$ as a factor. We have shown then the following theorem.

THEOREM 3. *In a cuttable matrix the cofactor of an element* V_1 *or* V_2 *is given by the product of the cofactors of the cut elements of its row and column, in their bordered submatrices. Therefore the ratios of cofactors of any elements in the same row of* V_1 (V_2) *will be independent of* Â (B̂), *and the ratios of elements of the same column (row) will be independent of* B̂ (Â).

PROOF: Since the cofactor of any element of V_1 will contain V_2, and vice versa, these cofactors will be reducible. Choose, an element of V_1, i.e. $i<c<j$, then

$$(-1)^{i+j}|M_{ij}|=[(-1)^{i+c}|\hat{A}_{ic}|][(-1)^{c+j}|\hat{B}_{cj}|]. \tag{5}$$

Similarly, for an element V_2, $j<c<i$, then

$$(-1)^{i+j}|M_{ij}|=[(-1)^{c+j}|\hat{A}_{cj}|][(-1)^{i+c}|\hat{B}_{ic}|]. \tag{6}$$

This completes the proof of Theorem 3.

The cofactors of elements in A and B are cuttable matrices, with blocks A_{ij} and B for elements in A, and B_{ij} and A for elements in B. Using eq (3)

$$\begin{aligned} |M_{ij}| &= |A_{ij}||\hat{B}|+|\hat{A}_{ij}||B|-m_{cc}|A_{ij}||B| && \text{for } i,j<c \\ |M_{ij}| &= |B_{ij}|\hat{A}|+|A||\hat{B}_{ij}|-m_{cc}|A||A_{ij}| && \text{for } c<j,\ i. \end{aligned} \tag{7}$$

It has now been shown that all cofactors and the determinant of a cuttable matrix can be simply expressed in terms of these entities in the blocks and bordered blocks. Obviously then, the inverse can be so expressed. The application of these properties to the analysis of linear compartmental systems will be discussed in section 3.

First let us consider briefly operations with cuttable matrices. Referring to figure 1, we have said that a cuttable matrix has vanishing submatrices V_1 and V_2 of dimensions $(n-1-r)xr$. If for two $n\times n$ cuttable matrices M and N, r has the same value, we will say that they are correspondingly cuttable with respect to c, where $c=n-r$, and denote the relation by $M\cong N$. It follows immediately, for M, N, correspondingly cuttable matrices, and D, Δ, diagonal matrices

$$M\cong N\rightarrow M\pm N\cong M\cong N \tag{8}$$

$$M^*\cong M$$

$$M \pm D \cong M$$

$$DM\Delta \cong M$$

MN is in general not cuttable.

It follows from these relations that

$$M - \lambda I \cong M \tag{9}$$

THEOREM 4. *If* M *is a cuttable matrix with blocks* A *and* B

1. *If* λ *is a root of both* A *and* B, *it is a root of* M.
2. *If* $\rho(\hat{A} - \lambda I) < r - 1$, $(\rho(\hat{B} - \lambda I) < r - 1)$ *where* r *is the rank of* $\hat{A}(\hat{B})$, *then* λ *is a root of* M.

PROOF: Since eq (9) shows that if M is cuttable, so is $M - \lambda I$, the characteristic equation of M can be written, with the dimensions of the I's suitably chosen, as

$$|M - \lambda I| = |B - \lambda I|\,|\hat{A} - \lambda I| + |A - \lambda I|\,|\hat{B} - \lambda I| - (m_{cc} - \lambda)\,|A - \lambda I|\,|B - \lambda I|. \tag{10}$$

It can be seen that if λ is a root of both A and B it is also a root of M. Similarly, any root common to the pairs A and $\hat{A}$, or to B and $\hat{B}$, will be a root of M. If $\rho(\hat{A} - \lambda I) < r - 1$, where r is the rank of $\hat{A}$, then λ is a root of A, as well as $\hat{A}$, and therefore of M.

3. Applications to Linear Compartmental Systems

In linear compartmental analysis, a system of n compartments is described by an $n \times n$ matrix M, in which the elements m_{ij} represent the fractional flow rate from compartment j to compartment i, and the elements m_{ii}, the fractional turnover rates, represent the fractional loss from compartment i to all other parts of the system. (In some disciplines, m_{ij} denotes a flow from i to j, so the present convention must be noted particularly).

If we define the transfer function x_{ij} to be the Laplace transform of the response in compartment i to a unit impulse function in compartment j at $t = 0$, with initial conditions equal to 0 in all other compartments, then it can be shown that

$$X = (sI - M)^{-1}. \tag{11}$$

We will call j the input and i the output in discussing the transfer function x_{ij}, and define $W = (sI - M)$.

We have shown that if M is cuttable, W is cuttable, and so the simplifications in computing the inverse of a cuttable matrix can be utilized in the computation of X. In addition the reducibility of some of the cofactors implies some important physical properties of systems containing cut points. Note first that if x_{ij} and x_{kq} are two transfer functions, then

$$\frac{x_{ij}}{x_{kq}} = \frac{(-1)^{i+j} W_{ji}}{(-1)^{k+q} W_{qk}}. \tag{12}$$

That is, the ratio of two transfer functions is given by the cofactors of the corresponding elements in W^T.

The ratio of x_{ij} to x_{ji} is of particular interest. Let $f_{ij}(t)$ and $f_{ji}(t)$ be the Laplace transforms of x_{ij} and x_{ji}. If x_{ij}/x_{ji} is independent of s, then $f_{ij}(t)/f_{ji}(t)$ is independent of time, or $f_{ij}(t)$ is a multiple of $f_{ji}(t)$. If this relation holds for all i and j, the system is said to possess quasi-reciprocity. If x_{ij}/x_{ji} equals unity, for all i and j the system possesses reciprocity.

In a system which does not exhibit quasi-reciprocity or reciprocity for all compartments, it may be the case that for one or more pairs of compartments i and j, x_{ij}/x_{ji} is equal to unity or some other constant. This property is called local reciprocity or quasi-reciprocity, and the types of systems which exhibit it have been studied from a graph theoretical viewpoint (Marimont, to appear in Bull. Math. Biophys. June, 1969). Local reciprocity in cuttable and cut-reducible systems will be discussed in the following sections.

Let us now relate the cuttable matrix discussed earlier to a physical system. Let $\hat{A}$ be the matrix corresponding to a system consisting of a set of compartments A, plus compartment c, with arbitrary connections among its members. Consider B, the matrix corresponding to a separate system with arbitrary connections among its members. Let the systems be connected by any number of flows in either direction between c and any members of B. Then c is a cutpoint, since all flows between A and B must pass through c, and therefore deletion of c would leave the two systems A and B disconnected. The elements in the cut row represent flows to c—those bordering block A are flows to c from members of A, and those bordering block B are flows to c from members of B. Similarly the cut column represents flows from c to other compartments. The elements of the zero submatrices are direct flows between members of A and members of B, all of which are zero, since all such flows must pass through c. We can now state the physical properties implied by Theorems 2 and 3.

Property 1 (From Theorem 2)

If in a compartmental system with cutpoint c and blocks A and B, i and j (k and q) are both members of A (B), then the ratio of any two transfer functions of the form x_{ic}, x_{ci}, x_{jc}, or x_{cj} (x_{kc}, x_{ck}, x_{qc}, or x_{cq}) is independent of the block B (A), and is given by the ratio of the corresponding transfer functions in the system $\hat{A}$ ($\hat{B}$). If local reciprocity or quasi-reciprocity obtains between two members of a system $\hat{A}$, it will not be affected by the addition of any other systems which are connected through either or both of these members as cutpoints.

Equations (5) and (6) yield results concerning transfer functions with input in one block and output in the other. Let i and j be members of A, and k and q be members of B. Then, from (5)

$$\frac{x_{ik}}{x_{qj}}=\frac{[(-1)^{i+c}|\hat{A}_{ic}|]\,[(-1)^{c+k}|\hat{B}_{ck}|]}{[(-1)^{j+c}|\hat{A}_{jc}|]\,[(-1)^{c+q}|\hat{B}_{cq}|]}. \tag{13}$$

And from (6)

$$\frac{x_{ik}}{x_{jq}}=\frac{[(-1)^{c+i}|\hat{A}_{ci}|]\,[(-1)^{k+c}|\hat{B}_{kc}|]}{[(-1)^{c+j}|\hat{A}_{cj}|]\,[(-1)^{q+c}|\hat{B}_{qc}|]}. \tag{14}$$

Several conclusions can be drawn immediately from eq (13) and (14).

Property 2 (From Theorem 3)

The ratio of two transfer functions with inputs in A (B) and outputs in B (A) may be expressed as a product of two ratios—that of the transfer functions in $\hat{A}$ ($\hat{B}$) of the two inputs, with the cutpoint as a common output, and that of the two outputs in $\hat{B}$ ($\hat{A}$), with the cutpoint as a common input.

Property 3 (From Theorem 3)

The ratio of two transfer functions with a common input in A (B) and separate outputs in B (A), is independent of A (B), and is given by the ratio of the transfer functions of those outputs in $\hat{B}$ ($\hat{A}$), with the cutpoint as a common input.

Property 4 (From Theorem 3)

The ratio of two transfer functions with separate inputs in A (B) and a common output in B (A) is independent of B (A), and is given by the ratio of the transfer functions in $\hat{A}$ ($\hat{B}$) of those inputs, with the cutpoint as a common output.

4. Cut-reducible Matrices

If in the matrix of Figure 2 the entries of either V_1 or V_2, but not of both, are arbitrary, the matrix is cut-reducible. An equivalent definition is that there is a diagonal element m_{cc} such that the co-factor of m_{cc} is reducible. Physically, this matrix describes a system with two blocks A and B such that all flow from A to B (B to A) must pass through point c, but that flow from B to A (A to B) is arbitrary. The point is called a cut-reduction point. Yet another physical characterization is that the deletion of point c makes flow between A and B unidirectional. Figure 2 shows a cut-reducible system and its matrix.

Let us assume for concreteness that V_1 is arbitrary and V_2 zero, as in figure 2. Then all flow from A to B must pass through c. The cut-reducible matrix differs from the cuttable in the following ways:

1. Not all cofactors of elements in the cut row and cut column are reducible. Cofactors of the cut column bordering A, and of the cut row bordering B are reducible, but since not all in any row or column are, the determinant of a cut-reducible matrix cannot be expressed in the simple form of eq (3), and therefore the results concerning the characteristic equations and roots do not hold.

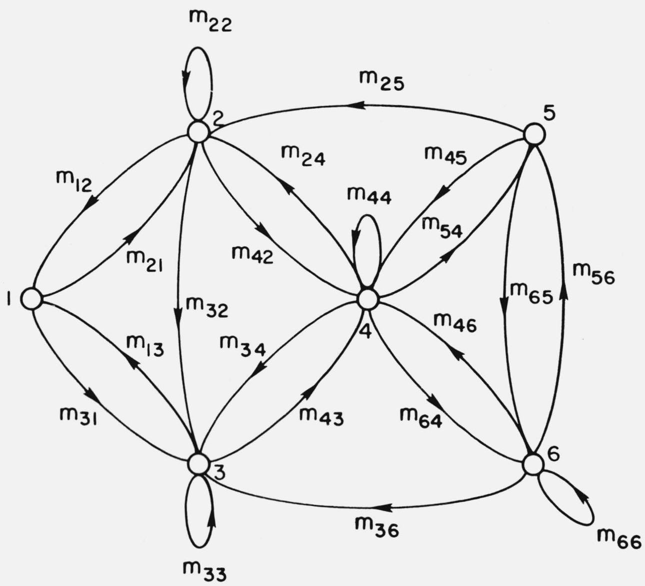
2. Cofactors of all elements in V_1 are reducible, but those in V_2 are not.

The analogues of Theorems 2 and 3 may immediately be stated for the cut-reducible matrix of figure 2.

THEOREM 5. *In the cut-reducible matrix of figure 2, the ratio of cofactors of any elements of the cut column (row) bordering* A (B) *is independent of* B (A). *The ratio is that of their cofactors in* $\hat{\mathrm{A}}$ ($\hat{\mathrm{B}}$).

THEOREM 6. *In the cut-reducible matrix of figure 2, the cofactor of any element of* V_1 *is given, by the product of the cofactors of the cut elements of its row and column in their bordered sub-matrices. The ratios of cofactors of any elements in the same row will be independent of* $\hat{\mathrm{A}}$, *and these in the same column independent of* $\hat{\mathrm{B}}$.

The physical properties of cut reducible systems can be easily deduced from these theorems, as were those for cuttable systems from Theorems 2 and 3.



$$
\begin{array}{cccccc}
 & A & & & & V_1 \\
 & \downarrow & & & & \downarrow \\
0 & m_{12} & m_{13} & 0 & 0 & 0 \\
m_{21} & m_{22} & 0 & m_{24} & m_{25} & 0 \\
m_{31} & m_{32} & m_{33} & m_{34} & 0 & m_{36} \\
\hline
0 & m_{42} & m_{43} & m_{44} & m_{45} & m_{46} \\
\hline
0 & 0 & 0 & m_{54} & 0 & m_{56} \\
0 & 0 & 0 & m_{64} & m_{65} & m_{66} \\
 & \uparrow & & & & \uparrow \\
 & V_2 & & & & B
\end{array}
$$

FIGURE 2. *A system with a cut reduction point and its cut-reducible matrix. All flow from block* A, *consisting of points 1, 2, and 3, to block* B, *consisting of points 5 and 6, passes through point 4, although flow from* B *to* A *is unrestricted. Deletion of point 4 therefore eliminates flow from* A *to* B. *The matrix expression of this connectivity is that the cofactor of* m_{44} *is reducible, with blocks* A *and* B.

A property similar to property 3 for cut-reducible chemical kinetic systems was described by Hearon (1955, unpublished). His term link variable corresponds to our cut-reduction point.

5. The General Cuttable or Cut-Reducible System

So far the simplest types of cuttable and cut-reducible systems and their matrices have been described—namely those with a single distinguished point (either cut-point or cut-reduction point) which divides the system into two subsystems. A cut-point may divide the system into more than two disjoint subsystems, in which case matrix B is block diagonal. There may be more than one cutpoint, in which case B is cuttable. The system may have both cut and cut-reduction points. In general, the greater the complexity of the system, the greater the advantage of the cut simplifications, since the submatrices dealt with are smaller relative to the whole matrix than they are in the case of two submatrices.

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