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# A Note on the T-Transformation of Lubkin*

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This paper is concerned with a sequence-to-sequence transformation studied extensively by Samuel Lubkin [J. Res. NBS **48**, 228–254 (1952)]. Lubkin has studied the rate of convergence of the transformed sequence, $\{T_n\}$, versus the original sequence, $\{S_n\}$. In this respect, the authors have shown that a more accurate evaluation of the transformation is achieved by the comparison of $\{T_n\}$ with $\{S_{n+1}\}$ instead of $\{S_n\}$. The main theorems proved are rate-of-convergence comparisons between $\{T_n\}$ and $\{S_{n+1}\}$ where $\{S_n\}$ is the sequence of partial sums of a convergent series whose terms are of constant sign or else are alternating.

Key Words: Convergence acceleration techniques; epsilon-transformation; nonlinear series transformation; numerical methods; series summability methods.

The sequence-to-sequence $T$-transformation (defined later by (1.1)) introduced by Aitken [1][1] has been studied extensively by Lubkin [4] and more recently by Shanks [6], Wynn [7], Marx [5], Gray and Atchison [3] and others.

If a sequence, $S_n$, converges then Lubkin has shown that, under certain conditions, its image sequence, $T(S_n)=T_n$, converges to this same limit. Moreover, he has shown that, under certain conditions, the sequence $T_n$ converges more rapidly than the sequence $S_n$. Consequently, this transform may be used to accelerate the convergence of some infinite series.

In applying this transform, however, the authors discovered a seeming paradox. The following theorem is stated and proved as Theorem 4 on page 231 of [3].

THEOREM A: *If* $S_n=\sum_{k=0}^{n} a_k$ *converges and if* $R_n=\frac{a_{n+1}}{a_n}$ *is of constant sign and converges to 0 then the sequence* $T(S_n)=T_n$ *converges more rapidly than the sequence* $S_n$.

The authors of this paper assumed that if a series satisfied the hypothesis of this theorem then the transform could be applied with good results. However, this proved to be an erroneous assumption and thus the seeming paradox.

It is the purpose of this paper to clarify the above mentioned irregularity and introduce some theorems that give a more valid evaluation of the transform. It is the contention of the authors that the sequence $T_n$ should be compared with $S_{n+1}$ instead of with $S_n$. Although the sequences $S_n$ and $S_{n+1}$ are "essentially" the same sequences, they do not necessarily converge at the same rate. It is this point that has evidently been overlooked.

Looking back at Theorem A, it should be noted that series satisfying the hypotheses are either series having terms of constant sign or else alternating series. In either case, since $R_n \to 0$, the sequence $|a_k|$ must be eventually decreasing. Consequently this paper will be concerned, in the main, with these two types of series.

The usual definitions for rates of convergence are used and are quoted for completeness.

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[1] Figures in brackets indicate the literature references at the end of this paper.

If $a_n \to a$ and $b_n \to b$ then;

**Definition I:** $a_n$ *converges more rapidly than* $b_n$ *if and only if* $\lim_{n\to\infty} \frac{a - a_n}{b - b_n} = 0$. *This may also be stated as* $b_n$ *converges less rapidly than* $a_n$.

**Definition II:** $a_n$ *converges with the same order of rapidity as* $b_n$ *if and only if there exist constants* A *and* B *such that* $0 < A < \left|\frac{a - a_n}{b - b_n}\right| < B$ *for* n *sufficiently large. Note that if* $\frac{a - a_n}{b - b_n} \to A \neq 0$ *then* $a_n$ *converges with the same order of rapidity as* $b_n$.

The next three theorems are of utmost importance to this paper and also provide a "L'Hospital's Rule" for certain types of sequences and series.

**Theorem 1:** *If* $a_n \to 0$, $b_n \to 0$ *and* $b_n$ *is monotone and if* $\lim_{n\to\infty} \frac{a_n - a_{n-1}}{b_n - b_{n-1}}$ *exists, finite or infinite, then* $\lim_{n\to\infty} \frac{a_n}{b_n}$ *exists and* $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{a_n - a_{n-1}}{b_n - b_{n-1}}$.

This is proved by Bromwich [2] for the case $b_n$ decreasing. The proof for $b_n$ increasing is analagous.

Throughout this paper $S_n$ will denote the partial sums of the convergent series $\sum_{n=0}^{\infty} a_n$, $R_n = \frac{a_{n+1}}{a_n}$ and $R = \lim_{n\to\infty} R_n$ if this limit exists.

**Theorem 2:** *If* $a_n$ *is of constant sign,* $S_n \to S$, *and* $R_n \to R$ *then* $\frac{S - S_{n+1}}{S - S_n} \to R$.

$$\text{i.e., } \lim_{n\to\infty} \frac{\sum_{k=n+2}^{\infty} a_k}{\sum_{k=n+1}^{\infty} a_k} = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}.$$

**Proof:** Let $A_n = S - S_{n+1}$ and $B_n = S - S_n$ and apply Theorem 1 to these sequences. $B_n$ is monotone since $a_n$ is of constant sign.

**Note:** If $R = 0$ then $S_{n+1}$ converges more rapidly than $S_n$.

**Theorem 3:** *If* $a_n = (-1)^n c_n$ *and* $c_n > c_{n+1} > 0$ *and if* $R_n \to R \neq -1$ *then* $\frac{S - S_{n+1}}{S - S_n} \to R$. *Moreover, if* $R = -1$ *and* $\frac{1 + R_{n+1}}{1 + R_n} \to 1$ *then the above result holds.*

**Proof:** In this case, the sequences $S_{2n}$ and $S_{2n+1}$ are monotone since $c_n > c_{n+1}$ and thus $S - S_{2n}$ and $S - S_{2n+1}$ are also monotone. Consider first the sequences $A_n = S - S_{2n+1}$ and $B_n = S - S_{2n}$. We have then that $\frac{A_n - A_{n-1}}{B_n - B_{n-1}} = R_{2n-1}\left[\frac{1 + R_{2n}}{1 + R_{2n-1}}\right]$ and this converges to $R$. Hence $\frac{S - S_{2n+1}}{S - S_{2n}} \to R$. Similarly, the odd subsequence of $\frac{S - S_{n+1}}{S - S_n}$ converges to $R$ and hence the result.

We are now ready to consider the $T$-transform which is defined by:

$$T_n = \frac{S_n^2 - S_{n-1}S_{n+1}}{2S_n - S_{n-1} - S_{n+1}} \tag{1.1}$$

where $S_n$ is any sequence for which the denominator is not zero.

In this paper $S_n$ will be a sequence of partial sums and thus $T_n$ may be written,

$$T_n = S_n + \frac{a_{n+1}}{1 - R_n} = \frac{S_{n+1} - R_n S_n}{1 - R_n}.$$

For completeness, we will state two theorems proved by Lubkin as Theorems 1 and 2 on pages 230–1 of [4].

THEOREM 4: *If* $S_n$ *converges to* S *and if* $T_n$ *converges then* $T_n$ *converges to* S.

THEOREM 5: *If* $S_n$ *converges and if there exists* k *such that* $|1-R_n|>k>0$ *for* n *sufficiently large then* $T_n$ *converges. As a special case, we note that if* R *exists and* $R\neq 1$ *then* $T_n$ *converges.*

Referring once again to Theorem A and to Theorems 2 and 3, we see that for series satisfying the hypotheses of the first theorem, $S_{n+1}$ converges more rapidly than $S_n$. If perchance $S_{n+1}$ and $T_n$ converge with the same order of rapidity, or if $T_n$ converges less rapidly than $S_{n+1}$, one would feel that the transform does not yield very good results—since $S_{n+1}$ must be computed in order to compute $T_n$. Thus the authors feel that $T_n$ should be compared with $S_{n+1}$ from the outset, rather than with $S_n$. The remainder of the paper makes this comparison.

LEMMA 1: *If* $\frac{R_n}{R_{n+1}}\to A$, $|A|\neq 1$ *(A finite or infinite) then* $R_n\to 0$ *or else* $|R_n|\to\infty$. *Moreover if* $|A|<1$ *then* $|R_n|\to\infty$ *and if* $|A|>1$ *then* $R_n\to 0$.

LEMMA 2: *If* $S_n\to S$ *and* $\frac{R_n}{R_{n+1}}\to A$, $|A|\neq 1$ *then* $R_n\to 0$ *and from Lemma 1 it follows that* $|A|>1$.

THEOREM 6: *If* $R_n\to R\neq 1, 0$ *and* $\frac{S-S_{n+1}}{S-S_n}\to R$ *then* $T_n$ *converges more rapidly than* $S_{n+1}$.

PROOF: By Theorem 5, $T_n$ converges to $S$. Also

$$\frac{S-T_n}{S-S_{n+1}}=\frac{1}{1-R_n}\left[1-\frac{R_n(S-S_n)}{S-S_{n+1}}\right]$$

and this converges to 0 when $R\neq 0, 1$.

THEOREM 7: *If* $a_n$ *is of constant sign,* $S_n\to S$ *and*

$$\frac{R_n}{R_{n+1}}\to A$$

*then*

(a) *If* $A=1$ *and there exists* k *such that* $|1-R_n|>k>0$ *for* n *sufficiently large then* $T_n$ *converges more rapidly than* $S_{n+1}$ *and*

(b) *If* $1<A<\infty$ *then* $T_n$ *converges with the same order of rapidity as* $S_{n+1}$ *and*

(c) *If* $A=\infty$ *then* $T_n$ *converges less rapidly than* $S_{n+1}$.

PROOF: Let $A_n=S-T_n$ and $B_n=S-S_{n+1}$. Then $A_n\to 0$ and $B_n$ converges monotonically to 0 since $a_n$ is of constant sign. Also

$$\frac{A_n-A_{n-1}}{B_n-B_{n-1}}=\frac{\left[1-\frac{R_{n-1}}{R_n}\right]}{(1-R_n)(1-R_{n-1})}. \tag{1.2}$$

In the case of (a), the absolute value of (1.2) is bounded by $\frac{\left|1-\frac{R_{n-1}}{R_n}\right|}{k^2}$ which converges to 0 and thus (1.2) converges to $0=1-A$. In cases (b) and (c), it is seen by Lemma 1 that $R_n\to 0$ and (1.2) converges to $1-A$. Thus by Theorem 1, $\frac{A_n}{B_n}=\frac{S-T_n}{S-S_{n+1}}\to 1-A$ in each case and the result in each case then follows.

THEOREM 8: *If* $a_n=(-1)^n c_n$ *where* $c_n>c_{n+1}>0$[2] *and if*

$$S_n\to S \text{ and } \frac{R_n}{R_{n+1}}\to A$$

*then*

(a) *If* $A=1$ *and there exists* k *such that* $|1+R_n|>k>0$ *for* n *sufficiently large then* $T_n$ *converges more rapidly than* $S_{n+1}$ *and*

[2] This can be shown to hold without the hypothesis $c_n>c_{n+1}$.

(b) *If* $1 < A < \infty$ *then* $T_n$ *converges with the same order of rapidity as* $S_{n+1}$ *and*
(c) *If* $A = \infty$ *then* $T_n$ *converges less rapidly than* $S_{n+1}$.

PROOF: This will be proved by showing that the even and odd subsequences of $\frac{S-T_n}{S-S_{n+1}}$ both converge to $1-A$ in each of the above cases. Let $A_n = S - T_{2n}$ and $B_n = S - S_{2n+1}$. Then $A_n \to 0$ and $B_n$ decreases to 0 since $c_n > c_{n+1}$. Also,

$$\frac{A_n - A_{n-1}}{B_n - B_{n-1}} = \frac{a_{2n-1}\dfrac{R_{2n-1}-R_{2n-2}}{(1-R_{2n-1})(1-R_{2n-2})} + a_{2n}\dfrac{R_{2n}-R_{2n-1}}{(1-R_{2n})(1-R_{2n-1})}}{a_{2n}+a_{2n+1}}$$

$$= \frac{1-\dfrac{R_{2n-2}}{R_{2n-1}}}{(1-R_{2n-1})(1-R_{2n-2})(1+R_{2n})} + \frac{R_{2n}-R_{2n-1}}{(1-R_{2n})(1-R_{2n-1})(1+R_{2n})}. \tag{1.3}$$

Since $R_n < 0$ for all $n$ it follows that $|1-R_n| > 1$. Moreover, $|R_n| < 1$. In case (a), the absolute value of (1.3) is bounded by

$$\frac{\left|1-\dfrac{R_{2n-2}}{R_{2n-1}}\right|}{k} + \frac{\left|1-\dfrac{R_{2n-1}}{R_{2n}}\right|}{k}$$

which converges to 0 and thus (1.3) converges to $0 = 1 - A$. In cases (b) and (c), $R_n \to 0$ and (1.3) converges to $1-A$. A similar argument applies to the odd subsequence and the result in each case then follows.

The following example will illustrate the considerations of this paper.

EXAMPLE I: *Let*

$$a_n = \frac{1}{n!}, \qquad b_n = \frac{1}{n!2^{n^2}}$$

*and*

$$C_n = \frac{1}{n!2^{n^3}}.$$

*Then*

$$R_n = \frac{a_{n+1}}{a_n} = \frac{1}{n+1}, \qquad V_n = \frac{b_{n+1}}{b_n} = \frac{1}{(n+1)2^{2n+1}}$$

*and*

$$W_n = \frac{C_{n+1}}{C_n} = \frac{1}{(n+1)2^{3n^2+3n+1}}.$$

*Also*

$$\frac{R_n}{R_{n+1}} = \frac{n+2}{n+1} \to 1, \qquad \frac{V_n}{V_{n+1}} = \frac{(n+2)}{(n+1)}4 \to 4$$

*and*

$$\frac{W_n}{W_{n+1}} = \frac{(n+2)2^{6n+6}}{(n+1)} \to \infty.$$

All three of the series associated with these sequences satisfy the hypotheses of Theorem A and thus it was felt that the transform should have given good results. This is however not the case. Theorem 7 can be invoked here to give a more accurate evaluation of the transform. It is seen then that the first series can be transformed with suitable results while the last two cannot.

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## References

[1] Aitken, A. C., "On Bernoulli's numerical solution of algebraic equations," Proc. Roy. Soc. (Edinburgh), **46**, (1926.

[2] Bromwich, T. J., An Introduction to the Theory of Infinite Series, p. 413 (Macmillan & Co. Ltd. New York, Second Edition, 1926).

[3] Gray, H. L., and Atchison, T. A., "Non-linear transformations related to the evaluation of improper integrals," Siam J. of Numerical Analysis, **4**, No. 3 (Sept. 1967).

[4] Lubkin, Samuel, A method of summing infinite series, J. Res. NBS **48**, 228–254 (RP2310).

[5] Marx, I., Remark concerning a non-linear sequence-to-sequence transform, "J. Math and Phys," **42**, 1963).

[6] Shanks, D., Non-linear transformations of divergent and slowly convergent sequences, Dissertation, University of Maryland (1954).

[7] Wynn, P., On the Convergence and Stability of the Epsilon Algorithm, J. Siam Numerical Analysis, **3**, No. 1 (1966).

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