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The Probability of an Equilibrium Point[1]

K. Goldberg, A. J. Goldman, and M. Newman

Institute for Basic Standards, National Bureau of Standards, Washington, D.C. 20234

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A formula is derived for the probability that a "random" m-by-n two-person noncooperative game has an equilibrium-point solution in pure strategies. The limit of this probability as $m, n \to \infty$ is shown to be 1-1/e. The probability is tabulated for $m, n \leqslant 10$.

Key Words: Equilibrium, probability, theory of games.

1. Introduction

The concept of a *mixed strategy*, i.e., a probabilistic mixture of alternative courses of action, is of great theoretical importance in the mathematical analysis of conflict situations ("games"). The reason is related to the way in which "solutions" of a game are defined; the typical situation is that the theory guarantees that a game has a solution in terms of mixed strategies, but not necessarily in unmixed or "pure" strategies.

In many prospective applications, however, the introduction of mixed strategies appears dubious at best, e.g., because adoption of such a strategy as *modus operandi* is simply not acceptable to the decision-maker concerned. This *may* for example reflect a natural identification of "probabilities" with "limiting relative frequencies of outcomes in indefinitely long sequences of repetitions of identical situations." For a decision-maker constrained to function in an ever-changing environment, the notion of such sequences of perfect repetitions may be intolerable even as a hypothetical construct.

Whatever the reason, this distaste for mixed strategies leads naturally to the question of assessing the likelihood that a game chosen "at random" will in fact possess a solution in pure strategies. Attention will be restricted here to noncooperative games with just *two* players; extension of the results to the case of more than two players would be desirable, but at present is obstructed by combinatorial complications.

The underlying model is formulated in section 2. Section 3 contains the derivations of two alternative formulas, (3.8) and (3.9), for the probability that a random two-player game (with a specified number of pure strategies for each player) *fails* to possess a solution in pure strategies.

It is natural to inquire about the limiting behavior for *large* games, those in which both players have many alternative courses of action. In section 5 it is shown that the probability of a solution in pure strategies for large games is surprisingly far from negligible, in fact converging to

$$1-1/e=0.632^{+}.$$

This is in sharp contrast with the situation when attention is restricted to games in which the players' interests are in direct conflict; for such games, as is shown in section 4, the analogous limiting probability is *zero*.

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These observations suggest that it may be worthwhile to attempt to define a measure C of the degree of "direct conflict" implicit in a given game, and to investigate the probability of a solution in pure strategies for a game chosen "at random" among those with a given value of C. One such measure is the fraction of pairs of game outcomes for which the two players' preferences run in opposite directions. With this measure, section 4's results show that the limiting probability for large games is 0 if $C=1$, and 1 if $C=0$; there is clearly an interesting intermediate range to be filled in.

2. Formulation

A 2-player noncooperative game, in which Players 1 and 2 have m and n pure strategies respectively, can be represented by two real $m \times n$ matrices A_1 and A_2. The entries $A_1(i, j)$ and $A_2(i, j)$ are the respective payoffs to Players 1 and 2 if Player 1 selects his ith pure strategy and Player 2 selects *his* jth.

The solution concept to be employed is the customary Nash *equilibrium point* (abbreviation: EP); the pair (i, j) is an EP of the game if $A_1(i, j)$ is a maximal entry of the jth column of A_1, and $A_2(i, j)$ is a maximal entry of the ith row of A_2. That is, if Player 1 tentatively selects his ith strategy and Player 2 *his* jth, then neither has any incentive to change to another strategy in the absence of a strategy change by the opponent.

Our notion of a "random game" is specified by the following model:

(a) the $2mn$ matrix entries of A_1 and A_2 are independent random variables;

(b) the entries of each column of A_1 have the same continuous cumulative distribution function (possibly differing from column to column);

(c) the entries of each row of A_2 have the same continuous cumulative distribution function (possibly differing from row to row).

For any particular specification of the distributions in (b) and (c), the quantity we wish to evaluate . . . the probability that the game has an EP . . . is well-defined. The formula for this quantity, derived in the next section, shows it to be independent of the choice of distributions.

For most of what will follow, assumptions (a)-(c) are actually somewhat more restrictive than is necessary. If we define a *line* to be either a column of A_1 or a row of A_2, then for the most part it suffices to assume that with probability 1 each of the $n+m$ lines has a unique maximum, whose location is uniformly distributed over the line and is independent of the locations of the maxima in all other lines.

3. Derivation

Consider the following events:

D: the $2mn$ entries of A_1 and A_2 are distinct.
E: the game has an EP.
$E(i, j)$: the game has (i, j) as EP.
$E(S)$: every $(i, j) \in S$ is an EP.

Then the desired probability is

$$p_{mn} = Pr(E) = Pr\{\bigcup_{i,j} E(i, j)\}. \tag{3.1}$$

Let S_k be the family of all sets S of pairs (i, j) such that S has cardinality k. Then the exclusion-inclusion principle gives

$$p_{mn} = \sum_{k=1} (-1)^{k+1} \sum \{Pr\{E(S)\} : S \in S_k\}.$$

Since the continuity assumptions imply $Pr\{D\}=1$, this can be rewritten

$$p_{mn}=\sum_{k=1}(-1)^{k+1}\sum\{Pr\{E(S)\cap D\}:S\epsilon S_k\}. \tag{3.2}$$

Consider any $S\epsilon S_k$, say

$$S=\{(i_1,j_1),(i_2,j_2),\ .\ .\ .,(i_k,j_k)\}.$$

It will be shown to follow, from the definition of an EP, that $E(S)\cap D$ is possible only if

$$(i_1,i_2,\ .\ .\ .,i_k)\text{ are distinct,} \tag{3.3}$$

$$(j_1,j_2,\ .\ .\ .,j_k)\text{ are distinct.} \tag{3.4}$$

Suppose for example that $i_1=i_2=i$. Then $E(S)$ implies $E(i,j_1)\cap E(i,j_2)$, which in turn implies that

$$A_2(i,j_1)=A_2(i,j_2)=\max_j A_2(i,j),$$

contradicting the occurrence of D.

Let T_k be the family of all sets S in S_k which satisfy (3.3) and (3.4), so that (3.2) becomes

$$p_{mn}=\sum_{k=1}(-1)^{k+1}\sum\{Pr\{E(S)\cap D\}:S\epsilon T_k\}. \tag{3.5}$$

For any $S=\{(i_1,j_1),(i_2,j_2),\ .\ .\ .,(i_k,j_k)\}\epsilon T_k$, the equation

$$E(S)=\bigcap_{i=1}^{k}E(i_s,j_s)$$

expresses $E(S)$ as an intersection of events which are *independent*, since they involve disjoint sets of matrix entry positions. Hence

$$Pr\{E(S)\cap D\}=Pr\{E(S)\}=\prod_{s=1}^{k}Pr\{E(i_s,j_s)\}$$

$$=\prod_{s=1}^{k}Pr\{E(i_s,j_s)\cap D\}=(1/mn)^k.$$

If t_k denotes the cardinality of T_k, then it follows from (3.5) that

$$p_{mn}=\sum_{k=1}(-1)^{k+1}t_k(1/mn)^k. \tag{3.6}$$

To evaluate t_k, note that the set $(i_1,i_2,\ .\ .\ .,i_k)$ can be chosen in $\binom{m}{k}$ ways, the set $(j_1,j_2,\ .\ .\ .,j_k)$ can be chosen (independently) in $\binom{n}{k}$ ways, and then the two can be paired off in $k!$ ways. Thus the final result is

$$p_{mn}=\sum_{k=1}(-1)^{k+1}\binom{m}{k}\binom{n}{k}k!(1/mn)^k. \tag{3.7}$$

TABLE 1. *Probability of an EP in pure strategies for two* M × N *payoff matrices* [a]

$N=$	2	3	4	5	6	7	8	9	10
$M=2$	0.875	0.833	0.812	0.800	0.792	0.786	0.781	0.788	0.755
3		.786	.764	.751	.743	.737	.732	.729	.727
4			.742	.729	.721	.715	.711	.708	.705
5				.717	.709	.703	.699	.696	.694
6					.701	.696	.692	.688	.686
7						.690	.686	.683	.681
8							.682	.679	.677
9								.676	.674
10									.672

[a] This computation was programmed by Sally Cosgrove.

Some values of p_{mn} are given in table 1. It is simpler to deal instead with the (complementary) probability

$$q_{mn} = 1 - p_{mn},$$

of *no* EP, which from (3.7) is given by

$$q_{mn} = \sum_{k=0} (-1)^k \binom{m}{k} \binom{n}{k} k!(1/mn)^k. \tag{3.8}$$

For some purposes, the representation (3.8) of q_{mn} as an alternating sum is inconvenient. We therefore derive an alternative formula, whose symmetry in m and n is amusingly nonobvious. This formula could be obtained from (3.8), but it seems more informative to derive it from the underlying model.

Let $A = (a_1, a_2, \ldots, a_m)$ denote a generic m-vector of nonnegative integers summing to n, and let

$$(n; A) = n!/a_1!, a_2!, \ldots, a_m!$$

denote the associated multinomial coefficient. The alternative formula is

$$q_{mn} = m^{-n} \sum_{A} (n; A) \prod_{i=1}^{m} (1 - a_i/n). \tag{3.9}$$

For the derivation, observe that any one placement of the n column maxima of A_1 has probability m^{-n}. To each such placement we can associate the A defined by

$$a_i = \text{number of columns of } A_1 \text{ with maximum in row } i.$$

The number of placements yielding a particular A is just $(n; A)$.

For any one placement (of the n column maxima of A_1) which gives rise to a particular A, an EP will *fail* to occur iff for each i, the placement of the ith row maximum in A_2 is *not* in one of the a_i positions corresponding to a column maximum in A_1. The probability of this, for a single i, is simply $1 - a_i/n$. Combining these considerations yields (3.9).

4. Comparison With Direct-Conflict Case

A game is called *zero-sum* if $A_1 + A_2 = 0$. Some years ago the second author observed[2] that the probability, that a random zero-sum game has an EP involving only pure strategies, is given by

$$p'_{mn} = m!n!/(m+n-1)! = (m+n) \Big/ \binom{m+n}{m}, \tag{4.1}$$

[2] A. J. Goldman, The Probability of a Saddlepoint, Amer. Math. Monthly **64** (1957). See also R. M. Thrall and J. E. Falk, Some Results Concerning the Kernel of a Game, SIAM Review **7**, 359–375 (1965).

where m and n have the same meanings as before.

We introduce the notation

$$M = \max(m, n), \qquad \mu = \min(m, n). \tag{4.2}$$

If $\mu > 1$ then

$$2 \leqslant M \leqslant m+n-2$$

and so

$$\binom{m+n}{M} \geqslant \binom{m+n}{2} = (m+n)(m+n-1)/2$$

and

$$p'_{mn} = (m+n) \Big/ \binom{m+n}{M} \leqslant 2/(m+n-1),$$

from which it follows that

$$p'_{mn} \to 0 \text{ as } M \to \infty \text{ and } \mu > 1. \tag{4.3}$$

In particular, "large" zero-sum games (those in which at least one player has many pure strategies) are quite unlikely to have solutions in pure strategies.

It was anticipated that the same conclusion would hold for general games (i.e., not necessarily zero-sum), but this proved false. For an elementary analysis, consider the product

$$\prod_{i=1}^{m} (1 - a_i/n)$$

in the Ath term of (3.9). It is an instance of $\prod_{i=1}^{m} (1-x_i)$ subject to $\sum_{i=1}^{m} x_i = 1$ and all $x_i \geqslant 0$; the constrained maximum of this product is $(1-1/m)^m$, corresponding to all $x_i = 1/m$, so that (3.9) yields

$$q_{mn} \leqslant m^{-n}(1-1/m)^m \sum_A (n; A).$$

But by the multinomial theorem,

$$\sum_A (n; A) = (1+1+\ldots+1)^n = m^n,$$

so that

$$q_{mn} \leqslant (1-1/m)^m.$$

Symmetric forms are

$$q_{mn} \leqslant (1-1/M)^M, \; q_{mn} \leqslant (1-1/\mu)^\mu, \tag{4.4}$$

where M and μ are as in (4.2). Thus

$$\limsup q_{mn} \leqslant 1/e \qquad (M \to \infty) \tag{4.5}$$

and so

$$\liminf p_{mn} \geqslant 1 - 1/e > 0 \qquad (M \to \infty),$$

in contrast with (4.3).

Moreover, for each m and n

$$p_{mn} \geqslant p'_{mn}, \tag{4.6}$$

i.e., the probability of an EP is in general reduced if attention is restricted to zero-sum games. Equality holds if $\mu = 1$. For the proof when $\mu > 1$, observe from (4.4) that

$$p_{mn} \geqslant 1 - (1 - 1/m)^m. \tag{4.7}$$

On the other hand, from

$$p'_{m,\,n+1}/p'_{mn} = (m+1)/(m+n) < 1$$

we see that p'_{mn} is decreasing in n, so that since $n > 1$,

$$p'_{mn} \leqslant p'_{m2} = 2/(m+1).$$

From this and (4.7), we see that (4.6) can be proved by showing that

$$1 - (1 - 1/m)^m \geqslant 2/(m+1),$$

or equivalently

$$(1 - 1/m)^m \leqslant (m-1)/(m+1). \tag{4.8}$$

But clearly

$$(1 - 1/m)^m \leqslant (m-1)^2/m^2 = (m-1)(m^2-1)/m^2(m+1) < (m-1)/(m+1),$$

and the proof is complete.

The defining characteristic of a zero-sum game, $A_1 + A_2 = 0$, clearly implies that the interests of the two players are in direct conflict. Jordan [3] has defined a broader class of games in terms of a characteristic which appears more accurately (less restrictively) to capture the "direct conflict" notion. His condition, on the matrix entries $A_1(i,j)$ and $A_2(i,j)$, is that

$$A_1(i_1, j_1) < A_1(i_2, j_2) \text{ iff } A_2(i_2, j_2) < A_2(i_1, j_1).$$

That is, the players have *opposite* preference orders over the set of possible outcomes (i, j). It is easy to show that formula (4.1) remains valid for Jordan's *cutthroat games*, so that (4.3) can be viewed as an (intuitively plausible) property of direct-conflict situations.

Before leaving this topic, we should consider the opposite case in which the players' interests are exactly *parallel*, i.e.,

$$A_1(i_1, j_1) < A_1(i_2, j_2) \text{ iff } A_2(i_1, j_1) < A_2(i_2, j_2).$$

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[3] S. L. Jordan, Cutthroat Games (Abstract), Bull. Oper. Res. Soc. Amer. **14** (1966), Supplement 1, p. B–68.

Here a largest entry in A_1 will necessarily occur in the same position (i, j) as a largest entry in A_2, so that (i, j) will be an EP, and hence (in this case) the analog of p_{mn} has value 1.

5. Asymptotic Analysis

In this section we sharpen the inequality (4.5) and show that

$$q_{mn} \to 1/e \qquad (\mu \to \infty), \tag{5.1}$$

so that for games with many courses of action for *both* players, p_{mn} is close to $1-1/e=0.632^+$. For the proof, rewrite (3.8) as

$$q_{mn} = \sum_{k=0} ((-1)^k/k!) \prod_{i=1}^{k-1} (1-i/m) \prod_{j=1}^{k-1} (1-j/n),$$

so that

$$e^{-1} - q_{mn} = \sum_{k=0} ((-1)^k/k!)\,\theta_k(m, n) \tag{5.2}$$

where

$$\theta_k(m, n) = 1 - \prod_{i=1}^{k-1} (1-i/m) \prod_{j=1}^{k-1} (1-j/n). \tag{5.3}$$

Also, set

$$K(\mu) = [\mu]^{1/3}. \tag{5.4}$$

Let any $\delta > 0$ be given. Since $0 < \theta_k < 1$ and $\sum_{k=0} (-1)^k/k!$ is convergent, the same holds for (5.2); since $K(\mu) \to \infty$ as $\mu \to \infty$, for all sufficiently large μ we have the "tail estimate"

$$\left| \sum_{k>K(\mu)} ((-1)^k/k!)\,\theta_k(m, n) \right| < \delta/2. \tag{5.5}$$

For each $k \leqslant K = K(\mu)$, by Bernoulli's Inequality

$$\begin{aligned} \prod_{i=1}^{k-1} (1-i/m) \prod_{j=1}^{k-1} (1-j/n) &> (1-K/m)^K (1-K/n)^K \\ &> (1-K^2/m)(1-K^2/n) \\ &\geqslant (1-K^2/\mu)^2 \geqslant (1-1/K)^2 \\ &\geqslant 1-2/K \end{aligned}$$

so that $0 < \theta_k(m, n) \leqslant 2/K$ and thus

$$\left| \sum_{k=0}^{K} ((-1)^k/k!)\,\theta_k(m, n) \right| \leqslant (2/K) \sum_{k=0}^{K} 1/k! < 2e/K.$$

If μ is so large that $K(\mu) > 4e/\delta$ and (5.5) holds, then

$$|e^{-1}-q_{mn}| < \delta;$$

hence, (5.1) is proved.

6. A Related Limit

The payoff matrices A_1 and A_2 have in common their set of mn "positions" corresponding to the ordered pairs (i, j). Consider the situation in which first a subset S of m of these positions is chosen "at random," and then (independently) a subset T of n of the mn positions is chosen. We are interested in the probability Q_{mn} that S and T have no common elements, and also in the complementary probability $P_{mn}=1-Q_{mn}$.

If we regard S as the set of positions of row maxima in A_2 and T as the set of positions of column maxima in A_1, then we have a version of the situation just described which is *constrained*, in the sense that the members of S (of T) must lie in distinct rows (columns) and the meaning of "at random" must be understood accordingly. (Our model ascribes zero probability to "ties" for a row or column maximum.) It is readily seen that the analogs of Q_{mn} and P_{mn}, for this constrained problem, are just q_{mn} and p_{mn}. But here we will deal with the much simpler "unconstrained" version.

Q_{mn} is quite easy to evaluate, since

$$Q_{mn}=\sum_{S} Pr\{S \text{ is chosen}\} Pr\{S\cap T=\emptyset | S \text{ is chosen}\}.$$

The event to which the conditional probability refers occurs if and only if the n elements of T are all among the $mn-m$ positions comprising the complement of S. Thus

$$Q_{mn}=\sum_{S}\binom{mn}{m}^{-1}\binom{mn-m}{n}\Big/\binom{mn}{n}$$

$$=\binom{mn}{m}^{-1}\binom{mn-m}{n}\binom{mn}{n}^{-1}\sum_{S}1$$

$$=\binom{mn}{m}^{-1}\binom{mn-m}{n}\binom{mn}{n}^{-1}\binom{mn}{m},$$

or finally

$$Q_{mn}=\binom{mn-m}{n}\Big/\binom{mn}{n}$$

$$=(mn-m)!(mn-n)!/(mn-m-n)!(mn)!. \tag{6.1}$$

Values of P_{mn} are given in table 2.

TABLE 2. *Analogous Probability for "Uncontrained" Problem*[a]

$N=$	2	3	4	5	6	7	8	9	10
$M=2$	0.833	0.800	0.786	0.773	.0773	0.769	0.767	0.765	0.763
3		.762	.745	.736	.730	.726	.723	.721	.719
4			.728	.718	.712	.708	.705	.702	.700
5				.708	.702	.697	.694	.691	.689
6					.695	.691	.687	.685	.683
7						.686	.683	.680	.678
8							.679	.676	.674
9								.674	.672
10									.670

[a]Calculated by L. S. Joel.

The limit of Q_{mn} as $\mu \to \infty$ can be found by applying Stirling's formula, but we prefer the more elementary argument obtained by rewriting (6.1) as

$$Q_{mn} = Q_{\mu M} = \prod_{k=1}^{\mu} (1 - M/(M\mu - \mu + k)).. \tag{6.2}$$

The denominators in (6.2) satisfy

$$M(\mu - 1) < M\mu - \mu + k \leq M\mu,$$

so that (6.2) implies

$$(1 - 1/(\mu - 1))^{\mu} > Q_{mn} > (1 - 1/\mu)^{\mu}. \tag{6.3}$$

This clearly yields

$$Q_{mn} \to 1/e = \lim q_{mn} \qquad (\mu \to \infty). \tag{6.4}$$

From (6.5) and (5.1) we have

$$Q_{mn} - q_{mn} \to 0 \qquad (\mu \to \infty) \tag{6.5}$$

which is equivalent to

$$P_{mn} - p_{mn} \to 0 \qquad (\mu \to \infty). \tag{6.6}$$

Our numerical results indicate that the convergence in (6.5) and (6.6) is considerably more rapid than (6.4) and (5.1), to an extent not fully explained by the consequence

$$q_{mn} < Q_{mn} \tag{6.7}$$

of (6.3) and (4.4).

It would appear very desirable to be able to prove (6.5) *directly*. This might provide the key to verifying the quite plausible conjecture that for the p-player case ($p > 2$), the analog of Q_{mn} (which arises from a *relatively* simple combinatorial problem) will again yield a good approximation to the analog of q_{mn}, as well as the same limiting value as $\mu \to \infty$.

ADDENDUM: M. Pearl (unpublished) has recently shown that for the 3-player case, the analog of Q_{mn} has the same limit $1/e$ as found above for the 2-player case.

(Paper 72B2–262)