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The Second Orthogonality Conditions in the Theory of Proper and Improper Rotations. III. The Conjugacy Theorem*

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The second orthogonality conditions are used to provide a concise proof of the theorem that two rotation matrices connected by an orthogonal similarity transformation have the same angles of rotations. This theorem is discussed in the context of its applicability to the problem of decomposing the real orthogonal group 0(3) into its classes of conjugate elements.

Key words: Conjugacy; conjugacy theorem; conjugate rotations; matrices; orthogonal group; orthogonal transformations; rotation; rotation group; rotation-inversions.

1. Introduction

As is well known, rotations belong to the group 0 (3) of real orthogonal transformations in Euclidean three-dimensional space. This group also contains the coordinate inversion with respect to the origin, and the rotation-inversions, that is, the products of the inversion with all proper rotations. An important theorem for 0 (3) states that all rotations through the same angle, regardless of their axes belong to the same class.[1] This means that the conjugate of any rotation by an arbitrary member of 0 (3) is another rotation by the same amount as the original rotation.

In the literature, the various proofs of this theorem are based on the representation of 0 (3) by real orthogonal three-by-three matrices. It is shown that the similarity transform of any given rotation matrix R by an arbitrary orthogonal matrix A, represents a rotation R' by the same angle as R. That is, with $R=R_{\mathbf{n}}(\alpha)$ given, and R' defined by

$$R' \equiv AR_{\mathbf{n}}(\alpha)A^{-1}=AR_{\mathbf{n}}(\alpha)\tilde{A}, \tag{1}$$

R' is expressible as $R_{\mathbf{n}'}(\alpha)$, a matrix of rotation by the same angle α, about an axis $\mathbf{n}'$ which differs from $\mathbf{n}$ but is related to it. The statement that the angle which apperas in R' is the same as that which appears in R, together with the relation between n_i and n_i' constitute what we shall refer to as the conjugacy theorem. (It has no specific name in the literature.)

One proof of the conjugacy theorem consists in showing that a given matrix can, by a suitable similarity transformation, be re-expressed as a matrix representing a rotation by the same angle about one of the coordinate axes [1, 2].[2] It then follows that any two matrices of rotation by the same angle can be transformed into each other by similarity transformation. This proof does not explicitly construct the relationship between the axes of rotation.

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[1] We remind the reader that the class of a group element consists of all of the distinct results of forming the conjugate of that element with all other elements in the group. The conjugate of a group element g by another element h is $g' \equiv hgh^{-1}$.

[2] Figures in brackets indicate the literature references at the end of this paper.

In a second proof [3], it is found convenient to regard the matrices R and R' as active, rather than as passive transformations. This means that R and R' are thought of as the matrix representatives in an initial and final coordinate system respectively, of an operator (on vectors) which is defined in *all* coordinate systems. The matrix A is still regarded as a passive transformation of coordinates between the initial and final systems. It is then shown that the matrix elements R_{ij} computed in the initial system of basis vectors ($\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{b}_3$) are identical to the corresponding matrix elements R'_{ij} computed in the final basis, that is, in the basis ($\mathbf{b}'_1$, $\mathbf{b}'_2$, $\mathbf{b}'_3$) given by

$$\mathbf{b}'_i = A_{ij}\mathbf{b}_j. \tag{2}$$

It is then argued that the identity of the matrix elements implies the equality of α and α'. This argument is substantially correct. However, it is not sufficiently emphasized in reference [3] that this argument fixes the transformation relation between $\mathbf{n}$ and $\mathbf{n}'$. As pointed out in paper II (ref. [4]) the choice of an angle of rotation is correlated with the sense of the axis. Therefore, arguing that $\alpha' = \alpha$ follows from $R'_{ij} = R_{ij}$, and assuming that α' is correlated with $\mathbf{n}'$ in a definite way, one arrives at an unambiguous relation between $\mathbf{n}$ and $\mathbf{n}'$, since α is assumed to be correlated with $\mathbf{n}$ in a definite way.

The purpose of this paper is to present a new proof of the conjugacy theorem, which follows concisely and elegantly from the second orthogonality conditions. In addition to its conciseness it has the advantage that it does not use a different interpretation (i.e. active versus passive) for R and A. By explicitly requiring that the convention for defining n with respect to R be the same as that for defining n' with respect to R', one immediately obtains both the equality of α' and α, and the relations

$$n'_i = (\det A) A_{ij} n_j \tag{3}$$

as the transformation law for the axis of rotation.

2. Proof of the Conjugacy Theorem

We consider the arbitrary real orthogonal three-by-three matrices G and A. The similarity transform of G by A is

$$G' \equiv AGA^{-1} = AG\tilde{A}. \tag{4}$$

We denote the intrinsic vector of G by $\mathbf{W}$ and that of G' by $\mathbf{W}'$. These vectors are

$$W_i = \epsilon_{ijk} G_{jk}, \tag{5a}$$

$$W'_i = \epsilon_{ijk} G'_{jk}. \tag{5b}$$

Inserting the right side of eq (4) for G'_{jk} in eq (5b), and using the cyclic property (I 10) [5] of the Levi-Civita symbol, we have

$$W'_i = (A_{rj} A_{sk} \epsilon_{rsi})\ G_{jk}. \tag{6}$$

For the parenthetic expression in eq (6) we can use the second orthogonality conditions in the form (I 9b). This leads to

$$W'_i = \frac{p}{p'} A_{ir} \epsilon_{rjk} G_{jk},$$

where we have again used the cyclic property of the Levi-Civita symbol. Here, the handedness factors p and p' refer to the coordinate systems connected by A. From eq (5a) we recognize the combination $\epsilon_{rjk} G_{jk}$ as W_r. Therefore [3]

[3] If G and G' were to be regarded as active transformations and A as a passive transformation then eq (7) says that the intrinsic vector of an active transformation matrix has the transformation properties of an axial vector.

$$W_i' = \frac{p}{p'} A_{ir} W_r = (\det A)\, A_{ir} W_r. \tag{7}$$

where in the last step we have used in fact that $\det A = p'/p = p/p'$. The transformation (7) preserves the length of $\mathbf{W}$, so that $W' = W$. This follows from the first orthogonality conditions (I 7).

We now specialize the development to matrices G which represent proper rotations, using R to denote such a matrix. From the result that the determinant of any orthogonal matrix is preserved under the similarity transformation (4) we have that $\det R' = \det R = +1$. Therefore $AR\tilde{A}$ represents a proper rotation if R does. The trace of an orthogonal transformation matrix is also preserved under the transformation (4). Hence $\operatorname{tr} R' = \operatorname{tr} R$. It then follows from eq (II 4) that $\cos\alpha' = \cos\alpha$.

We now impose the requirement that the convention for relating the sense of $\mathbf{n}'$, the axis of rotation of R', to that of $\mathbf{W}'$, be the same as the convention for relating the sense of $\mathbf{n}$, the axis of rotation of R, to that of $\mathbf{W}$, regardless of whether A involves a change of handedness or not. Thus if $\mathbf{n} = \lambda(\mathbf{W}/W)$ where $\lambda = \pm 1$, then we define[4] $\mathbf{n}' \equiv \lambda(\mathbf{W}'/W')$. From eq (7) this means that the required transformation for the axis of rotation is given by eq (3).

The sine of α, the angle of rotation of R is given by eq (II 17) to be

$$\sin\alpha = \tfrac{1}{2}\,\mathbf{n}\cdot\mathbf{W} = \tfrac{1}{2}\,\lambda W. \tag{8}$$

Similarly the sine of α', the angle of rotation of R' is

$$\sin\alpha' = \frac{1}{2}\mathbf{n}'\cdot\mathbf{W}' = \frac{1}{2}\lambda\frac{W'^2}{W'} = \frac{1}{2}\lambda W, \tag{9}$$

where we have used the invariance of the length of $\mathbf{W}$ under the transformation (7). We see as a result that $\sin\alpha' = \sin\alpha$. Combining this result with the fact that $\cos\alpha' = \cos\alpha$ we have that $\alpha' = \alpha$ *unambiguously*. This latter result together with eq (3) completes the proof of the conjugacy theorem.

The conjugacy theorem allows one to classify not only the proper rotations but the improper ones as well. This follows from the fact that any improper rotation is expressible as a rotation-inversion, that is, as a coordinate inversion preceded or followed by a proper rotation. The inversion would appear on both sides of eq (4) and be cancelled. The conjugacy theorem then shows that eq (3) and the result $\alpha' = \alpha$ hold for the proper rotations of the rotation-inversions which would appear on the two sides of eq (4). It then follows that the class of a given improper rotation consists of all rotation-inversions of the same angle regardless of their axes.

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3. References

[1] Wigner, E. P., Group Theory and its Application to the Quantum Mechanics of Atomic Spectra, sec. 14–7. (Academic Press, New York, 1959).

[2] Hamermesh, M., Group Theory, sec. 9–2. (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1962).

[3] Lyubarskii, G. Ya., The Application of Group Theory in Physics, secs. 5, 6. (Pergamon Press, Oxford, 1960).

[4] Gelman, H., The Second Orthogonality Conditions in the Theory of Proper and Improper Rotations, II. The Intrinsic Vector, J. Res. NBS 73B (Math. Sci.) No. 2 (1969).

[5] Gelman, H., The second orthogonality conditions in the theory of proper and improper rotations, I. Derivation of the Conditions and of Their Main Consequences, J. Res. NBS **72B** (Math. Sci.) No. 3, 229 (1968).

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[4] Note that if $W = 0$, $\mathbf{n}$ and $\mathbf{n}'$ cannot be defined in this way. But in that case $\sin\alpha' = \sin\alpha = 0$.