On the Signs of the $\nu$-Derivatives of the Modified Bessel Functions $I_\nu(x)$ and $K_\nu(x)^*$

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It is proved that $\partial I_\nu(x)/\partial \nu$ is negative and $\partial K_\nu(x)/\partial \nu$ is positive when $x > 0$ and $\nu > 0$.

Key Words: Bessel functions; $\nu$-derivatives.

1. Introduction

The properties of the modified Bessel functions of the third kind have been investigated for many years. In a private communication, F. W. J. Olver pointed out that $\partial K_\nu(x)/\partial \nu > 0$ when $x > 0$ and $\nu > 0$. This follows immediately from the integral representation for $K_\nu(x)$

$$K_\nu(x) = \int_0^\infty e^{-x \cosh t} \cosh (\nu t) dt.$$  

(1)

It is the purpose of this paper to show that a similar fundamental property holds for $I_\nu(x)$, namely, $\partial I_\nu(x)/\partial \nu < 0$ when $x > 0$ and $\nu > 0$.

The author wishes to acknowledge his discussions with F. Oberhettinger concerning this result.

2. Proof

Consider the following known integral $^1$

$$I_\nu(x)K_\nu(x) = \frac{2}{\pi^2} \int_0^\infty \frac{\lambda \sinh (\pi \lambda)}{\lambda^2 + \nu^2} K_\nu^2(\lambda) d\lambda \quad \text{Re}\nu > 0.$$  

(2)

Now differentiate the above formula with respect to $\nu$ to obtain an expression for $\partial I_\nu(x)/\partial \nu$,

$$\frac{\partial I_\nu(x)}{\partial \nu} = -\frac{1}{K_\nu(x)} \left[ I_\nu(x) \frac{\partial K_\nu(x)}{\partial \nu} + 4\nu \int_0^\infty \frac{\lambda \sinh (\pi \lambda)}{(\lambda^2 + \nu^2)^2} K_\nu^2(\lambda) d\lambda \right].$$  

(3)

It is easily seen from (1) that $K_\nu(x)$ is real when $\lambda$ is real and $x > 0$; hence for $\nu > 0$, the integral in (3) is nonnegative. Therefore, since $K_\nu(x)$, $I_\nu(x)$, and $\partial K_\nu(x)/\partial \nu$ are positive for $\nu > 0$ and $x > 0$, it follows immediately that

$$\frac{\partial I_\nu(x)}{\partial \nu} < 0 \quad x > 0, \quad \nu > 0.$$  

(4)

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$^1$This integral is given in [1, p. 176, No. 8] for $\nu$ a positive integer, but it has been shown [2] that the expression is valid for $\text{Re}\nu > 0$. Figures in brackets indicate the literature references at the end of this paper.
3. References
