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Subgroups of $SL(t, Z)$

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It is shown that if $t \geqslant 3$, then no subgroup of $SL(t, Z)$ of finite index is free (in fact is not even the free product of cyclic groups). Here $SL(t, Z)$ is the multiplicative group of $t \times t$ matrices over the integers of determinant 1.

Key words: Congruence subgroup property; free groups; free products; higher modular groups.

It is known that the classical modular group $\Gamma = PSL(2, Z)$ possesses many subgroups which are free: In fact (see [4])[1] every normal subgroup of Γ is free, apart from the three exceptions Γ, Γ^2, Γ^3, (Γ^p is the fully invariant subgroup of Γ generated by the pth powers of the elements of Γ). The reason for this is that Γ is the free product of two cyclic groups, one of order 2 and the other of order 3, and the Kurosh subgroup theorem then implies that a subgroup of Γ is free if and only if it contains no elements of finite order. This result at once implies that every normal subgroup of $SL(2, Z)$ which does not contain $-I$ must be free.

It is natural to ask if a simular situation exists for the higher modular groups. The purpose of this note is to point out that just the opposite is true. In fact we will show that no normal subgroup nor any subgroup of $\Gamma_t = SL(t, Z)$ of finite index can be free for $t \geqslant 3$.

If n is any positive integer, then $\Gamma_t(n)$ is the principal congruence subgroup of Γ_t of level n; that is, the totality of matrices $A \in \Gamma_t$ such that $A \equiv I \bmod n$. E_{pq} stands for the $t \times t$ matrix with a 1 in position (p, q) and 0 elsewhere.

The proof makes use of a deep result proved recently by Mennicke [2], and by Bass, Lazard, and Serre [1], together with some elementary remarks on free groups.

We assume throughout the following that $t \geqslant 3$. We first prove

LEMMA 1. *Let* F *be a free group. Then any two commuting elements of* F *are each powers of the same element of* F.

PROOF. Suppose that x, y are commuting elements of F. Put $G = \{x, y\}$. Then G is an abelian subgroup of F of rank 1 or 2. Furthermore G is free, by the Kurosh subgroup theorem. Thus G cannot be of rank 2, since a free group of rank 2 is not abelian. Hence G must be of rank 1 and therefore cyclic: $G = \{z\}$. Thus x and y are each powers of z, and the lemma is proved.

LEMMA 2. *Let* n *be any positive integer.* Put $A = I + nE_{12}$, $B = I + nE_{13}$. *Then* A, B *are commuting elements of* $\Gamma_t(n)$ *which are not powers of the same element* C, *for any* $C \in \Gamma_t(n)$.

PROOF. The fact that A and B are commuting elements of $\Gamma_t(n)$ is clear, since $E_{12}E_{13} = E_{13}E_{12} = 0$. Suppose that there is an element $C \in \Gamma_t(n)$ such that $A = C^r$, $B = C^s$, for some integers r, s. Then $rs \neq 0$, and $A^s = B^r = C^{rs}$. Since $E_{12}^2 = E_{13}^2 = 0$, we have

$$(I + nE_{12})^s = (I + nE_{13})^r,$$

$$I + snE_{12} = I + rnE_{13},$$

$$sE_{12} = rE_{13},$$

[1] Figures in brackets indicate the literature references at the end of this paper.

an impossibility. Hence the lemma is proved.

Clearly, these lemmas imply

LEMMA 3. *If* $t \geqslant 3$ *and* n *is any positive integer then* $\Gamma_t(n)$ *is never free.*

From this lemma we deduce

THEOREM 1. *Suppose that* $t \geqslant 3$. *Then no normal subgroup nor any subgroup of* Γ_t *of finite index is free.*

PROOF. It was shown in [1] and [2] that if G is a subgroup of Γ_t of finite index, then G is a congruence group; i.e., G contains $\Gamma_t(n)$ for some appropriate positive integer n. By Lemma 3 the group $\Gamma_t(n)$ is not free; and so then neither is G.

It was also shown in the papers referred to above that a normal subgroup G of Γ_t is either central, or of finite index and hence a congruence group. In either case G is not free. This completes the proof.

We remark that a simular argument shows that for $t \geqslant 3$, no normal subgroup nor any subgroup of finite index of Γ_t can be the free product of cyclic groups. Here the relevant fact is that two commuting elements in a free product of cyclic groups must belong to a common cyclic subgroup.

Finally, Mennicke has shown in [3] that the symplectic group $Sp(2t, Z)$ also has the "congruence subgroup property" for $t \geqslant 2$. If we note that

$$A=\begin{pmatrix} I & nE_{11} \\ 0 & I \end{pmatrix}, B=\begin{pmatrix} I & nE_{22} \\ 0 & I \end{pmatrix}$$

are commuting elements of $Sp(2t, Z)$ which are not powers of the same element C of $Sp(2t, Z)$ we get

Theorem 2. *Suppose that* $t \geqslant 2$. *Then no normal subgroup nor any subgroup of finite index of* $Sp(2t, Z)$ *is free.*

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