Thickness and Connectivity in Graphs

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A graph G has *thickness* t if and only if t is the smallest number of planar subgraphs of G whose union is G. A *thickness-minimal*, or *t-minimal*, graph is a graph of thickness t with no proper subgraph of thickness t. In this paper we show that, for every t, a cut set in a t-minimal graph can have no fewer than t edges. This theorem suggests that the connectivity of a t-minimal graph is bounded from below by t. We disprove this suggestion for every $t > 2$ by constructing a graph which contains a 2-connected t-minimal graph. Two theorems extend this result and develop additional properties of the construction method. In addition, we show that the connectivity of a graph of thickness t is at most $6t-1$ and that the thickness of K_{22} is 4. Finally, we make four conjectures about graphs of thickness t.

Key Words: Conjectures, connectivity, cut-set, graph theory, mathematics, minimality, thickness of graphs.

1. Definitions and Previous Results

A *graph* is a finite set of points, called *vertices*, together with a finite set of line segments, called *edges*, such that each end point of an edge is a vertex, no edge meets more than two vertices, and edges meet only at end points. If two edges share an end point, they are *adjacent*. The end points of an edge are said to be *joined* by the edge. A *subgraph* of a graph G is a subset of the edges and vertices of G which is again a graph.

A graph is *complete* if and only if every pair of vertices is joined by an edge; a complete graph with n vertices is denoted by K_n. A graph G is *bipartite* if and only if there is a separation of the vertices of G into two disjoint sets A and B such that every edge of G joins a vertex in A with a vertex in B. A bipartite graph is *complete* if and only if every vertex in A is joined to every vertex in B by an edge; such a graph is denoted by $K_{m,n}$, where m is the number of vertices in the set A and n is the number of vertices in the set B.

A graph is *planar* if and only if it can be drawn in the plane in such a way that no two distinct points of the graph occupy the same point of the plane. The *union* of subgraphs $H_1, H_2, \ldots, H_n$ of a graph G is the subgraph of G consisting of all edges and all vertices from the H_i. The *thickness* $\theta(G)$ of a graph G is defined [5] as the minimum number of planar subgraphs whose union is G. A *thickness-minimal*, or *t-minimal*, graph is a graph of thickness t which contains no proper subgraph of thickness t.

A (simple) *path* in G is a vertex, a single edge, two adjacent edges, or a subset $e_1, e_2, \ldots, e_n$, $n > 2$, of the edges of G such that e_{i+1} joins an end point of e_i with an end point of e_{i+2}, $i = 1, 2, \ldots, n-2$, and no vertex is the end point of more than two edges of the path. The path is said to *join* any two vertices met by edges of the path. A *circuit* is a path such that e_1 and e_n are adjacent. A graph or subgraph G is *connected* if and only if for any two vertices of G there exists a path in G joining the two vertices. The *girth* of a graph G is the smallest integer n such that there exists a circuit in G having n edges.

A *component* of a graph G is a maximal connected subgraph of G. The *connectivity* of a connected graph G is the smallest number of vertices of G whose removal (together with all edges meeting at least one of the vertices) leaves a subgraph of G which has no edges or which is not connected. If G is not connected, it has connectivity 0. An *articulation point* of a graph G is a vertex whose removal from G (together with all incident edges) increases the number of components in the remaining graph. A *cut set* of a graph G is a subset S of the edges of G whose removal disconnects G and such that no proper subset of S has the same property.

W. T. Tutte [5][1] proved several theorems which we have found very useful. We state those theorems here with their original numbering:

(3.1) Let G be a graph of thickness t. Let G' be formed from G by deleting either a single edge or a single vertex with all its incident edges. Then the thickness of G' is either t or $t-1$. (In particular, if G is t-minimal, then $\theta(G')=t-1$.)

(3.2) Let k be a nonnegative integer and let G be a graph of thickness $t \geq k$. Then there exists a k-minimal subgraph of G.

(3.3) Every t-minimal graph is connected and nonseparable. (A graph is nonseparable if and only if it is connected and contains no articulation points.)

(5.3) For each integer $t \geq 2$ there exist infinitely many nonisomorphic t-minimal graphs of upper valency $2t-1$, and of girth greater than any specified integer N.

2. Connectivity and Thickness

The *valency* of a vertex of a graph G is the number of edges incident on the vertex. Let v be the least integer occurring as the valency of a vertex of G. W. T. Tutte has shown [5] that if G is a t-minimal graph, $t > 0$, then $v \geq t$. The following generalizes Tutte's result:

THEOREM 1: *Let* G *be* t-*minimal,* t > 0, *and let* n *be the number of edges in a cut set of* G. *Then* n $\geq$ t.

PROOF: Suppose G has a cut set S with $k < t$ edges in it. Let the vertices of G be partitioned into the sets A and B by S. Obtain G' by removing one edge $e = (a, b) \epsilon S$ from G. Since G is t-minimal, $\theta(G')=t-1$, and we can find $t-1$ planar subgraphs $H_1, H_2, \ldots, H_{t-1}$ of G' whose union is G'. We may assume that no edge of G' appears in more than one of the H_i. There are only $k-1 < t-1$ edges of S in G', so one of the H_i, say H_1, must not contain any edge of S. Because S is a cut set of G, H_1 (which we may assume contains all the vertices of G) must be disconnected with vertex a in one component and vertex b in some other component. Using stereographic projection, we may draw the component containing vertex a in the plane so that a is on the boundary of the infinite region of the component, and similarly with the component containing vertex b. Then a new planar subgraph H_1' can be formed from H_1 by joining vertices a and b by edge e. But the union of H_1' together with $H_2, \ldots, H_{t-1}$ is G, thus giving a partition of G into $t-1$ planar subgraphs, contrary to hypothesis.   Q.E.D.

Given the result of Theorem 1, we might conjecture that the connectivity of a t-minimal graph is bounded from below by t. However, for all $t > 2$, the following propositions show this conjecture to be false.

Let us define an operation on two disjoint graphs G_1 and G_2 as follows: choose an edge $e_i = (a_i, b_i)$ from graph G_i, delete edge e_i from G_i, and identify a_1 with a_2 and b_1 with b_2, obtaining a graph $J(G_1, G_2)$ called the *join* of G_1 and G_2. Note that if G_1 and G_2 each have connectivity at least 2, then $J(G_1, G_2)$ has connectivity exactly 2. In general there are many different joins of two graphs, depending on the choice of edges e_1 and e_2. However, in this paper we are concerned only with those properties of the join which are independent of the choice of the two edges. Let us denote the vertex resulting from identifying a_1 and a_2 by a, and that from b_1 and b_2 by b. The subgraph of $J(G_1, G_2)$ corresponding to $G_1 - \{e_1\}$ we denote by G_1' and that corresponding to $G_2 - \{e_2\}$ by G_2'. Since the thickness of a graph is reduced at most 1 by the removal of an edge, it is clear that $\theta(J(G_1, G_2)) \geq \max (\theta(G_1), \theta(G_2)) - 1$.

[1] Figures in brackets indicate the literature references at the end of this paper.

LEMMA: *If* $\theta(G_1) = \theta(G_2) = t$, *then any join of* G_1 *and* G_2 *has thickness* t *or* t $+1$.

PROOF: Remove vertices a and b and all incident edges from $J(G_1, G_2)$. In the resulting graph every component is a proper subgraph of G_1' or G_2' and can therefore be partitioned into t or fewer planar subgraphs. Since $K_{2,m}$ is planar for every integer m, we can place a and b with their incident edges in one additional plane. Thus, $\theta(J(G_1, G_2)) \leqslant t+1$.

But suppose $J(G_1, G_2)$ can be partitioned into $t-1$ planar subgraphs $H_1, \ldots, H_{t-1}$. The subgraph G_i' of $J(G_1, G_2)$ together with the additional edge (a, b) is isomorphic to G_i for each i. Since G_i has thickness t, $H_1 \cap G_1'$ together with edge (a, b) cannot be planar. Hence in any embedding of $H_1 \cap G_1'$ in the plane there is a circuit C composed entirely of edges and vertices of $H_1 \cap G_1'$ such that a and b are in different components of the plane if C is removed from it.

Supposing that a and b are not joined by a path in $H_1 \cap G_2'$ they must be in separate components thereof. We can then draw each in the plane in such a way as to place a and b on the boundaries of their components and join them by an edge. Thus we obtain a partition of G_2 into $t-1$ planar subgraphs. Therefore, a path P having more than one edge joins a with b and lies entirely within $H_1 \cap G_2'$. Now P must cross C in H_1. By the Jordan curve theorem, P and C must share a point, and by the definition of planar embedding of graphs, that point must be a vertex of G other than a and b. But a and b are the only vertices shared by G_1' and G_2'. Hence, $\theta(J(G_1, G_2)) \geqslant t$. Q.E.D.

THEOREM 2: *For every* t $\geqslant 2$, *there exists a* t-*minimal graph with connectivity 2.*

PROOF: If G_i is t-minimal, G_i' cannot include a t-minimal subgraph. But since $\theta(J(G_1, G_2)) \geqslant t$, there exists a t-minimal subgraph K contained in $J(G_1, G_2)$ (Theorem 3.2 of [5]). Clearly this t-minimal subgraph includes one or both of a and b, as well as other vertices from each of the G_i'. But Theorem 3.3 of [5] states that K cannot have an articulation point. Therefore, both a and b are contained in K, and K has connectivity 2.   Q.E.D.

The results of Tutte's paper [5] allow us to extend the above theorem much farther. Denote the girth of G_i by $g(G_i)$.

THEOREM 3: *For each integer* t $\geqslant 2$ *and* N $\geqslant 2$, *there exists a* t-*minimal graph of connectivity 2 with girth greater than* N.

PROOF: $J(G_1, G_2)$ cannot have girth less than min $(g(G_1), g(G_2))$, since the only circuits of G_i affected by the join are those including edge (a_i, b_i), and those circuits are increased by at least one edge in the join. Thus, for all graphs G_1 and G_2, $g(J(G_1, G_2)) \geqslant$ min $(g(G_1), g(G_2))$. By Theorem 5.3 of [5], given an integer N, there exists a t-minimal graph G with girth greater than N. Then $g(J(G, G)) \geqslant g(G) > N$. Since any subgraph of a graph of girth m has girth at least m, for any t-minimal subgraph K contained in $J(G, G)$, $g(K) > N$, and, by Theorem 2, the connectivity of K is 2.   Q.E.D.

COROLLARY: *For every* t $\geqslant 2$ *there exists an infinite class of nonisomorphic* t-*minimal graphs, each having connectivity 2.*

THEOREM 4: *Let* G_1 *be* t-*minimal and* $\theta(G_2) = $ t. *Then* $\theta(J(G_1, G_2)) = $ t.

PROOF: Partition G_2 into t planar subgraphs H_i such that a_2 and b_2 are on the boundary of the same region in one of the subgraphs, say H_1 (this is possible since G_2 has thickness t with edge (a_2, b_2) present). Remove vertices a_1 and b_1 and all incident edges from G_1. Since G_1 is t-minimal, the resulting graph G_1'' has thickness $t-1$ or less. Partition G_1'' into $t-1$ planar subgraphs and assign one of these subgraphs to each of the $t-1$ planar subgraphs of G_2 other than H_1. Now it is trivial to embed $K_{2,n}$ in the plane so that the two vertices in the first subset are on the boundary of the infinite region. Hence we may identify a_1 with a_2 and b_1 with b_2 in subgraph H_1 of G_2, placing all edges joined to either a_1 or b_1 in that subgraph. The result is an embedding of $J(G_1, G_2)$ (together with edge (a, b)!) in t planes, and by Theorem 2, $\theta(J(G_1, G_2)) = t$.   Q.E.D.

Having answered the question of the lower bound for the connectivity of a t-minimal graph, it is natural for us to ask for an upper bound for the connectivity of a graph of thickness t. Such an upper bound can be easily derived by the use of Euler's formula for planar graphs as follows. Let $E(G)$ be the set of edges of a graph G and $V(G)$ the set of vertices of G. Denote the number of elements of a set S by $|S|$.

THEOREM 5: *Let* $\theta(G) = $ t > 0. *Then there is a vertex of valency less than or equal to* 6t -1 *in* G.

241

PROOF: Suppose the smallest valency in G is m. Then

$$|E(G)| \geq \tfrac{1}{2} m |V(G)|. \tag{1}$$

Since $\theta(G) = t$, there exist t planar subgraphs $H_1, \ldots, H_t$ of G whose union is G. By Euler's formula in each planar subgraph,

$$|E(H_i)| \leq 3|V(H_i)| - 6 = 3|V(G)| - 6,$$

since we may assume that every H_i contains all of the vertices of G. Thus,

$$|E(G)| \leq 3t|V(G)| - 6t. \tag{2}$$

Combining (1) and (2),

$$\tfrac{1}{2} m |V(G)| \leq 3t |V(G)| - 6t.$$

Rearranging terms,

$$|V(G)|(\tfrac{1}{2} m - 3t) \leq -6t.$$

Since $|V(G)|$ and t are both positive for $t > 0$,

$$\tfrac{1}{2}m - 3t < 0,$$

or

$$m \leq 6t - 1. \qquad \text{Q.E.D.}$$

COROLLARY: *The connectivity of a graph of thickness* t *is less than or equal to* 6t − 1.

3. Conjectures

1. There are no G_1 and G_2 with $\theta(G_1) = \theta(G_2) = t$ such that $\theta(J(G_1, G_2)) = t + 1$. This conjecture is certainly true for $t = 1$. For $t > 1$, we have been unable to find a counterexample.

2. We have shown in a previous paper [3] that $K_{4t-5, \, 4t-5}$ is t-minimal for all $t \geq 2$. By taking $J(K_{4t-5, \, 4t-5}, \, K_{4t-5, \, 4t-5})$ and finding a t-minimal subgraph, we obtain a t-minimal bipartite graph which is not a complete bipartite graph. It would be interesting for each t to characterize the class of all t-minimal bipartite graphs.

The 2-minimal bipartite graphs are easily characterized. Kuratowski [4] proved that every 2-minimal graph is homeomorphic to K_5 or $K_{3,3}$. If G is bipartite, let the vertices of G be partitioned into sets A and B such that all edges of G join a vertex in A with one in B. Clearly any path joining a vertex in A with a vertex in B contains an odd number of edges (= even number of vertices), while a path joining a vertex in one of the sets A or B with another vertex in the same set contains an even number of edges (= odd number of vertices). To generate all 2-minimal bipartite graphs, start with one of K_5 or $K_{3,3}$ and write down a partition of its vertices into two sets. With respect to the given partition, an edge joining two vertices in the same set may be replaced by a path consisting of valence 2 vertices and an even number of edges without destroying 2-minimality. Likewise edges joining vertices in different sets may be replaced by paths consisting of valence 2 vertices and an odd number of edges, or by a single edge. The set of all graphs generated by all such replacements in all such partitionings of both K_5 and $K_{3,3}$ constitutes the complete set of 2-minimal bipartite graphs.

3. Let $N(t)$ be the smallest integer such that there exists a t-minimal graph having $N(t)$ vertices. We know that $N(2)=5$ (reference [4]) and $N(3)=9$ (reference [6]). Beineke and Harary [1] have shown that $6t-8 \leq N(t) \leq 6t-7$ for all $t \geq 4$. Since $\theta(K_{22})=4$ (see appendix A), $N(5)=6(5)-7=23$. Beineke has shown [8] that $\theta(K_{28})=5$, from which it follows that $N(6)=29$. In accordance with Harary's conjecture [7] that $\theta(K_{16})=4$, we conjecture that $N(4)=16$ and $N(t)=6t-7$ for all $t \geq 5$.

4. For all $n \geq N(t)$, there exists a t-minimal graph having exactly n vertices.

This conjecture is certainly true for $t=2$. For $t>2$, an immediate consequence of Theorem 5.3 of [5] is that for every $n \geq N(t)$, there exists a t-minimal graph having at least n vertices. Further evidence about the conjecture is lacking, and it does not appear to be easily proven if true. Unfortunately, because of the difficulty in establishing the thickness and t-minimality of most graphs, the conjecture is not likely to be easily disproven either.

4. Appendix A

In their paper on the thickness of the complete graph [1], Beineke and Harary were unable to give the thickness of the graphs K_n for $n \geq 16$ with $n \equiv 4 \pmod 6$, except for the case of K_{28}
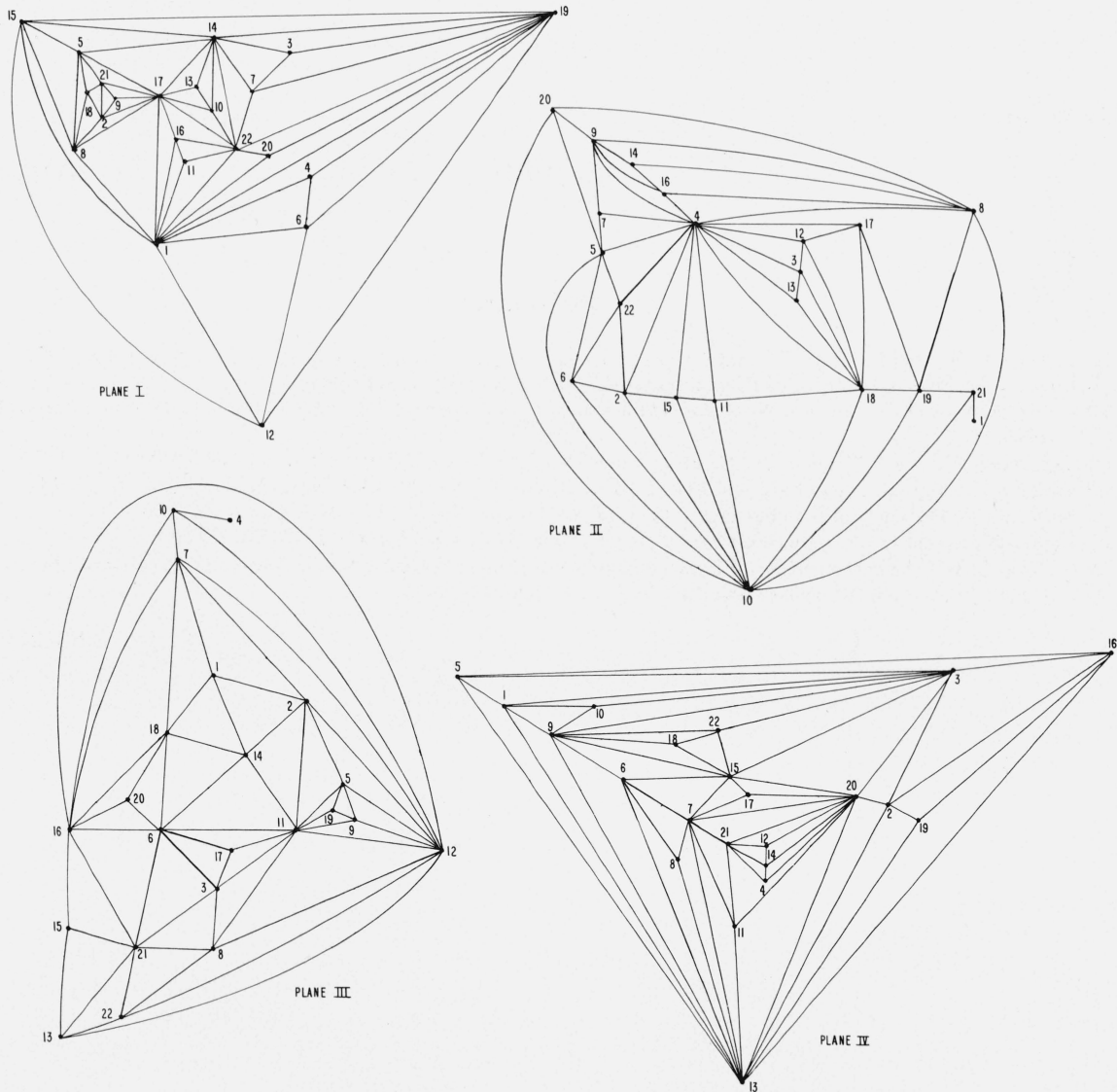

FIGURE 1. K_{22} *in four planes.*

243

whose thickness they found by trial and error to be 5. To provide additional data for those who are attempting to determine the thickness of this class of complete graphs, we include the following:

THEOREM: $\theta(K_{22}) = 4$.

PROOF: Figure 1 shows four planar graphs whose union is K_{22}. By Euler's formula,

$$|E(K_{22})| \leq 3t|V(K_{22})| - 6t = 60t.$$

But

$$|E(K_{22})| = 231.$$

Thus,

$$60t \geq 231,$$

or

$$t \geq 4.$$

Therefore,

$$\theta(K_{22}) = 4. \qquad \qquad \text{Q.E.D.}$$

5. References

[1] Beineke, L. W., and F. Harary, The thickness of the complete graph, Canadian J. Math. **17,** 850–859 (1965).

[2] Dirac, G. A., Short proof of Menger's graph theorem, Mathematika **13,** 42–44 (1966).

[3] Hobbs, A. M., and Grossman, J. W., A class of thickness-minimal graphs, J. Res. NBS **72B** (Math. Sci.) No. 2, 145–153 (1968).

[4] Kuratowski, C., Sur le Problème des Courbes Gauches en Topologie, Fund. Math. **15,** 271–283 (1930).

[5] Tutte, W. T., The thickness of a graph, Nederl. Akad. Wetensch. Proc. Ser. A66 Indag. Math. **25,** 567–577 (1963).

[6] Tutte, W. T., The non-biplanar character of the complete 9-graph, Canad. Math. Bull. **6,** 319–330 (1963).

[7] Harary, F., Recent results in topological graph theory, Acta Math. Acad. Sci. Hung. **15,** 405–412 (1964).

[8] Beineke, L. W., The decomposition of complete graphs into planar subgraphs, Graph Theory and Theoretical Physics, edited by F. Harary, Academic Press, New York, N.Y., 139–153 (1967).

(Paper 72B3–273)