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The Cylinder Problem in Thermoviscoelasticity *

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Solutions are obtained for several axially symmetric plane strain problems involving a hollow circular viscoelastic cylinder. The cylinder is assumed to be subject to planar, axially symmetric body force and temperature fields. Displacement, traction, and mixed boundary conditions are considered.

Key Words: Cylinder; Neumann series; thermoviscoelasticity.

1. Introduction

In this paper we study a hollow thick-walled right circular cylinder composed of an isotropic homogeneous viscoelastic material whose response to temperature is thermorheologically simple.

We solve the axially symmetric plane strain problem for this cylinder subject to three types of boundary conditions:

(i) zero displacement at the inner and outer walls,

(ii) zero traction on both walls, and

(iii) zero traction inside, zero displacement outside.

The latter mixed problem was chosen to suggest the solid fuel core of a rocket. Our analysis can be easily modified to deal with the case of a solid cylinder or with nonvanishing boundary conditions.

Many problems in isothermal viscoelasticity are readily solved by means of Laplace transforms. This technique has also been applied with some success in thermoviscoelasticity, for example to sphere problems by Muki and Sternberg [1][1] and Sternberg and Gurtin [2], and to a contact problem by Graham [3].

In fact Valanis and Lianis [9] use Laplace transforms to obtain the solution of a thermoviscoelastic cylinder problem under the assumption, unnecessary in the present paper, of a constant hereditary Poisson ratio.

Our method is fundamentally different from the above. It is a generalization of that used in [4] to obtain existence for the displacement problem of isothermal viscoelasticity. The analysis in [4] led to a Neumann expansion representation for the viscoelastic displacement field which involved the inverse E^{-1} of an elastic operator E. Also, convergence of the expansion depended on a Schauder estimate for E. In order to assure existence of E^{-1} and of the estimate we were obliged in that paper to confine ourselves to the displacement problem. For the solution of the cylinder problem we obtain an analogous Neumann expansion. However, due to the extremely simple geometry of the present problem an explicit representation of E^{-1} is easily derived for a variety of boundary conditions. Thus, our solution is altogether explicit, and moreover, estimates which suffice for the convergence of the expansion are immediate.

Finally we remark that no attempt has been made to expound on the derivation of the constitutive equations set forth below. Thorough expositions of the theory of linear viscoelasticity and

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[1] Figures in brackets indicate the literature references at the end of this paper.

thermorheologically simple materials are given in the papers [5, 6] of Gurtin and Sternberg. If one accepts our field equations, then our analysis can be followed without consultation of any of the above references.

We plan a sequel to this paper in which the effects of ablation are considered and numerical examples are given.

2. Statement of the Problems

Consider a hollow circular cylinder with inner radius a and outer radius b. If x_1, x_2, and x_3 are orthogonal Cartesian coordinates oriented in such a way that the x_3-axis coincides with the axis of the cylinder, then a set of cylindrical coordinates ρ, φ, z is defined by

$$x_1 = \rho \cos\varphi, \qquad x_2 = \rho \sin\varphi, \qquad x_3 = z$$

where

$$0 \leq \rho < \infty, \qquad 0 \leq \varphi < 2\pi, \qquad -\infty < z < \infty.$$

The cylinder is composed of isotropic homogeneous viscoelastic material subject to a temperature field[2] $T(\rho, t)$, where t stands for time, and a body force vector

$$\mathbf{F} = (F_\rho, F_\varphi, F_z)$$

such that

$$F_z = F_\varphi = 0, \qquad F_\rho = F(\rho, t).$$

We assume that the displacement field

$$\mathbf{u} = (u_\rho, u_\varphi, u_z)$$

is axially symmetric and planar, i.e.,

$$u_z = u_\varphi = 0, \qquad u_\rho = u(\rho, t). \tag{2.1}$$

Let $\sigma_{\rho\rho}, \sigma_{\varphi\varphi}$ . . . and $\epsilon_{\rho\rho}, \epsilon_{\varphi\varphi}$, . . . denote respectively the components of stress and strain in cylindrical coordinates. Then from (2.1) and the usual strain-displacement relations for these coordinates ([7], p. 183) it follows that

$$\epsilon_{\rho\rho}(\rho, t) = \frac{\partial u}{\partial \rho}(\rho, t), \qquad \epsilon_{\varphi\varphi}(\rho, t) = \frac{u(\rho, t)}{\rho}, \qquad \epsilon_{zz} = 0$$

$$0 = \epsilon_{\rho\varphi} = \epsilon_{\varphi z} = \epsilon_{z\rho}$$

Let $G_1(t)$, $G_2(t)$ be respectively the relaxation functions in shear and isotropic compression corresponding to the base temperature T_0. We assume that the cylinder has been at rest at the reference temperature T_0 up to time $t = 0$ and that the material with which we are dealing is thermorheologically simple.

In order to express concisely the plane strain stress-strain relations for a thermorheologically simple material and to avoid needlessly detailed equations in the sequel, we introduce the following operator notation. Given any differentiable function $G(t)$ defined on $[0, \infty)$ we associate with it the thermoviscoelastic operator V_G by defining[3]

$$V_G f(\rho, t) = G(0) f(\rho, t) + \int_0^t G^{(1)}\left(\int_{t'}^t \varphi(T(\rho,\tau))\,d\tau)\varphi(T(\rho, t')\right) f(\rho, t')\,dt'. \tag{2.3}$$

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[2] In this paper, we have not attempted to determine the minimum sufficient smoothness of the given data. All given functions are assumed to be several times continuously differentiable.

[3] $G^{(1)}(u) = \frac{\partial G}{\partial u}$.

Here, $T(\rho, t)$ is the temperature field introduced above, and $\varphi(T)$, the so-called shift factor, is a prescribed function which governs the change in material properties induced by a temperature field. The function φ has the properties

$$\varphi(T_0)=1,\ \varphi(T)>0,\ \frac{d\varphi}{dT}>0. \tag{2.4}$$

Notice that for $G(0)\neq 0$, V_G is a Volterra-type integral operator of the *second kind.* We shall assume that all the thermoviscoelastic operators which we encounter are of the second kind.

It is also convenient to define

$$M(t)=\tfrac{1}{3}(G_2(t)-G_1(t)) \tag{2.5}$$

$$G_1(0)=\gamma,\ M(0)=\mu \tag{2.6}$$

$$g(\rho,t,t')=G_1^{(1)}\left(\int_{t'}^{t}\varphi(T(\rho,\tau))d\tau\right)\varphi[T(\rho,t')] \tag{2.7}$$

$$m(\rho,t,t')=M^{(1)}\left(\int_{t'}^{t}\varphi(T(\rho,\tau))d\tau\right)\varphi[T(\rho,t')] \tag{2.8}$$

so that V_{G_1} and V_M take the form

$$V_{G_1}f(\rho,t)=\gamma f(\rho,t)+\int_0^t g(\rho,t,t')f(\rho,t')dt' \tag{2.9}$$

$$V_M f(\rho,t)=\mu f(\rho,t)+\int_0^t m(\rho,t,t')f(\rho,t')dt'. \tag{2.10}$$

Finally we introduce the function

$$\Phi(\rho,t)=\alpha_0 V_{G_2}\Theta(\rho,t). \tag{2.11}$$

The function Θ is defined by

$$\Theta(\rho,t)=\frac{1}{\alpha_0}\int_{T_0}^{T(\rho,t)}\alpha(T')dT',\ \alpha_0=\alpha(T_0)$$

and is known to thermoviscoelasticians as a "pseudotemperature." The prescribed material function $\alpha(T)$ is called the coefficient of linear thermal expansion.

The thermoviscoelastic stress-strain relations appear in many papers, for example Sternberg and Gurtin [2]. Written in terms of Cartesian coordinates and with the present operator notation they take the form

$$\sigma_{ij}(\rho,t)=V_{G_1}\epsilon_{ij}(\rho,t)+\delta_{ij}V_M\epsilon_{kk}(\rho,t)-\delta_{ij}\Phi(\rho,t). \tag{2.12}$$

In this equation, Latin subscripts take the range 1,2,3 and summation over repeated indices is implied. In the case of plane strain, Sternberg and Gurtin [2] apply Laplace transforms to solve for σ_{zz} in terms of the other stresses in the form

$$\sigma_{zz}=V_{\bar{\nu}}(\sigma_{\rho\rho}+\sigma_{\varphi\varphi})-\alpha_0 V_{\bar{E}}\Theta \tag{2.13}$$

where [4]

$$\bar{E}^*(\eta)=\frac{3G_1^*(\eta)G_2^*(\eta)}{G_1^*(\eta)+2G_2^*(\eta)}$$

$$\bar{\nu}^*(\eta)=\frac{G_2^*(\eta)-G_1^*(\eta)}{\eta(G_1^*(\eta)+2G_2^*(\eta))}.$$

[4] For f defined on $[0,\infty)$ $f^*(\eta)=\int_0^\infty e^{-\eta t}f(t)dt$.

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From equations (2.12), which are invariant under a change of coordinate systems, and (2.2) we obtain the plane strain axisymmetric stress-strain relations

$$\begin{aligned}
\sigma_{\rho\rho} &= V_{G_1}\left(\frac{\partial u}{\partial \rho}\right) + V_M\left(\frac{\partial u}{\partial \rho} + \frac{u}{\rho}\right) - \Phi. \\
\sigma_{\varphi\varphi} &= V_{G_1}\left(\frac{u}{\rho}\right) + V_M\left(\frac{\partial u}{\partial \rho} + \frac{u}{\rho}\right) - \Phi \\
\sigma_{\rho\varphi} &= \sigma_{\varphi z} = \sigma_{z\rho} = 0.
\end{aligned} \tag{2.14}$$

Due to (2.13 and (2.14), the quasistatic stress equations of equilibrium ([7], page 184) reduce to the single equation

$$\frac{\partial}{\partial \rho}\sigma_{\rho\rho} + \frac{1}{\rho}(\sigma_{\rho\rho} - \sigma_{\varphi\varphi}) + F = 0. \tag{2.15}$$

If we substitute from (2.14) into (2.15) and apply the notation (2.9), (2.10) then the quasistatic displacement equation of motion takes the form

$$\begin{aligned}
&\frac{\partial^2 u}{\partial \rho^2}(\rho, t) + \frac{\partial}{\partial \rho}\left(\frac{u(\rho, t)}{\rho}\right) \\
&\quad + \frac{1}{\gamma + \mu}\int_0^t \left\{(g(\rho, t, t') + m(\rho, t, t'))\left[\frac{\partial^2 u}{\partial \rho^2}(\rho, t') + \frac{\partial}{\partial \rho}\left(\frac{u(\rho, t')}{\rho}\right)\right]\right. \\
&\quad + \left(\frac{\partial g}{\partial \rho}(\rho, t, t') + \frac{\partial m}{\partial \rho}(\rho, t, t')\right)\frac{\partial u}{\partial \rho}(\rho, t') \\
&\quad \left. + \frac{\partial m}{\partial \rho}(\rho, t, t')\frac{u(\rho, t')}{\rho}\right\} dt' = H(\rho, t)
\end{aligned} \tag{2.16}$$

where

$$H(\rho, t) = \frac{1}{\gamma + \mu}\left(\left(\frac{\partial \Phi}{\partial \rho}\right)(\rho, t) - F(\rho, t)\right). \tag{2.17}$$

Three types of boundary value problems are considered, namely,
(i) The displacement problem (first boundary value problem)

$$u(a, t) = u(b, t) = 0, \ (t > 0) \tag{2.18}$$

(ii) the traction problem

$$\sigma_{\rho\rho}(a, t) = \sigma_{\rho\rho}(b, t) = 0, \text{ and} \tag{2.19}$$

(iii) the mixed problem

$$0 = \sigma_{\rho\rho}(a, t) = u(b, t). \tag{2.20}$$

Our task then is to find a function $u(\rho, t)$ which satisfies (2.16) for (ρ, t) in $[a, b] \times [0, \infty)$ subject consecutively to each of the boundary conditions (2.18), (2.19), (2.20) where $\sigma_{\rho\rho}$ is defined in terms of u by (2.14). Stresses are then computed via (2.13), (2.14).

3. The solution

We define the differential operator E and the integrodifferential operator B as follows:

$$\mathrm{E}f(\rho, t)=\frac{\partial^2 f}{\partial \rho^2}(\rho, t)+\frac{\partial}{\partial \rho}\left(\frac{f(\rho, t)}{\rho}\right), \tag{3.1}$$

$$\begin{aligned} Bf(\rho, t)=\frac{1}{\gamma+\mu}\int_0^t &\{(g(\rho, t, t')+m(\rho, t, t'))Ef(\rho, t') \\ &+\left(\frac{\partial g}{\partial \rho}(\rho, t, t')+\frac{\partial m}{\partial \rho}(\rho, t, t')\right)\frac{\partial f}{\partial \rho}(\rho, t') \\ &+\frac{\partial m}{\partial \rho}(\rho, t, t')\frac{f(\rho, t')}{\rho}\Big\}\, dt'. \end{aligned} \tag{3.2}$$

Then (2.16) reduces to

$$Eu+Bu=H. \tag{3.3}$$

Our plan of attack is as follows: consider the model equation

$$Ef+Bf=P \tag{3.4}$$

for which we seek a solution $f(\rho, t)$ in $[a, b]\times[0, \infty)$ subject to the boundary conditions

$$M_a f(a, t)=M_b f(b, t)=0 \ (t>0). \tag{3.5}$$

Here M_a and M_b are some sort of linear integrodifferential boundary operators. Let E^{-1} be that operator which assigns to a given function $\psi(\rho, t)$ a unique function f which satisfies

$$Ef=\psi \tag{3.6}$$

in $[a, b[\times[0, \infty)$ and (3.5) for $t \geqslant 0$. Then clearly we obtain a solution f for (3.4), (3.5) by first finding a solution ψ for the equation

$$\psi+BE^{-1}\psi=P \tag{3.7}$$

and then setting

$$f=E^{-1}\psi. \tag{3.8}$$

The advantage to be gained by dealing with (3.7) rather than (3.4) lies in the fact that (3.7) is a sort of generalized integral equation and may be solved by the method of successive approximations. This decomposition of a viscoelastic problem into an elastic problem and an integral equation was introduced by Fichera in [8].

The formal solution of (3.7) is

$$\psi=(I+BE^{-1})^{-1}P$$

$$\psi=\sum_{n=0}^{\infty}(-1)^n(BE^{-1})^n P \tag{3.9}$$

so that

$$f=\sum_{n=0}^{\infty}(-1)^n E^{-1}(BE^{-1})^n P. \tag{3.10}$$

We shall now set forth sufficient conditions for the uniform convergence of the series (3.9), (3.10) in any rectangle $[a, b] \times [0, t]$ (t finite).

To this end, we define the norms [5]

$$\|h\|(t) = \text{l.u.b.}\{|h(\rho, \tau)| : (\rho, \tau) \epsilon [a, b] \times [0, t]\} \tag{3.11}$$

$$\|h\|_1(t) = \left\|\frac{h}{\rho}\right\|(t) + \left\|\frac{\partial h}{\partial \rho}\right\|(t) \tag{3.12}$$

and postulate the existence of a nonnegative nondecreasing function $N(t)$ such that for any smooth function $\theta(\rho, t)$ and $t \geqslant 0$,

$$\|E^{-1}\theta\|_1(t) \leqslant N(t)\|\theta\|(t). \tag{3.13}$$

It is clear from (3.2) that for smooth g and m, B satisfies an inequality of the form

$$\|Bh\|(t) \leqslant Q(t) \int_0^t (\|Eh\|(t') + \|h\|_1(t'))dt'. \tag{3.14}$$

For example, one may take

$$Q(t) = \frac{1}{|\gamma+\mu|} \max \Bigg\{ \text{l.u.b.}\{|g(\rho, \tau, s) + m(\rho, \tau, s)| : (\rho, \tau, s) \epsilon [a, b] \times [0, t] \times [0, t]\},$$

$$\text{l.u.b.} \left\{ \left|\frac{\partial g}{\partial \rho}(\rho, \tau, s)\right| + \left|\frac{\partial m}{\partial \rho}(\rho, \tau, s)\right| : (\rho, \tau, s) \epsilon [a, b] \times [0, t] \times [0, t] \right\} \Bigg\}.$$

The function Q so defined is clearly nonnegative and nondecreasing.

By replacing h by $E^{-1}\theta$ in (3.14) and applying (3.13) we obtain the estimate

$$\|BE^{-1}\theta\|(t) \leqslant Q(t)(1+N(t)) \int_0^t \|\theta\|(t')dt'. \tag{3.15}$$

From (3.15), it follows by a standard iteration argument that

$$\|(BE^{-1})^n\theta\|(t) \leqslant \frac{(Q(t)(1+N(t))t)^n}{n!} \|\theta\|(t). \tag{3.16}$$

Thus the series (3.9) converges uniformly in any rectangle $[a, b] \times [0, t]$. Since (3.13) implies that E^{-1} is continuous with respect to the uniform norm in such a rectangle, it follows that (3.10) also converges uniformly in $[a, b] \times [0, t]$ and that interchange of E^{-1} and the sum sign is permissible. Thus, sufficient conditions for the uniform convergence of (3.9) and (3.10) are the existence of the nondecreasing bounding functions $N(t)$ and $Q(t)$. The boundedness of Q follows if we assume that G_1, G_2, φ, and T are sufficiently smooth. The function $N(t)$ depends on the latter functions together with the boundary conditions.

Each of the boundary value problems (i), (ii), (iii) may be reduced to a problem of the form (3.4), (3.5). However, the operators M_a and M_b appearing in (3.5) will be different in each of the three cases. Thus, we must derive three different inverse operators $E_1^{-1}, E_2^{-1}, E_3^{-1}$ corresponding respectively to (i), (ii), and (iii) and find nonnegative, nondecreasing functions N_1, N_2, N_3 such that for any function φ and $t \geqslant 0$, E_j^{-1} and N_j satisfy (3.13). Then by the above theory, the solution of each of the three problems will have a uniformly convergent representation of the form (3.10).

[5] θ and h denote arbitrary smooth functions of ρ, t.

(i) *The Displacement Problem*

Let $E_1^{-1}\theta$ be that function h which satisfies

$$Eh(\rho, t) = \theta(\rho, t) \tag{3.17}$$

in $[a, b] \times [0, \infty)$

and

$$h(a, t) = h(b, t) = 0 \tag{3.18}$$

for $t \geqslant 0$. By elementary techniques we obtain from (3.1), (3.17), and (3.18) the representation

$$E_1^{-1}\Theta(\rho, t) = \frac{1}{\rho}\int_a^\rho \Theta(y, t)\,\frac{(\rho^2 - y^2)}{2}\,dy - \frac{(\rho^2 - a^2)}{\rho(b^2 - a^2)}\int_a^b \Theta(y, t)\,\frac{(b^2 - y^2)}{2}\,dy. \tag{3.19}$$

It is obvious from (3.19) that a constant N_1 depending only on a and b exists such that

$$\|E_1^{-1}\Theta\|_1(t) \leqslant N_1\|\Theta\|(t). \tag{3.20}$$

Thus the displacement problem (2.18), (3.3) has the solution

$$u = \sum_{n=0}^{\infty}(-1)^n E_1^{-1}(BE_1^{-1})^n H, \tag{3.21}$$

where E_1^{-1} is given explicitly by (3.19).

(ii) *The Traction Problem*

For this problem, it is convenient to modify the operator notation (2.9), (2.10) as follows:

$$\begin{aligned} V_{G_1}^c[f(d, t')](t) &= \gamma f(d, t) + \int_0^t g(c, t, t') f(d, t')\,dt' \\ V_M^c[f(d, t')](t) &= \mu f(d, t) + \int_0^t m(c, t, t') f(d, t')\,dt'. \end{aligned} \tag{3.22}$$

Then by (3.3), (2.14), and (2.19) the traction problem takes the form

$$Eu + Bu = H \tag{3.3}$$

$$\Phi(a, t) = V_{G_1}^a\left[\frac{\partial u}{\partial \rho}(a, t')\right](t) + V_M^a\left[\frac{\partial u}{\partial \rho}(a, t') + \frac{u(a, t')}{a}\right](t) \tag{3.23}$$

$$\Phi(b, t) = V_{G_1}^b\left[\frac{\partial u}{\partial \rho}(b, t')\right](t) + V_M^b\left[\frac{\partial u}{\partial \rho}(b, t') + \frac{u(b, t')}{b}\right](t). \tag{3.24}$$

We seek a solution u for this problem in the form

$$u = v + w \tag{3.25}$$

where w is to be chosen in such a way that v satisfies a problem of the form (3.4), (3.5). Plugging (3.25) into (3.3) we obtain

$$Ev + Bv = H - Ew - Bw. \tag{3.26}$$

Also we demand of w that it satisfy the boundary conditions

$$\Phi(a,t)=V^a_{G_1}\left[\frac{\partial w}{\partial\rho}(a,t')\right](t)+V^a_M\left[\frac{\partial w}{\partial\rho}(a,t')+\frac{w(a,t')}{a}\right](t) \tag{3.27}$$

$$\Phi(b,t)=V^b_{G_1}\left[\frac{\partial w}{\partial\rho}(b,t')\right](t)+V^b_M\left[\frac{\partial w}{\partial\rho}(b,t')+\frac{w(b,t')}{b}\right](t). \tag{3.28}$$

Then u defined by (3.25) will satisfy (3.23), (3.24) provided v satisfies

$$0=V^a_{G_1}\left[\frac{\partial v}{\partial\rho}(a,t')\right](t)+V^a_M\left[\frac{\partial v}{\partial\rho}(a,t')+\frac{v(a,t')}{a}\right](t) \tag{3.29}$$

$$0=V^b_{G_1}\left[\frac{\partial v}{\partial\rho}(b,t')\right](t)+V^b_M\left[\frac{\partial v}{\partial\rho}(b,t')+\frac{v(b,t')}{b}\right](t). \tag{3.30}$$

The function w is by no means uniquely determined. We assume one of the form

$$w(\rho,t)=(\rho-a)^2\,\varphi_1(t)+(\rho-b)^2\,\varphi_2(t) \tag{3.31}$$

where the functions φ_1, φ_2 are to be determined by a substitution of (3.31) into (3.27) and (3.28). In this way we arrive at the equations

$$\Phi(a,t)=\left[2(\gamma+\mu)(a-b)+\mu\,\frac{(a-b)^2}{a}\right]\varphi_2(t)$$
$$+\int_0^t\left\{2(a-b)\,[g(a,t,t')+m(a,t,t')]+m(a,t,t')\,\frac{(a-b)^2}{a}\right\}\varphi_2(t')dt'$$

$$\Phi(b,t)=\left[2(\gamma+\mu)(b-a)+\mu\,\frac{(b-a)^2}{b}\right]\varphi_1(t)$$
$$+\int_0^t\left\{2(b-a)\,[g(b,t,t')+m(b,t,t')]+m(b,t,t')\,\frac{(b-a)^2}{b}\right\}\varphi_1(t')dt'.$$

Since these are, in general, Volterra-type integral equations of the second kind their solution presents no difficulties. We may thus regard w as known, so that the traction problem has been reduced to that of finding a function v which satisfies (3.26), (3.29) and (3.30). To this end, we define E_2^{-1} to be that operator which assigns to a given function $\theta(\rho,t)$ the function $h(\rho,t)$ which satisfies (3.17), (3.29), and (3.30). In order to derive E_2^{-1}, we write the general solution of (3.17) in the form

$$h(\rho,t)=\frac{1}{\rho}\int_a^\rho\theta(y,t)\left(\frac{\rho^2-y^2}{2}\right)dy+c_1(t)\rho+\frac{c_2(t)}{\rho}. \tag{3.32}$$

Then

$$\frac{\partial h}{\partial\rho}(\rho,t)=-\frac{1}{\rho^2}\int_a^\rho\theta(y,t)\left(\frac{\rho^2-y^2}{2}\right)dy+\int_a^\rho\theta(y,t)dy+c_1(t)-\frac{c_2(t)}{\rho^2}. \tag{3.33}$$

The functions $c_1(t)$, $c_2(t)$ are determined by a substitution of the expressions (3.32), (3.33) into (3.29), (3.30). They are governed by the equations

$$0=V^a_{G_1+2M}[c_1(t')](t)-\frac{1}{a^2}V^a_{G_1}[c_2(t')](t) \tag{3.34}$$

$$\Gamma(t)=V^b_{G_1+2M}[c_1(t')](t)-\frac{1}{b^2}V^b_{G_1}[c_2(t')](t) \tag{3.35}$$

where

$$\Gamma(t) = V^b_{G_1}\left[\frac{1}{b^2}\int_a^b \theta(y, t')\left(\frac{b^2-y^2}{2}\right)dy\right](t) - V^b_{G_1+M}\left[\int_a^b \theta(y, t')dy\right](t). \tag{3.36}$$

Except in singular cases, which we choose to disregard, all of the operators of the form V_G are Volterra-type integral operators of the second kind. Thus, they all have inverses, V_G^{-1}, which are operators of the same kind. Hence the system (3.34), (3.35) is readily solved, and we have

$$\begin{aligned} c_1(t) &= \frac{1}{a^2} V^{-1a}_{G_1+2M} V^a_{G_1}\left[\frac{1}{a^2} V^{-1a}_{G_1+2M} V^a_{G_1} - \frac{1}{b^2} V^{-1b}_{G_1+2M} V^b_{G_1}\right]^{-1} V^{-1b}_{G_1+2M}[\Gamma(t')](t) \\ c_2(t) &= \left[\frac{1}{a^2} V^{-1a}_{G_1+2M} V^a_{G_1} - \frac{1}{b^2} V^{-1b}_{G_1+2M} V^b_{G_1}\right]^{-1} V^{-1b}_{G_1+2M}[\Gamma(t')](t). \end{aligned} \tag{3.37}$$

An explicit representation for E_2^{-1} is obtained by combining (3.32), (3.37), and (3.36). It is clear from the form of these equations that there exists a nondecreasing function $N_2(t)$ such that

$$\|E_2^{-1}\theta\|_1(t) \leq N_2(t)\|\theta\|(t). \tag{3.38}$$

Consequently, we can write for the solution of the traction problem

$$u(\rho, t) = w(\rho, t) + \sum_{n=0}^{\infty} (-1)^n E_2^{-1}(BE_2^{-1})^n (H - Ew - Bw)(\rho, t). \tag{3.39}$$

(iii) *The Mixed Problem*

This problem is somewhat simpler than (ii). Thus, in the remainder of this paper we return to the earlier notation (2.9), (2.10).

We seek a function $u(\rho, t)$ which satisfies

$$Eu + Bu = H \tag{3.3}$$

$$\Phi(a, t) = V_{G_1}\left(\frac{\partial u}{\partial \rho}\right)(a, t) + V_M\left(\frac{\partial u}{\partial \rho} + \frac{u}{\rho}\right)(a, t) \tag{3.40}$$

$$0 = u(b, t). \tag{3.41}$$

In order to deal with the inhomogeneity in (3.40), we again assume a solution of the form

$$u = v + w \tag{3.42}$$

where

$$Ev + Bv = H - Ew - Bw \tag{3.43}$$

$$0 = V_{G_1}\left(\frac{\partial v}{\partial \rho}\right)(a, t) + V_M\left(\frac{\partial v}{\partial \rho} + \frac{v}{\rho}\right)(a, t) \tag{3.44}$$

$$0 = v(b, t) \tag{3.45}$$

$$\Phi(a, t) = V_{G_1}\left(\frac{\partial w}{\partial \rho}\right)(a, t) + V_M\left(\frac{\partial w}{\partial \rho} + \frac{w}{\rho}\right)(a, t)$$

$$0 = w(b, t).$$

An appropriate w is evidently given by

$$w(\rho, t) = (\rho - b)\varphi(t),$$

where φ satisfies

$$\Phi(a, t) = \left[\gamma + \mu\left(1 + \left(\frac{a-b}{a}\right)\right)\right]\varphi(t) + \int_0^t \left[g(a, t, t') + m(a, t, t')\left(1 + \left(\frac{a-b}{a}\right)\right)\right]\varphi(t')dt'.$$

Let E_3^{-1} be that operator which assigns to a given function $\theta(\rho, t)$ that function $h(\rho, t)$ which satisfies (3.17), (3.44), and (3.45).

We write the general solution of (3.17) in the form

$$h(\rho, t) = \frac{1}{\rho}\int_\rho^b \theta(y, t)\left(\frac{y^2-\rho^2}{2}\right)dy + c_1(t)\rho + \frac{c_2(t)}{\rho}.$$

Condition (3.45) implies the further restriction

$$h(\rho, t) = \frac{1}{\rho}\int_\rho^b \theta(y, t)\left(\frac{y^2-\rho^2}{2}\right)dy + c(t)\left(\rho - \frac{b^2}{\rho}\right). \tag{3.46}$$

Due to (3.44), $c(t)$ must satisfy the integral equation

$$\left[\left(1 + \frac{b^2}{a^2}\right)V_{G_1}^a + 2V_M^a\right][c(t')](t) = \frac{1}{a^2}V_{G_1}^a\left[\int_a^b \theta(y, t')\left(\frac{y^2-a^2}{2}\right)dy\right](t)$$

$$+ V_{G_1+M}^a\left[\int_a^b \theta(y, t')dy\right](t) \tag{3.47}$$

The existence of $N_3(t)$ is clear from (3.46), (3.47), so that the solution to the mixed problem takes the form

$$u(\rho, t) = w(\rho, t) + \sum_{n=0}^{\infty}(-1)^n E_3^{-1}(BE_3^{-1})^n(H - Ew - Bw)(\rho, t).$$

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