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On Spaces and Maps of Generalized Inverses

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Several classes of generalized inverses of a given $m \times n$ matrix are considered. A collection of continuous maps is given, each of which maps a class of generalized inverses onto a stronger class and the elements of the stronger class are the fixed points of the map. For the case of EPr matrices one of these maps is studied in more detail. The various classes of generalized inverses are characterized as subspaces of the space of all $n \times m$ matrices.

Key Words: Generalized inverse, linear algebra, matrix.

1. Introduction

Several classes of generalized inverses of a given $m \times n$ matrix A are considered.

In section 3 it is shown that the ability to construct a generalized inverse of the weakest class provides the ability to construct a generalized inverse in any one of the stronger classes. Then a collection of continuous maps is given each of which maps a class of generalized inverses onto a stronger class of generalized inverses that remains fixed under the map. In section 4 we characterize these classes of generalized inverses as subspaces of the space of all $n \times m$ matrices and examine in more detail one of the maps given in section 3.

2. Preliminaries and Definitions

We consider only matrices with complex entries. For any matrix M we denote by $\rho(M)$, $R(M)$, $N(M)$ and M^* the rank, range, null space and conjugate transpose of M. By I we denote an identity matrix the order of which will be clear from the context. By V^k we denote a k-dimensional vector space over the complex field. If S_1 and S_2 are any two sets we denote by $S_1 - S_2$ the set of all elements which are in S_1 and not in S_2; by $S_1 \cup S_2$ the union of S_1 and S_2; by $S_1 \cdot S_2$ the intersection of S_1 and S_2; and by $S_1 \leq S_2$ denote that S_1 is a subset of S_2. We recall that a homeomorphism is a continuous map which is one to one, onto and has a continuous inverse.

When the matrix A is nonsingular, we denote in the usual way by A^{-1} the inverse of A. For generalized inverses we adopt a special terminology as follows: We define five classes of generalized inverses. For a given matrix A, $C_1(A)$ is the set of all matrices B such that $ABA = A$; $C_2(A)$ is the set of all matrices $B \epsilon C_1(A)$ such that $BAB = B$; $C_3(A)$ is the set of all matrices $B \epsilon C_2(A)$ such that AB is Hermitian; $C_{3'}(A)$ is the set of all matrices such that $B \epsilon C_2(A)$ and BA is Hermitian; and $C_4(A)$ is the set of all matrices such that $B \epsilon C_3(A)$ and $B \epsilon C_{3'}(A)$. We call a matrix $B \epsilon C_i(A)$, a C_i-inverse of A, $i = 1, 2, 3, 3', 4$. This classification of generalized inverses has been used in previous work to which we will refer [4, 5, 6] [1] and related there to other systems of nomenclature which are in use. We note here that the $C_{3'}$-inverse is the Goldman-Zelen weak generalized inverse [3] and that $C_4(A) = C_3(A) \cdot C_{3'}(A)$ is a single matrix, the unique Moore-Penrose generalized inverse [10]. There are many statements regarding a C_3-inverse which of necessity hold for the $C_{3'}$-inverse (see [5], other examples occur in sec. 4). But there are contexts in which the role of elements of

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[1] Figures in brackets indicate the literature references at the end of this paper.

$C_3(A)$ and $C_{3'}(A)$ are quite different (see Theorem 7). It is clear that $C_4(A) \leq C_3(A) \leq C_2(A) \leq C_1(A)$ and that $C_4(A) \leq C_{3'}(A) \leq C_2(A) \leq C_1(A)$. We define four classes of *strict* generalized inverses as follows: A *strict* C_1-inverse of A is any matrix $B \epsilon (C_1(A) - C_2(A))$; a strict C_2-inverse of A is any matrix $B \epsilon (C_2(A) - C_3(A) \cup C_{3'}(A))$; a strict C_3-inverse of A is any matrix $B \epsilon (C_3(A) - C_4(A))$; a strict $C_{3'}$-inverse of A is any matrix $B \epsilon (C_{3'}(A) - C_4(A))$. We will sometimes say $B \epsilon C_i(A)$ and is strict, meaning that B is a strict C_i-inverse of A, and when it is clear from the context that B is in a given class we will simply say that B is strict when it is a strict member of that class. Consideration of strictness leads to alternative characterizations of the classes of generalized inverses defined above and studies of this type were first carried out by Rohde [11, 12]. For example, Lemma 1 below, gives necessary and sufficient conditions for $B \epsilon C_1(A)$ to be strict. If certain demands are made on A, and especially if demands are made on A and a generalized inverse of A, there may exist no strict generalized inverses in a certain class. Such cases are known [5] and others appear in section 4.

Finally we recall that a matrix A is called an EP matrix if $N(A) = N(A^*)$, in particular if A is EP and $\rho(A) = r$ we say that A is an EPr matrix [8], or merely that A is EPr. For ready reference we record the following known lemmas (one dealing with EP matrices) to which repeated reference will be made.

LEMMA 1. *The matrix* B *is a* C_1*-inverse of* A *if and only if* BA *is a projection and* $\rho(A) = \rho(BA)$; *and if and only if* AB *is a projection and* $\rho(A) = \rho(AB)$. *If* $B \epsilon C_1(A)$, *then* $\rho(B) \geq \rho(A)$ *with strict equality if and only if* $B \epsilon C_2(A)$.

The first statement of Lemma 1 is Corollary 1 of [4]. The second statement combines Lemmas 1 and 2 of [4], both proved in a different way by Rohde [12].

LEMMA 2. *Let* P *be an* $n \times m$ *matrix,* Q *and* R *be* $m \times n$ *matrices. If* PQ *is a projection such that* $\rho(PQ) = \rho(Q)$ *and* $N(R) = N(Q)$, *then* $RPQ = R$.

LEMMA 3. $P^* = P^*QP$ *if and only if* $Q \epsilon C_1(P)$ *and* $N(P) = N(P^*)$. *Further,* $P^* = P^*QP$ *if and only if* $P^* = PQP^*$.

Lemma 2 is Lemma 3 of [4] and Lemma 3 is Corollary 2 of [4].

3. Maps and Constructions

We take for granted the existence of, and known methods for constructing, a C_1-inverse of an arbitrary matrix[2] [1, 11]. The first theorem shows that the ability to construct a C_1-inverse of an arbitrary matrix gives us the ability to construct a matrix in any given class of the five classes of generalized inverses which we have defined.

THEOREM 1. *Let* A *be a given matrix. Define* $H = A^*A$, $J = AA^*$ *and let* $B_1 \epsilon C_1(A)$, $K \epsilon C_1(H)$, $L \epsilon C_1(J)$ *and* $M \epsilon C_{3'}(H)$. *Then*

(i) $B_2 = B_1AB_1$ *is in* $C_2(A)$ *and every matrix in* $C_2(A)$ *can be so expressed for some* $B_1 \epsilon C_1(A)$.

(ii) $B_3 = KA^*$ *is in* $C_3(A)$ *and every matrix in* $C_3(A)$ *can be so expressed for some* $K \epsilon C_1(H)$.

(iii) $B_{3'} = A^*L$ *is in* $C_{3'}(A)$ *and every matrix in* $C_{3'}(A)$ *can be so expressed for some* $L \epsilon C_1(J)$.

(iv) $B_4 = MA^*$ *is the* C_4*-inverse of* A.

(v) *Let* $\rho(A) = \rho(A^2)$. *Then* $B = AWA$ *is in* $C_2(A)$ *and commutes with* A *if and only if* $W \epsilon C_1(A^3)$.

PROOF. (i): That $B_2 = B_1AB_1$ is in $C_2(A)$ is a special case of a known theorem [4]. That every matrix in $C_2(A)$ can be so expressed is obvious. (ii): If $B_3 = KA^*$ we have $B_3A = KH$. Then by Lemma 1, B_3A is a projection, $\rho(B_3A) = \rho(H) = \rho(A)$ and thus $B_3 \epsilon C_1(A)$. By Lemma 1, $\rho(B_3) \geq \rho(A)$ but also $\rho(B_3) = \rho(KA^*) \leq \rho(A)$. Hence $\rho(B_3) = \rho(A)$ and, by Lemma 1, $B_3 \epsilon C_2(A)$. By Lemma 1, AB_3 is a projection with rank $\rho(A)$ and from $AB_3 = AKA^*$ it is clear that $N(AB_3) = N((AB_3)^*)$ and AB_3 is Hermitian. Thus $B_3 \epsilon C_3(A)$. Conversely if $B_3 \epsilon C_3(A)$ we have $B_3 = B_3AB_3 = B_3B_3^*A^*$ and we have to show that $B_3B_3^* \epsilon C_1(H)$. But by Lemma 1, B_3A is a projection with rank $\rho(A) = \rho(H)$, and $B_3A = B_3B_3^*H$ shows, by Lemma 1, that $B_3B_3^* \epsilon C_1(H)$. (iii): The proof of (iii) parallels that of (ii) in an obvious manner. (iv): Let $B_4 = MA^*$. Then by (ii) we have $B_4 \epsilon C_3(A)$. We now observe that $B_4A = MH$ is Hermitian and hence $B_4 \epsilon C_4(A)$. We note that conversely if $B_4 \epsilon C_4(A)$ we have $B_4 = B_4B_4^*A^*$, and $B_4B_4^* \epsilon C_4(H)$ [10]. (v): If $W \epsilon C_1(A^3)$, then by Lemma 1, $WA^3 = (WA^2)A$ is a projec-

[2] See also (i) of Theorem 4.

tion with rank $\rho(A^3)=\rho(A)$ and $WA^2\epsilon C_1(A)$. This being the case we have, by Lemma 1, $AWA^2=BA$ is a projection of rank $\rho(A)$ and $B\epsilon C_1(A)$. But $\rho(B)\leq\rho(A)$ and hence, by Lemma 1, $B\epsilon C_2(A)$. The projections $AB=A^2WA$ and $BA=AWA^2$ clearly both have null space $N(A)$ and range $R(A)$. Thus $AB=BA$. Now assume $B\epsilon C_2(A)$ and $AB=BA$. Then $A^3WA^3=A^2BA^2=A^3$ shows that $W\epsilon C_1(A^3)$. This completes the proof of the theorem.

REMARK: Goldman and Zelen [3] have proved that $B_{3'}=A^*N$ is in $C_{3'}(A)$ if $N\epsilon C_2(J)$, and that every matrix in $C_{3'}(A)$ can be so expressed for some $N\epsilon C_2(J)$. The above proof of (iii) shows that this theorem goes through under the weaker condition $N\epsilon C_1(J)$. However we note that if $B\epsilon C_3(A)$, then BB^* is in fact in $C_2(H)$. For, in the proof of (ii) we have shown $BB^*\epsilon C_1(H)$, and from Lemma 1 and $\rho(BB^*)=\rho(B)=\rho(A)=\rho(H)$, we have $BB^*\epsilon C_2(H)$. By the same kind of argument $B^*B\epsilon C_2(J)$ whenever $B\epsilon C_{3'}(A)$.

It is well known that for any matrix A, the C_4-inverse of A exists and is unique. The next theorem gives a set of mappings which map an arbitrary C_1-inverse onto a stronger class of inverse and fixes the stronger class of inverse. These maps are continuous and have differentiability properties of which considerable use is made in a subsequent paper [7].

We first give two lemmas which deal with the difference of two C_1-inverses.

LEMMA 4. *If* B_1 *and* B_2 *are in* $C_1(A)$ *then* $A(B_1-B_2)A=0$. *Conversely, if* $ADA=0$, *then for any* $B_1\epsilon C_1(A)$ *we have* $(B_1+D)\epsilon C_1(A)$.

PROOF. The proof is obvious from the definition of $C_1(A)$.

LEMMA 5. *If* $ADA=0$ *then* $D=P_NDP_R+D(I-P_R)$, *where* P_N *and* P_R *are any projections onto* $N(A)$ *and* $R(A)$, *respectively. Conversely, for any* D *of the form* $D=P_NZ_1+Z_2(I-P_R)$, *where* Z_1 *and* Z_2 *are arbitrary matrices such that the indicated products exist, we have* $ADA=0$.

PROOF. If $ADA=0$, then $P_NDP_R=DP_R$. We now have $D=DP_R+D(I-P_R)=P_NDP_R+D(I-P_R)$ and the first statement is proved. The converse is obvious since $(I-P_R)A=0$ and each column of P_NZ_1 is in $N(A)$.

THEOREM 2. *Let* A *be an* $m\times n$ *matrix,* B^+ *the* C_4*-inverse of* A, P_N *and* P_R *any projections onto* $N(A)$ *and* $R(A)$, *respectively, and* B *an* $n\times m$ *matrix. Then*

(i) $\varphi_1(B)=B^++P_N(B-B^+)P_R+(B-B^+)(I-P_R)$ *is in* $C_1(A)$ *and* $\varphi_1(B)=B$ *if and only if* $B\epsilon C_1(A)$.

(ii) $\varphi_2(B)=BAB$ *is in* $C_2(A)$ *whenever* $B\epsilon C_1(A)$. *In this case,* $\varphi_2(B)=B$ *if and only if* $B\epsilon C_2(A)$.

(iii) $\varphi_3(B)=BAB^+$ *is in* $C_3(A)$ *if and only if* $B\epsilon C_1(A)$. *In this case,* $\varphi_3(B)=B$ *if and only if* $B\epsilon C_3(A)$.

(iv) $\varphi_{3'}(B)=B^+AB$ *is in* $C_{3'}(A)$ *if and only if* $B\epsilon C_1(A)$. *In this case,* $\varphi_{3'}(B)=B$ *if and only if* $B\epsilon C_{3'}(A)$.

PROOF.

(i) Let $C=\varphi_1(B)-B^+$. Then by the second part of Lemma 5, $ACA=0$ and, by Lemma 4, $\varphi_1(B)=B^++C$ is in $C_1(A)$. If $B\epsilon C_1(A)$, then $A(B-B^+)A=0$ and, by the first part of Lemma 5, we have $C=B-B^+$, but then $B^++C=\varphi_1(B)=B$. Conversely if $\varphi_1(B)=B$ then $B\epsilon C_1(A)$.

(ii) If $B\epsilon C_1(A)$, that $\varphi_2(B)\epsilon C_2(A)$ is a special case of a known theorem [4]. If $B\epsilon C_1(A)$ then $\varphi_2(B)=B$ if and only if $B\epsilon C_2(A)$ follows from the definition of $C_2(A)$.

(iii) If $B\epsilon C_1(A)$, by a known theorem [4] we have $\varphi_3(B)\epsilon C_2(A)$. But $A\varphi_3(B)=AB^+$ shows that $A\varphi_3(B)$ is Hermitian and hence $\varphi_3(B)\epsilon C_3(A)$. Conversely, if $\varphi_3(B)\epsilon C_3(A)$, then $\varphi_3(B)A=BA$ is a projection with rank $\rho(A)$, and by Lemma 1, $B\epsilon C_1(A)$. If $B\epsilon C_3(A)$ then the projections AB and AB^+ are Hermitian and we have $N(B)=N(B^+)=N(A^*)$. From Lemma 2 it follows that $B=BAB^+=\varphi_3(B)$.

(iv) The proof of (iv) is an obvious parallel of that of (iii).

It is known that every square matrix of rank r has C_2-inverses which are EPr [4]. When A is EPr the construction of C_2-inverses which are EPr is particularly simple.

THEOREM 3. *Let* A *be* EPr. *Then* $\psi(B)=BA^*B^*$ *is in* $C_2(A)$ *and is* EPr *whenever* $B\epsilon C_2(A)$. *In that case* $\psi(B)=B$ *if and only if* B *is* EPr.

PROOF. That $\psi(B)$ is in $C_2(A)$ and is EPr when B is an arbitrary matrix in $C_2(A)$ is Lemma 5 of [4]. If B is EPr, then, since $B\epsilon C_2(A)$ implies $A\epsilon C_2(B)$, we can apply Lemma 3 to write $B=BA^*B^*=\psi(B)$.

4. Subspaces of Generalized Inverses

In this section we will characterize the classes of generalized inverses previously discussed as subspaces of the space of all $n \times m$ matrices with complex entries viewed as a vector space V^{nm} over the complex numbers.

THEOREM 4. *Let* A *be an* $m \times n$ *matrix of rank* r. *Then*

(i) *The collection of all* C_1*-inverses of* A *is an affine space in* V^{nm} *of dimension* $nm - r^2$.

(ii) *The collection of all* C_2*-inverses of* A *is an algebraic variety in* V^{nm} *which is homeomorphic to a linear space of dimension* $(nm - r^2) - (m - r)(n - r)$.

(iii) *The collection of all* C_3*-inverses of* A *is an affine space of* V^{nm} *of dimension* $(nm - r^2) - (n - r)r - (n - r)(m - r) = r(m - r)$.

(iv) *The collection of all* $C_{3'}$*-inverses of* A *is an affine space of* V^{nm} *of dimension* $r(n - r)$.

PROOF: Any matrix A can be written as QMR where Q and R are unitary and $M = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}$ with D diagonal and nonsingular and with $\rho(D) = \rho(A)$ [2]. It is easily seen that $B \epsilon C_i(A)$ if and only if $RBQ \epsilon C_i(M)$, $i = 1, 2, 3, 3', 4$. Let $RBQ = \begin{pmatrix} U & V \\ W & X \end{pmatrix}$. Then the following may be derived directly from the definitions of the types of generalized inverses considered.

(i) $B \epsilon C_1(A)$ if and only if $U = D^{-1}$.

(ii) $B \epsilon C_2(A)$ if and only if $B \epsilon C_1(A)$ and $X = WDV$.

(iii) $B \epsilon C_3(A)$ if and only if $B \epsilon C_2(A)$ and $V = 0$.

(iv) $B \epsilon C_{3'}(A)$ if and only if $B \epsilon C_2(A)$ and $W = 0$.

The dimension and nature of the subspaces given in the theorem then follows from the dimension of the matrices V, W, and X not fixed by the requirement that B belong to a particular class of generalized inverses, and from the nature of the stated relations among them.

In case A is $n \times n$ and is EPr we have the following theorem.

THEOREM 5. *Let* A *be* $n \times n$ *and* EPr. *Then the* EPr *matrices of* $C_2(A)$ *form an algebraic variety of* V^{n^2} *which is homeomorphic to a linear space of dimension* $r(n - r)$.

PROOF. Since A is EPr it is known [8] that there exists a unitary matrix Q such that

$$A = Q \begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix} Q^{-1}$$

where A_1 is $r \times r$ and nonsingular. It is easily seen that $B \epsilon C_2(A)$ if and only if

$$Q^{-1}BQ = \begin{pmatrix} A_1^{-1} & U \\ V & VA_1U \end{pmatrix}$$

where U and V are arbitrary. By applying the map ψ of Theorem 3 we see that $B \epsilon C_2(A)$ is EPr if and only if

$$Q^{-1}BQ = \begin{pmatrix} A_1^{-1} & A_1^{-1}A_1^{*}V^{*} \\ V & VA_1^{*}V^{*} \end{pmatrix}.$$

The theorem follows immediately from equating these two expressions.

When A is EPr the question of strictness of generalized inverses of A becomes rather special. The following lemma shows that when A is EPr, there exists no strict C_2-inverse which commutes with A, no strict C_3-inverse and no strict $C_{3'}$-inverse.

LEMMA 6. *Let* A *be* EPr. *Then the intersection of the set of all* EPr C_2*-inverses of* A *with* $C_3(A)$ *(or with* $C_{3'}(A)$*) is the* C_4*-inverse of* A *which in this case is the unique* C_2*-inverse of* A *which commutes with* A.

PROOF. If $B \epsilon C_3(A)$ and is EPr then we have that AB is Hermitian and $N(AB)=N(B)=N(A)$. Also $N(BA)=N(A)$ and $N((BA)^*)=N(B)$. Thus $N(BA)=N((BA)^*)$, BA is Hermitian and $B \epsilon C_4(A)$. An exactly parallel proof shows that if $B \epsilon C_{3'}(A)$ and is EPr then $B \epsilon C_4(A)$. It is known that for any G such that $\rho(G)=\rho(G^2)$ there exists a C_2-inverse which commutes with G and this matrix is uniquely determined by G [5]. If A is EPr, then $\rho(A)=\rho(A^2)$ [8], and there is a unique $B \epsilon C_2(A)$ which commutes with A. By a known theorem [9], a matrix commutes with its C_4-inverse if and only if the matrix is EPr; and, by this very theorem, the C_4-inverse of an EPr matrix is itself EPr. Thus, since the C_4-inverse of A is in $C_2(A)$ and commutes with A, it is the unique $B \epsilon C_2(A)$ which commutes with A and that unique $B \epsilon C_2(A)$ must be EPr. This completes the proof of the lemma.

REMARK. Lemma 6 shows that in Lemma 2 of [5] the condition of normality can be weakened to EPr.

We now use Lemma 6 to study the strictness of the EPr C_2-inverses of an EPr matrix as established by the following theorem.

THEOREM 7. *Let* A *be* EPr *and let* ψ *be the map of Theorem 3. Then* ψ *sends all of* $C_{3'}(A)$ *to* $C_4(A)$ *and the remainder of* $C_2(A)$ *into the set of all strict* C_2*-inverses of* A *which are* EPr.

PROOF. From Lemma 3 we have

$$A\psi(B)=ABA^*B^*=A^*B^*$$

$$\psi(B)A=BA^*B^*A=BA.$$

Since $\psi(B) \epsilon C_2(A)$ and is EPr, by Theorem 3, we have by Lemma 6 that if $\psi(B)$ is not strict, then $\psi(B) \epsilon C_4(A)$. It follows from the display that if $\psi(B)$ is not strict then $B \epsilon C_{3'}(A)$. Conversely if $B \epsilon C_{3'}(A)$ then the display shows that $\psi(B) \epsilon C_4(A)$. Thus ψ sends all of $C_{3'}(A)$ and only elements in $C_{3'}(A)$ to $C_4(A)$. The remainder of $C_2(A)$ are those elements in $C_2(A)$ which are strict and those in $C_3(A)$ which are strict. We now show that ψ sends these elements into strict C_2-inverses. If $B \epsilon C_3(A)$ and is strict, then BA is not Hermitian and the display shows that neither $A\psi(B)$ nor $\psi(B)A$ is Hermitian and $\psi(B)$ is strict. Assume that $\psi(B)$ is not strict. Then, by Lemma 6, $\psi(B) \epsilon C_4(A)$. But we have shown that ψ sends only elements of $C_{3'}(A)$ to $C_4(A)$ and therefore B is not strict. Thus if $B \epsilon C_2(A)$ and is strict, $\psi(B) \epsilon C_2(A)$ and is strict.

We observe that if A is EPr, then by the same kind of proof given for Theorem 3, $\psi_1(B)=B^*A^*B$ is in $C_2(A)$ and is EPr whenever $B \epsilon C_2(A)$. Moreover, when $B \epsilon C_2(A)$, $\psi_1(B)=B$ if and only if B is EPr. We then have the following parallel of Theorem 7: ψ_1 sends all of $C_3(A)$ to $C_4(A)$ and the remainder of $C_2(A)$ into the set of all strict C_2-inverses of A which are EPr.

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