JOURNAL OF RESEARCH of the National Bureau of Standards — B. Mathematical Sciences
Vol. 72B, No. 4, October–December 1968

On the Signs of the ν-Derivatives of the Modified Bessel Functions $I_\nu(x)$ and $K_\nu(x)$ *

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(September 17, 1968)

It is proved that $\partial I_\nu(x)/\partial\nu$ is negative and $\partial K_\nu(x)/\partial\nu$ is positive when $x > 0$ and $\nu > 0$.

Key Words: Bessel functions; ν-derivatives.

1. Introduction

The properties of the modified Bessel functions of the third kind have been investigated for many years. In a private communication, F. W. J. Olver pointed out that $\partial K_\nu(x)/\partial\nu > 0$ when $x > 0$ and $\nu > 0$. This follows immediately from the integral representation for $K_\nu(x)$

$$K_\nu(x) = \int_0^\infty e^{-x \cosh t} \cosh (\nu t) dt. \tag{1}$$

It is the purpose of this paper to show that a similar fundamental property holds for $I_\nu(x)$, namely, $\partial I_\nu(x)/\partial\nu < 0$ when $x > 0$ and $\nu > 0$.

The author wishes to acknowledge his discussions with F. Oberhettinger concerning this result.

2. Proof

Consider the following known integral [1]

$$I_\nu(x)K_\nu(x) = \frac{2}{\pi^2}\int_0^\infty \frac{\lambda \sinh (\pi\lambda)}{\lambda^2+\nu^2} K^2_{i\lambda}(x) d\lambda \quad \text{Re } \nu > 0. \tag{2}$$

Now differentiate the above formula with respect to ν to obtain an expression for $\partial I_\nu(x)/\partial\nu$,

$$\frac{\partial I_\nu(x)}{\partial\nu} = -\frac{1}{K_\nu(x)}\left[I_\nu(x)\frac{\partial K_\nu(x)}{\partial\nu} + \frac{4\nu}{\pi^2}\int_0^\infty \frac{\lambda \sinh (\pi\lambda)}{(\lambda^2+\nu^2)^2} K^2_{i\lambda}(x) d\lambda\right]. \tag{3}$$

It is easily seen from (1) that $K_{i\lambda}(x)$ is real when λ is real and $x > 0$; hence for $\nu > 0$, the integral in (3) is nonnegative. Therefore, since $K_\nu(x)$, $I_\nu(x)$, and $\partial K_\nu(x)/\partial\nu$ are positive for $\nu > 0$ and $x > 0$, it follows immediately that

$$\frac{\partial I_\nu(x)}{\partial\nu} < 0 \qquad x > 0, \qquad \nu > 0. \tag{4}$$

*An invited paper.

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[1] This integral is given in [1, p. 176, No. 8] for ν a positive integer, but is has been shown [2] that the expression is valid for Re $\nu > 0$. Figures in brackets indicate the literature references at the end of this paper.

3. References

[1] Erdélyi, A., Magnus, W., Oberhettinger, F., and Tricomi, F. G., Tables of Integral Transforms, Vol. **2** (McGraw-Hill Book Co., New York, 1954).
[2] Oberhettinger, F., On the Diffraction and Reflection of Waves and Pulses by Wedges and Corners, J. Res. NBS **61,** No. 5, 343–365 (1958) RP2906.

(Paper 72b4–278)