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# Principal Submatrices III: Linear Inequalities*

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Let $H$ be an $n \times n$ Hermitian matrix with eigenvalues $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$. Let $H(i|i)$ denote the principal submatrix of $H$ obtained by deleting row $i$ from $H$. Let $\xi_{i1} \leq \xi_{i2} \leq \ldots \leq \xi_{i,n-1}$ be the eigenvalues of $H(i|i)$. The famous Cauchy inequalities assert that $\xi_{i1}, \ldots, \xi_{i,n-1}$ interlace $\lambda_1, \ldots, \lambda_n$. It was recently proved by the present author that, for each fixed $j$, the arithmetic mean $n^{-1} \sum_{i=1}^{n} \xi_{ij}$ of the $\xi_{ij}$ lies between $(1-\theta)\lambda_j + \theta\lambda_{j+1}$ and $\theta\lambda_j + (1-\theta)\lambda_{j+1}$, where $\theta = 1/n$. In the present paper the cases of equality in these inequalities for the arithmetic mean of the $\xi_{ij}$ are discussed.

Key Words: Cauchy inequalities, Hermitian matrices, interlacing theorems, matrices, matrix inequalities, matrix theory, principal submatrices.

## 1. Introduction

This paper is the third in a series of papers in which the principal submatrices of a matrix are studied. In the first paper [1][1] in this series a large number of inequalities involving the eigenvalues of all of the principal $(n-1) \times (n-1)$ submatrices of a normal or Hermitian $n \times n$ matrix $H$ were derived. In the second paper [2] certain of the inequalities obtained in [1] for Hermitian $H$ were examined for cases of equality. The inequalities studied in [2] involved the eigenvalues of the principal submatrices in a quadratic fashion, hence we chose to call these inequalities *quadratic inequalities.* In [1] inequalities involving the eigenvalues of the principal $(n-1)$-square submatrices of Hermitian $H$ in a linear fashion were obtained (see (2) below). These are the *linear inequalities* referred to in the title of this paper. It is the purpose of this paper to discuss cases of equality in these linear inequalities. Most of our results are obtained under the assumption that $H$ has only simple eigenvalues. However, our most important result, Theorem 10, which characterizes those Hermitian $H$ for which every one of our linear inequalities can achieve equality, does not require the assumption that $H$ has simple eigenvalues. Theorem 10 produces a rather unusual condition which essentially requires that the eigenvalues of $H$ be roots of a polynomial closely linked to the Legendre polynomials.

Certain of our results are valid for real symmetric matrices. Our most important tools are Theorems 1 and 2 of [2]. In general, when our proofs use Theorem 2 of [2] we obtain results valid for both the Hermitian and the real symmetric situations. When Theorem 1 of [2] is required, we obtain results valid only in the Hermitian situation.

## 2. Notation

We assume throughout this paper that $H = UDU^{-1}$ where $U$ is unitary and $D = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$. Except in section 5, we suppose $\lambda_1 < \lambda_2 < \ldots < \lambda_n$. In section 5, we only require that $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$. Let $H(i|i)$ be the principal submatrix of $H$ obtained by deleting row $i$ and

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[1] Figures in brackets indicate the literature reference at the end of this paper.

column $i$ of $H$. Let $\xi_{i1} \leq \xi_{i2} \leq \ldots \leq \xi_{i,n-1}$ be the eigenvalues of $H(i|i)$. The famous Cauchy inequalities assert that

$$\lambda_1 \leq \xi_{i1} \leq \lambda_2 \leq \xi_{i2} \leq \ldots \leq \lambda_{n-1} \leq \xi_{i,n-1} \leq \lambda_n.$$

We say $\xi_{ij}$ belongs to the interval $[\lambda_j, \lambda_{j+1}]$. Let

$$_jA_{j+1} = n^{-1} \sum_{i=1}^{n} \xi_{ij}.$$

In [1] the following inequalities were derived:

$$(n-1)n^{-1}\lambda_j + n^{-1}\lambda_{j+1} \leq {_jA_{j+1}} \leq n^{-1}\lambda_j + (n-1)n^{-1}\lambda_{j+1}, \qquad 1 \leq j < n. \tag{2}$$

The quantities $_jA_{j+1}$ are functions of the unitary matrix $U$ and it is the purpose of this paper to determine which of the inequalities (2) can become equality as $U$ varies over all unitary matrices. (Thus $H$ and $U$ are to be variable matrices and only $D$ is constant.) Let $U = (u_{ij})_{1 \leq i, j \leq n}$. The following fundamental formula was derived both in [1] and [2].

$$f_{(i)}(\lambda) = \sum_{t=1}^{n} |u_{it}|^2 f(\lambda)/(\lambda - \lambda_t). \tag{3}$$

Here

$$f(\lambda) = \prod_{i=1}^{n} (\lambda - \lambda_i) \tag{4}$$

is the characteristic polynomial of $H$ and

$$f_{(i)}(\lambda) = \prod_{i=1}^{n-1} (\lambda - \xi_{ij}) \tag{5}$$

is the characteristic polynomial of $H(i|i)$.

We will always let $P$ and $Q$ be permutation matrices. The symbol $\dotplus$ denotes direct sum.

## 3. Individual Cases of Equality

THEOREM 1. *Let* j *be fixed,* $1 \leq j < n$. *Then unitary* U *exists such that*

$$_jA_{j+1} = (n-1)n^{-1}\lambda_j + n^{-1}\lambda_{j+1}, \tag{6}$$

*if and only if*

$$f''(\lambda_j) f'(\lambda_j) \leq 0. \tag{7}$$

PROOF. We first of all remind the reader of the derivation of (2). From (3) we obtain on setting $\lambda = \lambda_j$ that

$$|u_{ij}|^2 = \left\{ \prod_{t=1}^{j-2} \frac{\lambda_j - \xi_{it}}{\lambda_j - \lambda_t} \right\} \left\{ \frac{\lambda_j - \xi_{i,j-1}}{\lambda_j - \lambda_{j-1}} \right\} \left\{ \frac{\xi_{ij} - \lambda_j}{\lambda_{j+1} - \lambda_j} \right\} \left\{ \prod_{t=j+1}^{n} \frac{\xi_{it} - \lambda_j}{\lambda_{t+1} - \lambda_j} \right\}. \tag{8}$$

(Certain of the factors in (8) are absent if $j=1, 2, n-1$, or $n$.) Because of the Cauchy inequalities, each fraction in (8) lies between zero and one, hence deleting all but one of the fractions in (8) increases the value of the expression. Thus from $\sum_{i=1}^{n} |u_{ij}|^2=1$ we obtain

$$\sum_{i=1}^{n} \frac{\lambda_j-\xi_{i,j-1}}{\lambda_j-\lambda_{j-1}} \geqslant 1, \tag{9}$$

$$\sum_{i=1}^{n} \frac{\xi_{ij}-\lambda_j}{\lambda_{j+1}-\lambda_j} \geqslant 1. \tag{10}$$

These inequalities (10), (9) are equivalent, respectively, to the first and second of the inequalities (2) after rearrangement of the terms.

Now (6) will be true if and only if the inequality (10) is equality. If $\xi_{ij}-\lambda_j \neq 0$, then equality in (10) can hold only if each of the fractions deleted from (8) is one. Hence

$$\xi_{i1}=\lambda_1, \ . \ . \ ., \xi_{i,j-1}=\lambda_{j-1}, \xi_{i,j+1}=\lambda_{j+2}, \ . \ . \ ., \xi_{i,n-1}=\lambda_n.$$

Using (8), it is easy to see that $u_{i1}= . \ . \ . =u_{i,j-1}=u_{i,j+2}= . \ . \ . =u_{in}=0$. Thus row $i$ of $U$ is zero outside columns $j$ and $j+1$. If $\xi_{ij}-\lambda_j \neq 0$ for three values of $i$, say $i=i_1, i_2, i_3$, then rows $i_1, i_2, i_3$ of $U$ would behave as 2-tuples, hence would be dependent. This is a contradiction because $U$ is nonsingular. Thus $\xi_{ij}-\lambda_j \neq 0$ for at most two values of $i$, and when $\xi_{ij}-\lambda_j \neq 0$, row $i$ of $U$ is zero except for $u_{ij}$ and (perhaps) $u_{i,j+1}$.

CASE (i): $\xi_{ij}-\lambda_j \neq 0$ for exactly one value of $i$, say $i=i_1$. In this case we must have $|u_{i_1j}|=1$, hence $u_{i_1,j+1}=0$. Thus $U$ is essentially composed of a 1-square block and an $(n-1)$ square block. After passing from $H=UDU^{-1}$ to $PHP^{-1}=(PUQ)(Q^{-1}DQ)(PUQ)^{-1}$, as in [2], we may assume

$$H=\lambda_j \dotplus H_2 \tag{11}$$

where $H_2$ is $(n-1)$-square and has $\lambda_1, \ . \ . \ ., \lambda_{j-1}, \lambda_{j+1}, \ . \ . \ ., \lambda_n$ as eigenvalues. Moreover, we now have $\xi_{1j}-\lambda_j \neq 0$ and $\xi_{ij}-\lambda_j=0$ for all $i>1$. For $i>1$ it follows from the form (11) of $H$ that the eigenvalues of $H(i|i)$ are

$$\delta_{i1}, \delta_{i2}, \ . \ . \ ., \delta_{i,j-2}; \delta_i, \lambda_j; \delta_{i,j+1}, \ . \ . \ ., \delta_{i,n-1}, \tag{12}$$

where

$$\delta_{i1}, \delta_{i2}, \ . \ . \ ., \delta_{i,j-2}, \delta_i, \delta_{i,j+1}, \ . \ . \ ., \delta_{i,n-1} \tag{13}$$

interlace $\lambda_1, \lambda_2, \ . \ . \ ., \lambda_{j-1}, \lambda_{j+1}, \ . \ . \ ., \lambda_n$. Here in (12), the terms $\delta_{i1}, \ . \ . \ ., \delta_{i,j-2}, \delta_i$ are absent when $j=1$. When $j=2$ the terms $\delta_{i1}, \ . \ . \ ., \delta_{i,j-2}$ are absent but $\delta_i$ is present. When $j=n-1$, $\delta_i$ is present but $\delta_{i,j+1}, \ . \ . \ ., \delta_{i,n-1}$ are absent.

We now show that with $H$ in the form (11), the equation (6) for $j=1$ is valid. For with $j=1$ we have $\xi_{11}=\lambda_2$ and $\xi_{i1}=\lambda_1$ for $i>1$, and hence (6) holds for $j=1$.

For $j \neq 1$ we require $\xi_{ij}=\lambda_j$ for all $i>1$. From (12) it is clear that this can happen only if $\delta_i \leqslant \lambda_j$. Moreover if $\delta_i \leqslant \lambda_j$ and $H$ is in the form (11), the equation (6) is valid, since then $\xi_{ij}=\lambda_j$ for all $i \geqslant 2$, and $\xi_{1j}=\lambda_{j+1}$ (because $H(1|1)$ has eigenvalues $\lambda_1, \ . \ . \ ., \lambda_{j-1}, \lambda_{j+1}, \ . \ . \ ., \lambda_n$).

Thus in Case (i) for (6) to hold it is necessary and sufficient that $\delta_i \leqslant \lambda_j$ for all $i \geqslant 2$ when $j \neq 1$. By an application of [2, Theorem 2] to $H_2$ and the interval $[\lambda_{j-1}, \lambda_{j+1}]$, we find that a unitary similarity of $H_2$ exists such that (6) holds if and only if

$$\begin{cases} \lambda_j \geqslant \text{the root of } \dfrac{d}{d\lambda} \displaystyle\prod_{\substack{t=1 \\ t\neq j}}^{n} (\lambda-\lambda_t) \text{ in interval } (\lambda_{j-1}, \lambda_{j+1}), & j \neq 1; \\ \text{no condition if } j=1. & \end{cases} \tag{14}$$

Now, by a graphical argument using the fact that $\lambda_j \epsilon (\lambda_{j-1}, \lambda_{j+1})$, (14) holds if and only if

$$\begin{cases} \operatorname{sgn} \dfrac{d}{d\lambda} \prod_{\substack{t=1 \\ t\neq j}}^{n} (\lambda - \lambda_t)|_{\lambda=\lambda_j} = -\operatorname{sgn} \prod_{\substack{t=1 \\ t\neq j}}^{n} (\lambda - \lambda_t)|_{\lambda=\lambda_j} \text{ or } 0, \qquad j \neq 1; \\ \text{no condition if } j=1. \end{cases} \tag{15}$$

It is not difficult to see that (15) for $j \neq 1$ is equivalent to $\operatorname{sgn} f''(\lambda_j) = -\operatorname{sgn} f'(\lambda_j)$ or 0, which in turn is equivalent to (7) for $j \neq 1$. For $j=1$, (15) imposes no condition and (7) is true. This completes Case (i).

CASE (ii): Here $\xi_{ij} - \lambda_j \neq 0$ for exactly two values of $i$, say $i = i_1$ and $i = i_2$. Thus rows $i_1, i_2$ of $U$ are zero outside columns $j$ and $j+1$, hence $U$ essentially splits into a $2 \times 2$ block and an $(n-2) \times (n-2)$ block. If we pass from $H$ to $PHP^{-1} = (PUQ)(Q^{-1}DQ)(PUQ)^{-1}$ as before, we may (after a change of notation) take $H$ in the form

$$H = H_1 \dotplus H_2, \tag{16}$$

where

$$H_1 = \begin{bmatrix} a & b \\ \bar{b} & c \end{bmatrix} \tag{17}$$

has eigenvalues $\lambda_j, \lambda_{j+1}$, and $H_2$ has eigenvalues $\lambda_1, \ldots, \lambda_{j-1}, \lambda_{j+2}, \ldots, \lambda_n$. Moreover, we must now have $\xi_{ij} = \lambda_j$ for all $i > 2$. Since $H_1$ has eigenvalues $\lambda_j, \lambda_{j+1}$ we have $\lambda_j \leqslant a, c \leqslant \lambda_{j+1}$ and

$$a + c = \lambda_j + \lambda_{j+1}.$$

With $H$ in the form (16), the eigenvalues of $H(i|i)$ for $i > 2$ are

$$\delta_{i1}, \ldots, \delta_{i,j-2}; \delta_i, \lambda_j, \lambda_{j+1}; \delta_{i,j+2}, \ldots, \delta_{i,n-1}, \tag{18}$$

where $\delta_{i1}, \ldots, \delta_{i,j-2}, \delta_i, \delta_{i,j+2}, \ldots, \delta_{i,n-1}$ interlace $\lambda_1, \ldots, \lambda_{j-1}, \lambda_{j+2}, \ldots, \lambda_{n-1}$. For $j=1$, the terms $\delta_{i1}, \ldots, \delta_{i,j-2}, \delta_i$ are absent in (18). For $j=2$, $\delta_{i1}, \ldots, \delta_{i,j-2}$ are absent but $\delta_i$ is present. For $j=n-1$, $\delta_i, \delta_{i,j+2}, \ldots, \delta_{i,n-1}$ are all absent. For $j=n-2$, $\delta_i$ is present but $\delta_{i,j+2}, \ldots, \delta_{i,n-1}$ are all absent.

We now show that with $H$ in the form (16), the equation (6) for $j=1$ is valid. This is so because for $j=1$ we have $\xi_{11} = c$, $\xi_{21} = a$, $\xi_{i1} = \lambda_1$ for $i > 1$, and $c + a = \lambda_1 + \lambda_2$.

We note also that if $n=2$ we are dealing with the case $j=1$ (since $j \leqslant n-1$). So let $n > 2$.

We next show that for $j \neq 1$ in Case (*ii*) we must have $j \neq n-1$. For if $j = n-1$ we have from (18) that $\xi_{ij} = \xi_{i,n-1} = \lambda_{j+1}$, contradicting the requirement that $\xi_{ij} = \lambda_j$. So we may suppose $j \neq n-1$. For $j \neq 1, j \neq n-1$, it follows from (18) that $\xi_{ij} = \lambda_j$, $(i > 2)$, can hold if and only if $\delta_i \leqslant \lambda_j$. Moreover, if we have $H$ in the form (16) and $\delta_i \leqslant \lambda_j$ for all $i > 2$, then (6) is valid (since $\xi_{1j} = c$, $\xi_{2j} = a$, $\xi_{ij} = \lambda_j$ for $i > 2$ and $a + c = \lambda_j + \lambda_{j+1}$). By an application of [2, Theorem 1] to $H_2$ for the interval $[\lambda_{j-1}, \lambda_{j+2}]$, we find that a unitary similarity of $H_2$ exists such that (6) holds if and only if

$$\begin{cases} \lambda_j \geqslant \text{the root of } \dfrac{d}{d\lambda} \prod_{\substack{t=1 \\ t\neq j, j+1}}^{n} (\lambda - \lambda_t) \text{ in the interval } (\lambda_{j-1}, \lambda_{j+2}), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad j \neq 1, n-1; \\ \text{no condition if } j=1. \end{cases} \tag{19}$$

The condition (19) is therefore necessary and sufficient for equality (6) to hold in Case (ii).

We now show that the condition (19) of Case (ii) implies the condition (14) of Case (i). This is clear for $j=1$. For $j\neq 1, n-1$, we show

$$\begin{cases} \text{the root of } \dfrac{d}{d\lambda}\displaystyle\prod_{\substack{t=1\\ t\neq j, j+1}}^{n}(\lambda-\lambda_t) \text{ in interval } (\lambda_{j-1}, \lambda_{j+2}) \\ > \text{the root of } \dfrac{d}{d\lambda}\displaystyle\prod_{\substack{t=1\\ t\neq j}}^{n}(\lambda-\lambda_t) \text{ in interval } (\lambda_{j-1}, \lambda_{j+1}). \end{cases} \tag{20}$$

To see (20) let

$$g(\lambda)=\prod_{\substack{t=1\\ t\neq j}}^{n}(\lambda-\lambda_t), \qquad h(\lambda)=\prod_{\substack{t=1\\ t\neq j, j+1}}^{n}(\lambda-\lambda_t).$$

Then $g(\lambda)=(\lambda-\lambda_{j+1})h(\gamma)$. Let $\gamma$ be the root of $h'(\lambda)$ in the interval $(\lambda_{j-1}, \lambda_{j+2})$. Then $g'(\gamma)=h(\gamma)$, hence sgn $g'(\gamma)=$ sgn $h(\gamma)=-$ sgn $g(\lambda_j)$, which (by a graphical argument) forces $\gamma$ to lie strictly between the two roots of $g'(\lambda)$ in the interval $(\lambda_{j-1}, \lambda_{j+2})$. Thus (20) is proved.

Thus in Case (ii) a condition that implies (7) holds. Hence (7) is the necessary and sufficient condition.

COROLLARY 1.

(i) *Let* (7) *be satisfied. Then* (6) *holds when* $H=P(\lambda_j \dotplus H_2)P^{-1}$ *where* $H_2$ *has eigenvalues* $\lambda_1, \ldots, \lambda_{j-1}, \lambda_{j+1}, \ldots, \lambda_n$, *and* (*for* $j\neq 1$) *each principal* $(n-2)$*-square submatrix of* $H_2$ *has its eigenvalue belonging to the interval* $[\lambda_{j-1}, \lambda_{j+1}]$ *within the smaller interval* $[\lambda_{j-1}, \lambda_j]$.

(ii) *For* $j\neq n-1$, *if* $\lambda_j$ *satisfies the condition* (19), *stronger than* (7) *when* $j\neq 1$, *then* (6) *holds when* $H=P(H_1\dotplus H_2)P^{-1}$ *where* $H_1$ *has eigenvalues* $\lambda_j, \lambda_{j+1}$ *and* $H_2$ *has eigenvalues* $\lambda_1, \ldots, \lambda_{j-1}, \lambda_{j+2}, \ldots, \lambda_n$ *and* (*for* $j\neq 1$) *each principal* $(n-3)$*-square submatrix of* $H_2$ *has its eigenvalue belonging to interval* $[\lambda_{j-1}, \lambda_{j+2}]$ *within the smaller interval* $[\lambda_{j-1}, \lambda_j]$.

(iii) *In no ways other than those described in* (i) *and* (ii) *can* (6) *hold.*

An entirely similar argument will establish Theorem 2 and Corollary 2.

THEOREM 2. *Let* j *be fixed* $1<j\leq n$. *Then unitary* U *exists such that*

$${}_{j-1}A_j=(n-1)n^{-1}\lambda_j+n^{-1}\lambda_{j-1} \tag{21}$$

*if and only if*

$$f''(\lambda_j)f'(\lambda_j)\geq 0. \tag{22}$$

COROLLARY 2.

(i) *Let* (22) *be satisfied. Then* (21) *holds when* $H=P(\lambda_j\dotplus H_2)P^{-1}$, *where* $H_2$ *has eigenvalues* $\lambda_1, \ldots, \lambda_{j-1}, \lambda_{j+1}, \ldots, \lambda_n$ *and* (*for* $j\neq n$) *each principal* $(n-2)$*-square submatrix of* $H_2$ *has its eigenvalue belonging to the interval* $[\lambda_{j-1}, \lambda_{j+1}]$ *within the smaller interval* $[\lambda_j, \lambda_{j+1}]$.

(ii) *For* $j\neq 2$, *if* $\lambda_j$ *satisfies the condition*

$$\begin{cases} \lambda_j\leq \textit{the root of } \dfrac{d}{d\lambda}\displaystyle\prod_{\substack{t=1\\ t\neq j-1, j}}^{n}(\lambda-\lambda_t) \textit{ in interval } (\lambda_{j-2}, \lambda_{j+1}), \quad j\neq n \\ \textit{no condition if } j=n. \end{cases} \tag{23}$$

(*condition* (23) *is stronger than* (22) *when* $j\neq n$), *then* (21) *holds when* $H=P(H_1\dotplus H_2)P^{-1}$ *where* $H_1$ *has eigenvalues* $\lambda_{j-1}, \lambda_j$, *and* $H_2$ *has eigenvalues* $\lambda_1, \ldots, \lambda_{j-2}, \lambda_{j+1}, \ldots, \lambda_n$ *and* (*for* $j\neq n$) *each principal* $(n-3)$*-square submatrix of* $H_2$ *has its eigenvalue belonging to the interval* $[\lambda_{j-2}, \lambda_{j+1}]$ *within the smaller interval* $[\lambda_j, \lambda_{j+1}]$.

(iii) *Other than as described in* (i), (ii), *the equality* (21) *does not hold.*

## 4. Simultaneous Equalities

THEOREM 3. *Let* $1 < j < k < n$. *Then*

$$ {}_{j-1}A_j = (n-1)n^{-1}\lambda_j + n^{-1}\lambda_{j-1}, \tag{23} $$

$$ {}_kA_{k+1} = (n-1)n^{-1}\lambda_k + n^{-1}\lambda_{k+1}, \tag{24} $$

*cannot simultaneously occur.*

PROOF. Suppose $\lambda_j - \xi_{i,j-1} \neq 0$. Then from (23), as in the proof of Theorems 1 and 2,

$$ \xi_{i1} = \lambda_1, \ldots, \xi_{i,j-2} = \lambda_{j-2}, \xi_{i,j-1} \neq \lambda_j, \xi_{ij} = \lambda_{j+1}, \ldots, \xi_{ik} = \lambda_{k+1}, \ldots . $$

Hence $\xi_{ik} - \lambda_k \neq 0$ and hence, from (24),

$$ \xi_{i1} = \lambda_1, \ldots, \xi_{i,j-2} = \lambda_{j-2}, \xi_{i,j-1} = \lambda_{j-1}, \xi_{ij} = \lambda_j, \ldots, \xi_{i,k-1} = \lambda_{k+1}, \ldots . $$

We now have the contradiction $\xi_{ij} = \lambda_{j+1}$ and $\xi_{ij} = \lambda_j$. This proves the theorem.

THEOREM 4. *Let* j *be fixed,* $1 < j < n$. *Then*

$$ {}_{j-1}A_j = (n-1)n^{-1}\lambda_j + n^{-1}\lambda_{j-1}, \tag{25} $$

$$ {}_jA_{j+1} = (n-1)n^{-1}\lambda_j + n^{-1}\lambda_{j+1}, \tag{26} $$

*can both hold (separately or simultaneously) if and only if* $f''(\lambda_j) = 0$.

PROOF. By Theorem 1, (26) can hold for some $U$ if and only if $f''(\lambda_j)f'(\lambda_j) \leqslant 0$ and by Theorem 2, (25) can hold for some $U$ if and only if $f''(\lambda_j)f'(\lambda_j) \geqslant 0$. Since $f'(\lambda_j) \neq 0$ as the eigenvalues of $H$ are simple, it follows that $f''(\lambda_j) = 0$ is the necessary and sufficient for (25) and (26) to occur separately or simultaneously.

From Corollaries 1 and 2 and Theorem 4 we deduce Corollary 3.

COROLLARY 3. *Suppose* $1 < j < n$, j *fixed, and that* $f''(\lambda_j) = 0$. *Then unitary* U *exists such that both* (25) *and* (26) *hold, and in fact both* (25) *and* (26) *hold precisely when*

$$ H = P(\lambda_j \dotplus H_2)P^{-1}, $$

*where* $H_2$ *has engenvalues* $\lambda_1, \ldots, \lambda_{j-1}, \lambda_{j+1}, \ldots, \lambda_n$ *and each principal* $(n-2)$*-square submatrix of* $H_2$ *has* $\lambda_j$ *as an eigenvalue.*

Theorem 4 is the $k = j$ analogue of Theorem 3. For $k = j-1$ and $n > 2$, it cannot, of course, happen that simultaneously ${}_{j-1}A_j$ and ${}_kA_{k+1}$ achieve the values $(n-1)n^{-1}\lambda_j + n^{-1}\lambda_{j-1}$ and

$$ (n-1)n^{-1}\lambda_k + n^{-1}\lambda_{k+1}. $$

Here Theorem 5 answers the question of when ${}_{j-1}A_j = {}_kA_{k+1}$ varies over the maximal permissible set of values as $U$ varies over all unitary matrices.

THEOREM 5. *Let* j *be fixed,* $2 \leqslant j \leqslant n$. *Then as* U *varies over all unitary matrices,* ${}_{j-1}A_j$ *varies over the full interval*

$$ [(n-1)n^{-1}\lambda_{j-1} + n^{-1}\lambda_j,\ n^{-1}\lambda_{j-1} + (n-1)n^{-1}\lambda_j] \tag{27} $$

*of permissible values allowed by the linear inequalities if and only if*

$$ f''(\lambda_{j-1})f'(\lambda_{j-1}) \leqslant 0 \text{ and } f''(\lambda_j)f'(\lambda_j) \geqslant 0. \tag{28} $$

PROOF. Since the roots of a polynomial are continuous functions of the coefficients of the polynomial, the function ${}_{j-1}A_j$ is a continuous function from the arcwise connected set of unitary $n\times n$ matrices to the real numbers. Thus ${}_{j-1}A_j$ covers the full interval (27) if and only if the endpoints of (27) are achievable. By Theorems 1 and 2, the endpoints of (27) are achievable if and only if the conditions (28) are satisfied.

Theorem 6 advances further the examination of ${}_{j-1}A_j$ and ${}_kA_{k+1}$ for various values of $j$ and $k$ that was started in Theorem 3 and continued in Theorems 4 and 5.

THEOREM 6. *Let* k, j *be fixed,* $1\leqslant k, j\leqslant n$, $j\geqslant k+2$. *Then*

$$ {}_{j-1}A_j=(n-1)n^{-1}\lambda_j+n^{-1}\lambda_{j-1}, \tag{29}$$

$$ {}_kA_{k+1}=(n-1)n^{-1}\lambda_k+n^{-1}\lambda_{k+1}, \tag{30}$$

*can simultaneously hold for some* U *if and only if:*

$$\begin{cases}\{(\lambda_k-\lambda_j)f''(\lambda_k)-2f'(\lambda_k)\}f'(\lambda_k)\geqslant 0, & k\neq 1,\\ \textit{no condition if } k=1;\end{cases} \tag{31}$$

*and*

$$\begin{cases}\{(\lambda_j-\lambda_k)f''(\lambda_j)-2f'(\lambda_j)\}f'(\lambda_j)\geqslant 0, & j\neq n,\\ \textit{no condition if } j=n.\end{cases} \tag{32}$$

PROOF. From (30) we find, as in the proof of Theorem 1, that $\xi_{ik}-\lambda_k\neq 0$ for exactly one or two values of $i$, and that when $\xi_{ik}-\lambda_k\neq 0$ then $\xi_{i1}=\lambda_1,\ldots,\xi_{i,k-1}=\lambda_{k-1}$, $\xi_{i,k+1}=\lambda_{k+2},\ldots,\xi_{i,n-1}=\lambda_n$, and $u_{i1}=\ldots=u_{i,k-1}=u_{i,k+2}=\ldots=u_{in}=0$, $u_{ik}\neq 0$. Similarly $\lambda_j-\xi_{i,j-1}\neq 0$ for at most two values of $i$ and when $\lambda_j-\xi_{i,j-1}\neq 0$ we have $\xi_{i1}=\lambda_1,\ldots,\xi_{i,j-2}=\lambda_{j-2},\xi_{ij}=\lambda_{j+1},\ldots,\xi_{i,n-1}=\lambda_n$, and $u_{i1}=\ldots=u_{i,j-2}=u_{i,j+1}=\ldots=u_{in}=0$, $u_{ij}\neq 0$. There are four cases to be considered according as $\xi_{ik}-\lambda_k$ and $\lambda_j-\xi_{i,j-1}$ are each nonzero for exactly one or two values of $i$. Moreover since $j-1\geqslant k+1$, it follows from these remarks that when $\xi_{ik}-\lambda_k\neq 0$ we have $\lambda_j-\xi_{i,j-1}=0$ and when $\lambda_j-\xi_{i,j-1}\neq 0$ we have $\xi_{ik}-\lambda_k=0$.

CASE (i). Let $\xi_{ik}-\lambda_k\neq 0$ for exactly one value of $i$ and let $\lambda_j-\xi_{i,j-1}\neq 0$ for exactly one value of $i$. We may let $\xi_{ik}=\lambda_k$ for $i\neq i_1$ and $\lambda_j=\xi_{i,j-1}$ for $i\neq i_2$. Then $i_1\neq i_2$ and also $u_{ik}=0$ for $i\neq i_1$, $u_{ij}=0$ for $i\neq i_2$. Hence $|u_{i_1k}|=1$, $|u_{i_2j}|=1$, so that $u_{i_1,k+1}=0=u_{i_2,j-1}$. This means that $U$ breaks down into two $1\times 1$ blocks and an $(n-2)\times(n-2)$ block. After passing to

$$PHP^{-1}=(PUQ)(Q^{-1}DQ)(PUQ)^{-1}$$

and changing notation, we may assume

$$H=\lambda_k\dotplus\lambda_j\dotplus H_3 \tag{33}$$

where $H_3$ is $(n-2)$-square with eigenvalues

$$\lambda_1,\ldots,\lambda_{k-1},\lambda_{k+1},\ldots,\lambda_{j-1},\lambda_{j+1},\ldots,\lambda_n. \tag{34}$$

Moreover, we now have $\xi_{ik}=\lambda_k$ for $i\neq 1$, and $\xi_{i,j-1}=\lambda_j$ for $i\neq 2$.

With $H$ in the form (33), the eigenvalues of $H(1|1)$ are $\lambda_1,\ldots,\lambda_{k-1},\lambda_{k+1},\ldots,\lambda_j,\ldots,\lambda_n$, hence $\xi_{1k}=\lambda_{k+1}$ and $\xi_{1,j-1}=\lambda_j$. The eigenvalues of $H(2|2)$ are $\lambda_1,\ldots,\lambda_k,\ldots,\lambda_{j-1},\lambda_{j+1},\ldots,\lambda_n$, hence $\xi_{2k}=\lambda_k$ and $\xi_{2,j-1}=\lambda_{j-1}$. For $i\geqslant 3$, the eigenvalues of $H(i|i)$ are

$$\delta_{i1},\delta_{i2},\ldots,\delta_{i,k-2};\delta_{ik},\lambda_k;\delta_{i,k+1},\ldots,\delta_{i,j-2};\delta_{ij},\lambda_j;\delta_{i,j+1},\ldots,\delta_{i,n-1}. \tag{35}$$

Here the numbers

$$\delta_{i1}, \ldots, \delta_{i,k-2}, \delta_{ik}, \delta_{i,k+1}, \ldots, \delta_{i,j-2}, \delta_{ij}, \delta_{i,j+1}, \ldots, \delta_{i,n-1}$$

interlace the eigenvalues (34) of $H_3$. In (35), the numbers $\delta_{i1}, \ldots, \delta_{i,k-2}, \delta_{ik}$ are missing when $k=1$, and when $k=2$, the numbers $\delta_{i1}, \ldots, \delta_{i,k-2}$ are missing but $\delta_{ik}$ is present. When $j=n$, the numbers $\delta_{ij}, \delta_{i,j+1}, \ldots, \delta_{i,n-1}$ are missing, and when $j=n-1$ the numbers $\delta_{i,j+1}, \ldots, \delta_{i,n-1}$ are missing but $\delta_{ij}$ is present. When $k+1=j-1$ the numbers $\delta_{i,k+1}, \ldots, \delta_{i,j-2}$ are missing but both $\delta_{ik}$ and $\delta_{ij}$ are present.

We now show that with $H$ in the form (33), the equation (30) is valid for $k=1$. For as computed above, $\xi_{11}=\lambda_2$, $\xi_{i1}=\lambda_1$ for all $i>1$, and this is sufficient to imply (30) when $k=1$. We also show that with $H$ in the form (33), the equation (29) is valid for $j=n$. For as computed above, $\xi_{1,n-1}=\lambda_n$, $\xi_{2,n-1}=\lambda_{n-1}$, $\xi_{i,n-1}=\lambda_n$ for $i>2$, and this is sufficient to imply (29) when $j=n$.

For $k\neq 1$, it follows from (35) that $\xi_{ik}=\lambda_k$ for $i>2$ can hold if and only if $\delta_{ik}\leqslant\lambda_k$. For $j\neq n$ it follows from (35) that $\xi_{i,j-1}=\lambda_j$ can hold for $i>2$ if and only if $\delta_{ij}\geqslant\lambda_j$. Moreover if we have $\delta_{ik}\leqslant\lambda_k$ for all $i>2$ (when $k\neq 1$) or $\delta_{ij}\geqslant\lambda_j$ for all $i>2$ (when $j\neq n$), then using the values of $\xi_{ik}$ or $\xi_{i,j-1}$ obtained above, we find that (30) or (29) holds, respectively. By [2, Theorem 1] we can make a unitary similarity of $H_3$ such that $\delta_{ik}\leqslant\lambda_k$ ($k\neq 1$) holds for all $i$ and/or such that $\delta_{ij}\geqslant\lambda_j$ (for $j\neq n$) holds for all $i$, if and only if (36) and/or (37) hold:

$$\begin{cases} \lambda_k \geqslant \text{the root of } \prod_{\substack{t=1\\ t\neq k\\ t\neq j}}^{n} (\lambda-\lambda_t) \text{ in interval } (\lambda_{k-1}, \lambda_{k+1}), & k\neq 1, \\ \text{no condition if } k=1. \end{cases} \tag{36}$$

$$\begin{cases} \lambda_j \leqslant \text{the root of } \prod_{\substack{t=1\\ t\neq k\\ t\neq j}}^{n} (\lambda-\lambda_t) \text{ in interval } (\lambda_{j-1}, \lambda_{j+1}), & j\neq n, \\ \text{no condition if } j=n. \end{cases} \tag{37}$$

Thus in Case (i) the conditions (36) and (37) are the necessary and sufficient conditions for (29) and (30) both to be achievable.

We place here the first part of Corollary 4. Further parts of Corollary 4 will appear as the proof of Theorem 6 goes forward.

COROLLARY 4. (i) *Let* $\lambda_k$ *and* $\lambda_j$ *satisfy the conditions* (36) *and* (37). *Then both* (29) *and* (30) *hold when* $H=P(\lambda_k \dotplus \lambda_j \dotplus H_3)P^{-1}$, *where* $H_3$ *has eigenvalues* (34), *and each principal* $(n-3)$*-square submatrix of* $H_3$ *has its eigenvalues belonging to the intervals* $[\lambda_{k-1}, \lambda_{k+1}]$ *(for* $k\neq 1$*)*, $[\lambda_{j-1}, \lambda_{j+1}]$ *(for* $j\neq n$*) within the smaller intervals* $[\lambda_{k-1}, \lambda_k]$, $[\lambda_j, \lambda_{j+1}]$, *respectively.*

CASE (ii). Here we assume that $\xi_{ik}-\lambda_k\neq 0$ for exactly two values of $i$, say $i=i_1$, and $i=i_2$, and that $\lambda_j-\xi_{i,j-1}\neq 0$ for exactly one value of $i$, say $i=i_3$. By remarks at the beginning of the proof, $i_1$, $i_2$, $i_3$ are three distinct integers. We have $|u_{i_3 j}|=1$ as $u_{ij}=0$ for $i\neq i_3$, hence $u_{i_3 t}=0$ for $t\neq j$. Thus $U$ splits up into three nonzero blocks: a $2\times 2$ block in rows $i_1$, $i_2$ and columns $k$, $k+1$; a $1\times 1$ block at position $(i_3, j)$; and $(n-3)\times(n-3)$ block in the rows complementary to $i_1$, $i_2$, $i_3$ and the columns complementary to $k$, $k+1$, $j$. After passing to $PHP^{-1}=(PUQ)(Q^{-1}DQ)(PUQ)^{-1}$, and changing notation, we may assume

$$H=H_1 \dotplus \lambda_j \dotplus H_3 \tag{38}$$

where $H_1$ has eigenvalues $\lambda_k$, $\lambda_{k+1}$, and $H_3$ has eigenvalues

$$\lambda_1, \ldots, \lambda_{k-1}, \lambda_{k+2}, \ldots, \lambda_{j-1}, \lambda_{j+1}, \ldots, \lambda_n. \tag{39}$$

Moreover we now have $\xi_{ik}=\lambda_k$ for $i>2$ and $\xi_{i,\,j-1}=\lambda_j$ for $i\neq 3$. Let $H_1$ be given by (17). Then $\lambda_k \leq a$, $c \leq \lambda_{k+1}$ and $a+c=\lambda_k+\lambda_{k+1}$.

Observe that, with $H$ given by (38), we have $\xi_{1k}=c$, $\xi_{2k}=a$, $\xi_{1,\,j-1}=\lambda_j$, $\xi_{2,\,j-1}=\lambda_j$, $\xi_{3k}=\lambda_k$, $\xi_{3,\,j-1}=\lambda_{j-1}$. For $i>3$ the eigenvalues of $H(i|i)$ are

$$\delta_{i1}, \ldots, \delta_{i,\,k-2}; \delta_{ik}, \lambda_k, \lambda_{k+1}; \delta_{i,\,k+2}, \ldots, \delta_{i,\,j-2}; \delta_{ij}, \lambda_j; \delta_{i,\,j+1}, \ldots, \delta_{i,\,n-1}. \tag{40}$$

Here the numbers

$$\delta_{i1}, \ldots, \delta_{i,\,k-2}, \delta_{ik}, \delta_{i,\,k+2}, \ldots, \delta_{i,\,j-2}, \delta_{ij}, \delta_{i,\,j+1}, \ldots, \delta_{i,\,n-1}$$

interlace the numbers (39). When $k=1$, the numbers $\delta_{i1}, \ldots, \delta_{i,\,k-2}, \delta_{ik}$ are absent from (40). When $k=2$, the numbers $\delta_{i1}, \ldots, \delta_{i,\,k-2}$ are absent from (40) but $\delta_{ik}$ is present. When $j=n$, the numbers $\delta_{ij}, \delta_{i,\,j+1}, \ldots, \delta_{i,\,n-1}$ are absent from (40). When $j=n-1$, the numbers $\delta_{i,\,j+1}, \ldots, \delta_{i,\,n-1}$ are absent from (40) but $\delta_{ij}$ is present. When $j-1>k+2$ the central set $\delta_{i,\,k+2}, \ldots, \delta_{i,\,j-2}$ is present in (40). When $j-1=k+2$ the central set $\delta_{i,\,k+2}, \ldots, \delta_{i,\,j-2}$ are absent but $\delta_{ik}$ and $\delta_{ij}$ are separate entities, both being present (except: $\delta_{ik}$ is absent if $k=1$ and $\delta_{ij}$ is absent if $j=n$). When $j-1=k+1$ the central set $\delta_{i,\,k+2}, \ldots, \delta_{i,\,j-2}$ is absent, and $\delta_{ik}$ and $\delta_{ij}$ merge into one entity, $\delta_{ik}=\delta_{ij}$ being the eigenvalue of $H_3(i-3|i-3)$ belonging to the interval $[\lambda_{k-1}, \lambda_{j+1}]$ (except that for $k=1$ or $j=n$, $\delta_{ik}=\delta_{ij}$ is absent).

We now show that the equation (30) for $k=1$ is valid when $H$ has the form (38). This follows from the values $\xi_{11}, \xi_{21}, \xi_{31}$ computed above, the fact that $a+c=\lambda_1+\lambda_2$, and the fact that (from (40)), $\xi_{i1}=\lambda_1$ for all $i>3$. We also show that the equation (29) for $j=n$ is valid when $H$ has the form (38). This follows from the values of $\xi_{1,\,n-1}, \xi_{2,\,n-1}, \xi_{3,\,n-1}$ computed above, and the fact that (from (40)) $\xi_{i,\,n-1}=\lambda_n$ for all $i>3$.

Now note that when $n=3$ we in fact have $k=1$ and $j=3=n$. This is so since $j \geq k+2$. So assume $n>3$.

We now prove (assuming $n>3$) that $j \neq k+2$. We use (40). If $j-1=k+1$ we have: when $k=1$, $\xi_{i,\,j-1}=\xi_{i2}=\lambda_2=\lambda_{j-1}$ contrary to the requirement $\xi_{i,\,j-1}=\lambda_j$ for $i>3$; when $j=n$, $\xi_{ik}=\xi_{i,\,n-2}=\lambda_{k+1}$ contrary to the requirement $\xi_{ik}=\lambda_k$ for $i>3$; when $k\neq 1$ and $j \neq n$ in order to meet the requirement $\xi_{ik}=\lambda_k$ we must have (from (40)) that $\delta_{ik}\leq\lambda_k$, and then $\xi_{i,\,j-1}=\lambda_{k+1}\neq\lambda_j$, thus contradicting $\xi_{i,\,j-1}=\lambda_j$ for $i>3$. So we may assume $j-1>k+1$ and that $\delta_{ik}$ and $\delta_{ij}$ are distinct entities.

For $k\neq 1$ it now follows from (40) that in order for $\xi_{ik}=\lambda_k$ for $i>3$ it is necessary and sufficient that $\delta_{ik}\leq\lambda_k$. For $j\neq n$ it follows from (40) that in order for $\xi_{i,\,j-1}=\lambda_j$ for $i>3$ it is necessary and sufficient that $\delta_{ij}\geq\lambda_j$. Moreover with $H$ in the form (38) and using values $\xi_{1k}, \xi_{2k}, \xi_{3k}, \xi_{1,\,j-1}, \xi_{2,\,j-1}, \xi_{3,\,j-1}$ obtained above, if $\xi_{ik}=\lambda_k$ for all $i>3$, $(k\neq 1)$, we find that (30) is satisfied and if $\xi_{i,\,j-1}=\lambda_j$ for all $i>3$, $(j\neq n)$, we find that (29) is satisfied. Now by [1, Theorem 1] it follows that $\delta_{ik}\leq\lambda_k$ (for $k\neq 1$) and/or $\lambda_j\leq\delta_{ij}$ (for $j\neq n$) can be achieved by some matrix unitarily similar to $H_3$, if and only if

$$\begin{cases} \lambda_k \geq \text{the root of } \dfrac{d}{d\lambda}\displaystyle\prod_{\substack{t=1\\ t\neq k,\,k+1,\,j}}^{n}(\lambda-\lambda_t) \text{ in interval } (\lambda_{k-1}, \lambda_{k+2}),\ k\neq 1, \\ \text{no condition if } k=1, \end{cases} \tag{41}$$

and/or

$$\begin{cases} \lambda_j \leq \text{the root of } \dfrac{d}{d\lambda}\displaystyle\prod_{\substack{t=1\\ t\neq k,\,k+1,\,j}}^{n}(\lambda-\lambda_t) \text{ in interval } (\lambda_{j-1}, \lambda_{j+1}),\ j\neq n, \\ \text{no condition if } j=n. \end{cases} \tag{42}$$



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We have thus demonstrated that in Case (ii) the conditions (41) and (42), together with $j \neq k+2$ for $n > 3$, are the necessary and sufficient conditions in order that (29) and (30) can both hold. We now complete the proof in Case (ii) by showing that for $k \neq 1$ and $j > k+2$,

$$\text{the root of } \frac{d}{d\lambda} \prod_{\substack{t=1 \\ t\neq k,\, k+1,\, j}}^{n} (\lambda-\lambda_t) \text{ in interval } (\lambda_{k-1}, \lambda_{k+2})$$

$$> \text{the root of } \frac{d}{d\lambda} \prod_{\substack{t=1 \\ t\neq k,\, j}}^{n} (\lambda-\lambda_t) \text{ in interval } (\lambda_{k-1}, \lambda_{k+1}) \tag{43}$$

and for $j \neq n$ and $j > k+2$,

$$\text{the root of } \frac{d}{d\lambda} \prod_{\substack{t=1 \\ t\neq k,\, k+1,\, j}}^{n} (\lambda-\lambda_t) \text{ in interval } (\lambda_{j-1}, \lambda_{j+1})$$

$$< \text{the root of } \frac{d}{d\lambda} \prod_{\substack{t=1 \\ t\neq k,\, j}}^{n} (\lambda-\lambda_t) \text{ in interval } (\lambda_{j-1}, \lambda_{j+1}). \tag{44}$$

If we can demonstrate (43), (44) then the conditions (41), (42) of Case (ii) imply the conditions (36) and (37) in Case (i).

Let

$$g(\lambda) = \prod_{\substack{t=1 \\ t\neq k,\, j}}^{n} (\lambda-\lambda_t), \qquad h(\lambda) = \prod_{\substack{t=1 \\ t\neq k,\, k+1,\, j}}^{n} (\lambda-\lambda_t).$$

Then $g(\lambda) = (\lambda-\lambda_{k+1})h(\lambda)$. We use the fact that $k+2 \leqslant j-1$. Let $\gamma$ be the root of $h'(\lambda)$ in the interval $(\lambda_{k-1}, \lambda_{k+2})$. Then $g'(\gamma) = h(\gamma)$ and so $\operatorname{sgn} g'(\gamma) = \operatorname{sgn} h(\gamma) = -\operatorname{sgn} g(\lambda_k)$, which forces $\gamma >$ root of $g'(\lambda)$ in interval $(\lambda_{k-1}, \lambda_{k+1})$. This proves (43). Let $\beta$ be the root of $h'(\lambda)$ in the interval $(\lambda_{j-1}, \lambda_{j+1})$. Then $g'(\beta) = h(\beta)$, hence $\operatorname{sgn} g'(\beta) = \operatorname{sgn} h(\beta) = \operatorname{sgn} g(\beta)$, which forces $\beta <$ root of $g'(\lambda)$ in interval $(\lambda_{j-1}, \lambda_{j+1})$. This proves (44) and completes Case (ii).

COROLLARY 4. (ii) *Let* $j > k+2$ *except* $k=1$, $j=3$ *for* $n=3$. *Then the condition* (41) *for* $k \neq 1$ *is stronger than the condition* (36) *and the condition* (42) *for* $j \neq n$ *is stronger than the condition* (37). *If both* (41) *and* (42) *are satisfied then equalities* (29) *and* (30) *are both valid when* $H = P(H_1 \dotplus \lambda_j \dotplus H_3 P^{-1}$, *when* $H_1$ *has eigenvalues* $\lambda_k$, $\lambda_{k+1}$, $H_3$ *has eigenvalues* (39), *and each principal* $(n-4)$*-square submatrix of* $H_3$ *has its eigenvalues belonging to the intervals* $[\lambda_{k-1}, \lambda_{k+2}]$ (*for* $k \neq 1$), $[\lambda_{j-1}, \lambda_{j+1}]$ (*for* $j \neq n$) *within the smaller intervals* $[\lambda_{k-1}, \lambda_k]$, $[\lambda_j, \lambda_{j+1}]$, *respectively.*

CASE (iii). Here we assume $\xi_{ik} - \lambda_k \neq 0$ for exactly one value of $i$ and $\lambda_j - \xi_{i,j-1} \neq 0$ for exactly two values of $i$. This case is very similar to Case (ii). One gets $H$ into the form

$$H = P(\lambda_k \dotplus H_2 \dotplus H_3)P^{-1} \tag{45}$$

where $H_2$ has eigenvalues $\lambda_{j-1}$, $\lambda_j$ and $H_3$ has eigenvalues

$$\lambda_1, \ldots, \lambda_{k-1}, \lambda_{k+1}, \ldots, \lambda_{j-2}, \lambda_{j+1}, \ldots, \lambda_n. \tag{45$'$}$$

One proves that $j > k+2$ except for $k=1$, $j=3$ when $n=3$. In place of (41) and (42) one obtains

$$\begin{cases} \lambda_k \geqslant \text{the root of } \dfrac{d}{d\lambda} \displaystyle\prod_{\substack{t=1 \\ t\neq k,\, j-1,\, j}}^{n} (\lambda-\lambda_t) \text{ in interval } (\lambda_{k-1}, \lambda_{k+1}), \quad k \neq 1, \\ \text{no condition if } k=1; \end{cases} \tag{46}$$

$$\begin{cases} \lambda_j \leq \text{the root of } \dfrac{d}{d\lambda} \prod\limits_{\substack{t=1 \\ t\neq k,\, j-1,\, j}}^{n} (\lambda-\lambda_t) \text{ in interval } (\lambda_{j-2}, \lambda_{j+1}), \qquad j \neq n, \\ \text{no condition if } j=n; \end{cases} \tag{47}$$

and in place of (43), (44), one obtains (48) for $k \neq 1$ and (49) for $j \neq n$:

$$\text{the root of } \frac{d}{d\lambda} \prod_{\substack{t=1 \\ t\neq k,\, j-1,\, j}}^{n} (\lambda-\lambda_t) \text{ in interval } (\lambda_{k-1}, \lambda_{k+1})$$

$$> \text{the root of } \frac{d}{d\lambda} \prod_{\substack{t=1 \\ t\neq k,\, j}}^{n} (\lambda-\lambda_t) \text{ in interval } (\lambda_{k-1}, \lambda_{k+1}); \tag{48}$$

$$\text{the root of } \frac{d}{d\lambda} \prod_{\substack{t=1 \\ t\neq k,\, j-1,\, j}}^{n} (\lambda-\lambda_t) \text{ in interval } (\lambda_{j-2}, \lambda_{j+1})$$

$$< \text{the root of } \frac{d}{d\lambda} \prod_{\substack{t=1 \\ t\neq k,\, j}}^{n} (\lambda-\lambda_t) \text{ in interval } (\lambda_{j-1}, \lambda_{j+1}). \tag{49}$$

COROLLARY 4. (iii) *Let* $j > k+2$ *except* $k=1$, $j=3$ *for* $n=3$. *Then the condition* (46) *for* $k \neq 1$ *is stronger than the condition* (36) *and the condition* (47) *for* $j \neq n$ *is stronger than the condition* (37). *If both* (46) *and* (47) *are satisfied then equalities* (29) *and* (30) *both are valid when* H *has the form* (45), *where* $H_2$ *has eigenvalues* $\lambda_{j-1}$, $\lambda_j$, $H_3$ *has eigenvalues* (45′), *and where each principal* $(n-4)$*-square submatrix of* $H_3$ *has its eigenvalues belonging to the intervals* $[\lambda_{k-1}, \lambda_{k+1}]$ (*for* $k \neq 1$), $[\lambda_{j-2}, \lambda_{j+1}]$ (*for* $j \neq n$) *within the smaller intervals* $[\lambda_{k-1}, \lambda_k]$, $[\lambda_j, \lambda_{j+1}]$, *respectively*.

CASE (iv). In this case we have $\xi_{ik}-\lambda_k \neq 0$ for exactly two values of $i$ and $\lambda_j-\xi_{i,j-1} \neq 0$ for exactly two values of $i$. Moreover, as before, when $\xi_{ik}-\lambda_k \neq 0$ then $\lambda_j-\xi_{i,j-1}=0$ and when $\lambda_j-\xi_{i,j-1} \neq 0$ then $\xi_{ik}-\lambda_k=0$. So we may find four integers $i_1$, $i_2$, $i_3$, $i_4$ such that $\xi_{ik}-\lambda_k=0$ for $i \neq i_1, i_2$ and $\lambda_j-\xi_{i,j-1}=0$ for $i \neq i_3, i_4$. We then have $u_{i_1t}=0$ for $t \neq k, k+1$; $u_{i_2t}=0$ for $t \neq k, k+1$; $u_{i_3t}=0$ for $t \neq j-1, j$; $u_{i_4t}=0$ for $t \neq j-1, j$. We show: $k+1 \neq j-1$. For if $k+1=j-1$, rows $i_1$, $i_2$, $i_3$, $i_4$ of $U$ are zero except for column positions $k$, $k+1$, $j$, hence these four rows behave as 3-tuples, hence are linearly dependent. This contradiction shows that $k+1 \neq j-1$, hence $k+1 < j-1$. Now rows $i_1$, $i_2$ of $U$ are zero except for the $2 \times 2$ submatrix sitting in rows $i_1$, $i_2$ and columns $k$, $k+1$. Thus, as $U$ is unitary, columns $k$, $k+1$ are also zero except for rows $i_1$, $i_2$. Similarly rows $i_3$, $i_4$ and columns $j-1$, $j$ are zero except for the $2 \times 2$ submatrix sitting at the intersection of these rows and columns.

Thus we may pass to $PHP^{-1}=(PUQ)(Q^{-1}DQ)(PUQ)^{-1}$ and so, after a change of notation, assume

$$H=H_1+H_2+H_3 \tag{50}$$

where: $H_1$ has eigenvalues $\lambda_k$, $\lambda_{k+1}$; $H_2$ has eigenvalues $\lambda_{j-1}$, $\lambda_j$; $H_3$ has eigenvalues

$$\lambda_1, \ldots, \lambda_{k-1}, \lambda_{k+2}, \ldots, \lambda_{j-2}, \lambda_{j+1}, \ldots, \lambda_n. \tag{51}$$

We now have $\xi_{ik}=\lambda_k$ and $\xi_{i,j-1}=\lambda_j$ for $i>4$. Let $H_1$ be given by (17) and $H_2$ by

$$H_2=\begin{bmatrix} a' & b' \\ \bar{b}' & c' \end{bmatrix}.$$

Assuming $H$ in the form (50), one easily sees that $\xi_{1k}=c$, $\xi_{1,j-1}=\lambda_j$, $\xi_{2k}=a$, $\xi_{2,j-1}=\lambda_j$, $\xi_{3k}=\lambda_k$, $\xi_{3,j-1}=c'$, $\xi_{4k}=\lambda_k$, $\xi_{4,j-1}=a'$.

For $i>4$, the eigenvalues of $H(i|i)$ are

$$\delta_{i1},\ldots,\delta_{i,k-2};\ \delta_{ik},\ \lambda_k,\ \lambda_{k+1};\ \delta_{i,k+2},\ldots,\delta_{i,j-3};\ \delta_{ij},\ \lambda_{j-1},\ \lambda_j;\ \delta_{i,j+1},\ldots,\delta_{i,n-1}, \tag{52}$$

where

$$\delta_{i1},\ldots,\delta_{i,k-2},\ \delta_{ik},\ \delta_{i,k+2},\ldots,\delta_{i,j-3},\ \delta_{ij},\ \delta_{i,j+1},\ldots,\delta_{i,n-1}$$

interlace the numbers (51). When $k=1$, the numbers $\delta_{i1},\ldots,\delta_{i,k-2}$, $\delta_{ik}$ are absent in (52). When $k=2$ the numbers $\delta_{i1},\ldots,\delta_{i,k-2}$ are absent but $\delta_{ik}$ is present. When $j=n$ the numbers $\delta_{ij}$, $\delta_{i,j+1},\ldots,\delta_{i,n-1}$ are absent. When $j=n-1$ the numbers $\delta_{i,j+1},\ldots,\delta_{i,n-1}$ are absent but $\delta_{ij}$ is present. When $j=k+4$ the numbers $\delta_{i,k+2},\ldots,\delta_{i,j-3}$ are absent but $\delta_{ik}$ and $\delta_{ij}$ are present as separate entities (except: $\delta_{ik}$ is absent if $k=1$ and $\delta_{ij}$ is absent if $j=n$). When $j=k+3$ the numbers $\delta_{i,k+2},\ldots,\delta_{i,j-3}$ are absent and $\delta_{ik}=\delta_{ij}$ appear as a single entity (except that $\delta_{ik}=\delta_{ij}$ is absent if $k=1$ or $j=n$).

We now show that when $H$ has the form (50), equation (30) for $k=1$ is valid. This follows from the values of $\xi_{11},\ldots,\xi_{41}$ computed above, the fact that $a+c=\lambda_1+\lambda_2$, and the fact that $\xi_{i1}=\lambda_1$ for $i>4$ (which follows from (52) when $k=1$). We also show that equation (29) for $j=n$ is valid. This follows from the values of $\xi_{1,n-1},\ldots,\xi_{4,n-1}$ computed above, and the fact that $a'+c'=\lambda_{n-1}+\lambda_n$, and the fact that $\xi_{i,n-1}=\lambda_n$ for $i>4$ (which follows from (52) when $j=n$).

Observe also that when $n=4$ we in fact have $k=1$ and $j=4=n$, since $j\geqslant k+3$. So assume $n>4$.

We now show that $j=k+3$ is impossible for $n>4$. For let $j=k+3$. We use (52). If $k=1$ then $\xi_{i,j-1}=\xi_{i,3}=\lambda_3\neq\lambda_4$ contradicting the requirement that $\xi_{i,j-1}=\lambda_j$ for $i>4$. If $j=n$ then $\xi_{ik}=\xi_{i,n-3}=\lambda_{k+1}\neq\lambda_k$ contradicting the requirement that $\xi_{ik}=\lambda_k$ for $i>4$. If $k\neq 1$, $j\neq n$, then in order that $\xi_{ik}=\lambda_k$ we must have $\delta_{ik}\leqslant\lambda_k$, and then $\xi_{i,j-1}=\xi_{i,k+2}=\lambda_{j-1}\neq\lambda_j$, again a contradiction for $i>4$. Thus $j\geqslant k+4$ as claimed, and so $\delta_{ik}$ and $\delta_{ij}$ are separate entities in (52).

For $k\neq 1$ we find from (52) that $\xi_{ik}=\lambda_k$ holds for $i>4$ if and only if $\delta_{ik}\leqslant\lambda_k$. For $j\neq n$ we find from (52) that $\xi_{i,j-1}=\lambda_j$ holds for $i>4$ if and only if $\delta_{ij}\geqslant\lambda_j$. Moreover, using values of $\xi_{1k},\ldots,\xi_{4k}$, $\xi_{1,j-1},\ldots,\xi_{4,j-1}$ obtained above and $a+c=\lambda_k+\lambda_{k+1}$, $a'+c'=\lambda_{j-1}+\lambda_j$, we find for $k\neq 1$ or for $j\neq n$ that if $\delta_{ik}\leqslant\lambda_k$ for all $i>4$ or $\delta_{ij}\geqslant\lambda_j$ for all $i>4$, then (30) or (29) are satisfied, respectively.

Now by an application of [2, Theorem 1] to $H_3$, the necessary and sufficient conditions that $H_3$ be unitarily similar to a matrix for which $\delta_{ik}\leqslant\lambda_k$ $(k\neq 1)$ and/or $\delta_{ij}\geqslant\lambda_j$ $(j\neq n)$ are the following conditions (53) and/or (54):

$$\begin{cases}\lambda_k\geqslant \text{the root of } \dfrac{d}{d\lambda}\displaystyle\prod_{\substack{t=1\\ t\neq k,\,k+1\\ t\neq j-1,\,j}}^{n}(\lambda-\lambda_t) \text{ in the interval } (\lambda_{k-1},\lambda_{k+2}); \qquad k\neq 1,\\[2ex] \text{no condition if } k=1,\end{cases} \tag{53}$$

$$\begin{cases}\lambda_j\leqslant \text{the root of } \dfrac{d}{d\lambda}\displaystyle\prod_{\substack{t=1\\ t\neq k,\,k+1\\ t\neq j-1,\,j}}^{n}(\lambda-\lambda_t) \text{ in the interval } (\lambda_{j-2},\lambda_{j+1}); \qquad j\neq n,\\[2ex] \text{no condition if } j=n.\end{cases} \tag{54}$$

Thus the necessary and sufficient conditions in Case (iv) that (29) and (30) can both hold are (53), (54), together with $j\geqslant k+4$ for $n>4$ and $k=1$, $j=4$ for $n=4$.

We now show for $j \geqslant k+4$:

$$\text{the root of } \frac{d}{d\lambda} \prod_{\substack{t=1 \\ t\neq k,\, k+1 \\ t\neq j-1,\, j}}^{n} (\lambda-\lambda_t) \text{ in interval } (\lambda_{k-1}, \lambda_{k+2})$$

$$> \text{root of } \frac{d}{d\lambda} \prod_{\substack{t=1 \\ t\neq k,\, j}}^{n} (\lambda-\lambda_t) \text{ in interval } (\lambda_{k-1}, \lambda_{k+1}); \qquad k \neq 1; \tag{55}$$

$$\text{root of } \frac{d}{d\lambda} \prod_{\substack{t=1 \\ t\neq k,\, k+1 \\ t\neq j-1,\, j}}^{n} (\lambda-\lambda_t) \text{ in interval } (\lambda_{j-2}, \lambda_{j+1})$$

$$< \text{root of } \frac{d}{d\lambda} \prod_{\substack{t=1 \\ t\neq k,\, j}}^{n} (\lambda-\lambda_t) \text{ in interval } (\lambda_{j-1}, \lambda_{j+1}), \qquad j \neq n. \tag{56}$$

Once (55), (56) are demonstrated it will follow that the conditions of Case (iv) imply the conditions (36), (37) of Case (i), and this will complete the demonstration that (36) and (37) are necessary and sufficient.

Let

$$g(\lambda) = \prod_{\substack{t=1 \\ t\neq k,\, j}}^{n} (\lambda-\lambda_t), \qquad h(\lambda) = \prod_{\substack{t=1 \\ t\neq k,\, k+1 \\ t\neq j-1,\, j}}^{n} (\lambda-\lambda_t).$$

Then $g(\lambda)=(\lambda-\lambda_{k+1})(\lambda-\lambda_{j-1})h(\lambda)$. Let $\gamma$ be the root of $h'(\lambda)$ in the interval $(\lambda_{k-1}, \lambda_{k+2})$. If $\gamma \geqslant \lambda_{k+1}$ then (55) holds. If $\gamma < \lambda_{k+1}$ then

$$\operatorname{sgn} \frac{d}{d\lambda} (\lambda-\lambda_{k+1})(\lambda-\lambda_{j-1}) \Big|_{\lambda=\gamma} = -1,$$

hence $\operatorname{sgn} g'(\gamma) = -\operatorname{sgn} h(\gamma) = -\operatorname{sgn} g(\gamma)$, which implies (55). Thus (55) holds. Let $\beta$ be the root of $h'(\lambda)$ in the interval $(\lambda_{j-2}, \lambda_{j+1})$. If $\beta \leqslant \lambda_{j-1}$ then (56) holds. If $\beta > \lambda_{j-1}$ then

$$\operatorname{sgn} \frac{d}{d\lambda} (\lambda-\lambda_{k+1})(\lambda-\lambda_{j-1}) \Big|_{\lambda=\beta} = +1,$$

hence $\operatorname{sgn} g'(\gamma) = \operatorname{sgn} h(\gamma) = \operatorname{sgn} g(\gamma)$, which implies (56). This completes Case (iv).

COROLLARY 4. (iv) *Let* $j \geqslant k+4$ *except* $k=1$, $j=4$ *for* $n=4$. *Then the condition* (53) *for* $k \neq 1$ *is stronger than the condition* (36) *and the condition* (54) *for* $j \neq n$ *is stronger than the condition* (37). *If both* (53) *and* (54) *are satisfied then equalities* (29) *and* (30) *both hold when* $H = P(H_1 \dot{+} H_2 \dot{+} H_3)P^{-1}$, *where* $H_1$ *has eigenvalues* $\lambda_k$, $\lambda_{k+1}$, $H_2$ *has eigenvalues* $\lambda_{j-1}$, $\lambda_j$, $H_3$ *has eigenvalues* (51), *and where each principal* $(n-5)$*-square submatrix of* $H_3$ *has its eigenvalues belonging to the intervals* $[\lambda_{k-1}, \lambda_{k+2}]$ (*for* $k \neq 1$), $[\lambda_{j-2}, \lambda_{j+1}]$ (*for* $j \neq n$) *within the smaller intervals* $[\lambda_{k-1}, \lambda_k]$, $[\lambda_j, \lambda_{j+1}]$, *respectively.*

(v) *Except as described in Corollary* 4, (i)-(iv), *the equalities* (29), (30) *do not both hold.*

To complete the proof of Theorem 6, we have to show that conditions (36) and (37) are equivalent to (31) and (32). This proof follows.

By a graphical argument one sees, for $k \neq 1$, that the condition (36) is equivalent to

$$\{f(\lambda)(\lambda-\lambda_k)^{-1}(\lambda-\lambda_j)^{-1}\}'\{f(\lambda)(\lambda-\lambda_k)^{-1}(\lambda-\lambda_j)^{-1}\}\,\big|_{\lambda=\lambda_k} \leqslant 0.$$

But

$$\{f(\lambda)(\lambda-\lambda_j)^{-1}\}'\Big|_{\lambda=\lambda_k}=\prod_{\substack{\nu=1\\ \nu\neq j\\ \nu\neq k}}^{n}(\lambda_k-\lambda_\nu)=f(\lambda)(\lambda-\lambda_k)^{-1}(\lambda-\lambda_j)^{-1}\Big|_{\lambda=\lambda_k}.$$

Also

$$\{f(\lambda)(\lambda-\lambda_j)^{-1}\}''\Big|_{\lambda=\lambda_k}=2\sum_{\substack{\mu=1\\ \mu\neq j\\ \mu\neq \kappa}}^{n}\prod_{\substack{\nu=1\\ \nu\neq\mu\\ \nu\neq j\\ \nu\neq k}}^{n}(\lambda_k-\lambda_\nu)$$

$$=\{2f(\lambda)(\lambda-\lambda_k)^{-1}(\lambda-\lambda_j)^{-1}\}'\Big|_{\lambda=\lambda_k}.$$

But

$$\{f(\lambda)(\lambda-\lambda_j)^{-1}\}'\Big|_{\lambda=\lambda_k}=f'(\lambda_k)(\lambda_k-\lambda_j)^{-1},$$

$$\{f(\lambda)(\lambda-\lambda_j)^{-1}\}''\Big|_{\lambda=\lambda_k}=\{(\lambda_k-\lambda_j)f''(\lambda_k)-2f'(\lambda_k)(\lambda_k-\lambda_j)^{-2}.$$

Combining all these facts and using $\lambda_k-\lambda_j<0$, the equivalence of (36) and (31) follows. Similarly one establishes the equivalence of (37) and (32). This completes the proof of Theorem 6.

COROLLARY 5. *Let* $\mathrm{t}\neq 1$, n. *If for some* $\mathrm{s}\geq \mathrm{t}+2$ *we have simultaneously*

$$_{\mathrm{t}}\mathrm{A}_{\mathrm{t}+1}=(\mathrm{n}-1)\mathrm{n}^{-1}\lambda_{\mathrm{t}}+\mathrm{n}^{-1}\lambda_{\mathrm{t}+1},$$

$$_{\mathrm{s}-1}\mathrm{A}_{\mathrm{s}}=(\mathrm{n}-1)\mathrm{n}^{-1}\lambda_{\mathrm{s}}+\mathrm{n}^{-1}\lambda_{\mathrm{s}-1}, \qquad (57)$$

*then for* $\mathrm{s}_1\leq \mathrm{t}-2$ *we never have simultaneously*

$$_{\mathrm{t}-1}\mathrm{A}_{\mathrm{t}}=(\mathrm{n}-1)\mathrm{n}^{-1}\lambda_{\mathrm{t}}+\mathrm{n}^{-1}\lambda_{\mathrm{t}-1},$$

$$_{\mathrm{s}_1}\mathrm{A}_{\mathrm{s}_1+1}=(\mathrm{n}-1)\mathrm{n}^{-1}\lambda_{\mathrm{s}_1}+\mathrm{n}^{-1}\lambda_{\mathrm{s}_1+1}, \qquad (58)$$

*If for some* $\mathrm{s}_1\leq \mathrm{t}-2$ *we have* (58) *simultaneously then for* $\mathrm{s}\geq\mathrm{t}+2$ *we never have* (57) *simultaneously. If* $\mathrm{f}''(\lambda_{\mathrm{t}})=0$ *then for no* $\mathrm{s}\geq\mathrm{t}+2$ *can* (57) *both hold and for no* $\mathrm{s}_1\leq\mathrm{t}-2$ *can* (58) *both hold.*

PROOF. If both of eqs (57) hold then by Theorem 6 we have

$$\{(\lambda_t-\lambda_s)f''(\lambda_t)-2f'(\lambda_t)\}f'(\lambda_t)\geq 0, \qquad (59)$$

and if both of eqs (58) hold then by Theorem 6 we have

$$\{(\lambda_t-\lambda_{s_1})f''(\lambda_t)-2f'(\lambda_t)\}f'(\lambda_t)\geq 0. \qquad (60)$$

If $f'(\lambda_t)>0$ we obtain the contradiction

$$0<2f'(\lambda_t)(\lambda_t-\lambda_{s_1})^{-1}\leq f''(\lambda_t)\leq 2f'(\lambda_t)(\lambda_t-\lambda_s)^{-1}<0$$

and a similar contradiction is obtained if $f'(\lambda_t)<0$. If $f''(\lambda_t)=0$ it is clear that both (59) and (60) are false.

THEOREM 7. *Let* $1 < j < k \leq n$. *Then*

$$ {}_{j-1}A_j = (n-1)n^{-1}\lambda_j + n^{-1}\lambda_{j-1}, \tag{61} $$

$$ {}_{k-1}A_k = (n-1)n^{-1}\lambda_k + n^{-1}\lambda_{k-1} \tag{62} $$

*never happen simultaneously.*

PROOF. If (62) holds then from $\xi_{i,k-1} \neq \lambda_k$ we get $\xi_{i,j-1} = \lambda_{j-1}$, hence $\lambda_j - \xi_{i,j-1} \neq 0$, hence from (61) follows $\xi_{i,k-1} = \lambda_k$. We have a contradiction. This completes the proof.

Similarly one proves Theorem 8.

THEOREM 8. *Let* $1 \leq j < k < n$. *Then*

$$ {}_jA_{j+1} = (n-1)n^{-1}\lambda_j + n^{-1}\lambda_{j+1}, \tag{63} $$

$$ {}_kA_{k+1} = (n-1)n^{-1}\lambda_k + n^{-1}\lambda_{k+1} \tag{64} $$

*never happen simultaneously.*

Combining several of our results we have Theorem 9.

THEOREM 9. *In the linear inequalities the maximum number of equality signs that can appear for a given* U *is two.*

PROOF. If more than two equality signs appeared in the set of linear inequalities, there would have to be a pair of the types (61), (62), or a pair of the types (63), (64). This shows the maximum number of equality signs for any given $U$ is $\leq 2$. Theorem 6 shows that for any given $H$ we can achieve two equality signs for an appropriate $U$. In particular we can always achieve ${}_{n-1}A_n = (n-1)n^{-1}\lambda_n + n^{-1}\lambda_{n-1}$ and ${}_1A_2 = (n-1)n^{-1}\lambda_1 + n^{-1}\lambda_2$ simultaneously.

This completes our discussion of simultaneous equalities in the linear inequalities.

## 5. Conditions Under Which Each ${}_{j-1}A_j$ Covers Its Full Interval of Permissible Values as $U$ Varies

In section 5 we do not require that $\lambda_1, \ldots, \lambda_n$ be distinct. The linear inequalities (2) are valid for matrices with nondistinct eigenvalues.

THEOREM 10. *Suppose that, as* U *varies over all unitary matrices, each interval*

$$ [(n-1)n^{-1}\lambda_{j-1} + n^{-1}\lambda_j,\ n^{-1}\lambda_{j-1} + (n-1)n^{-1}\lambda_j] $$

*is completely filled out by* ${}_{j-1}A_j$, $j = 2, 3, \ldots, n$. *Then*

(i) H *is scalar; or*

(ii) H *has eigenvalues* $a + b\gamma_i$, $i = 1, 2, \ldots, n$, a, b *real constants, where* $\gamma_1, \gamma_2, \ldots, \gamma_n$ *are the roots of the polynomial*

$$ (\lambda^2 - 1)\frac{d}{d\lambda}P_{n-1}(\lambda). $$

*Here* $P_{n-1}(\lambda)$ *is the Legendre polynomial of degree* $n-1$. *Conversely, in each of Case* (i), *Case* (ii), *each interval* (65) *is completely covered by* ${}_{j-1}A_j$ *as* U *varies over all unitary matrices*, $j = 2, 3, \ldots, n$.

PROOF. If $H = \gamma I$ is scalar then each interval (65) consists of the single point $\gamma$ and each ${}_{j-1}A_j = \gamma$, hence the result is trivial in Case (i). Suppose $H$ has at least two distinct eigenvalues. Let $\mu_1 < \mu_2 < \ldots < \mu_s$ with multiplicities $e_1, e_2, \ldots, e_s$ be the distinct eigenvalues of $H$. We now use the notation of [1]. By [1, (19) and (20)],

$$ \theta_{i\alpha} \leq (\mu_\alpha - \xi_{i,\alpha-1})/(\mu_\alpha - \mu_{\alpha-1}), \qquad \alpha \neq 1, $$

$$ \theta_{i\alpha} \leq (\xi_{i\alpha} - \mu_\alpha)/(\mu_{\alpha+1} - \mu_\alpha), \qquad \alpha \neq n, $$

and $\Sigma_{i=1}^{n}\theta_{i\alpha}=e_\alpha$. Thus by summation we get stronger linear inequalities:

$$_{\alpha-1}A_\alpha \leq (n-e_\alpha)n^{-1}\mu_\alpha + e_\alpha n^{-1}\mu_{\alpha-1}, \qquad \alpha \neq 1, \tag{66}$$

$$_{\alpha}A_{\alpha+1} \geq (n-e_\alpha)n^{-1}\mu_\alpha + e_\alpha n^{-1}\mu_{\alpha+1}, \qquad \alpha \neq n. \tag{67}$$

Now if the full interval (65) is covered by $_{j-1}A_j$ for each $j$, then for some $U$ we achieve

$$_{\alpha-1}A_\alpha = (n-1)n^{-1}\mu_\alpha + n^{-1}\mu_{\alpha-1}, \qquad \alpha \neq 1 \tag{68}$$

and for some other $U$ we get

$$_{\alpha}A_{\alpha+1} = (n-1)n^{-1}\mu_\alpha + n^{-1}\mu_{\alpha+1}, \qquad \alpha \neq n. \tag{69}$$

Combining (66) and (68) yields $(e_\alpha-1)\mu_\alpha \leq (e_\alpha-1)\mu_{\alpha-1}$, a contradiction unless $e_\alpha=1$. Combining (67) and (69) yields $(e_\alpha-1)\mu_{\alpha+1} \leq (e_\alpha-1)\mu_\alpha$, again a contradiction except if $e_\alpha=1$. Thus $e_\alpha=1$ for every $\alpha$. That is, the eigenvalues of $H$ are simple.

Now we are in a position to apply the results of section 4. In the present situation back to back equalities of the type described in Theorem 4 are possible, hence $f''(\lambda_j)=0$ for all $j\neq 1, n$. Thus

$$(\lambda-\lambda_1)(\lambda-\lambda_n)f''(\lambda)=n(n-1)f(\lambda). \tag{70}$$

After translation and change of scale to bring the points $\lambda_1$, $\lambda_n$ to $-1$, 1 respectively, the differential equation (70) becomes

$$(\lambda^2-1)f''(\lambda)=n(n-1)f(\lambda). \tag{71}$$

From (71) $\lambda^2-1$ is clearly a factor of $f(\lambda)$ so put $f(\lambda)=(\lambda^2-1)g(\lambda)$. Then the differential equation for $g(\lambda)$ is

$$(1-\lambda^2)g''(\lambda)-4\lambda g'(\lambda)+(n-2)(n+1)g(\lambda)=0. \tag{72}$$

The differential equation for the Legendre polynomial $P_{n-1}(\lambda)$ is

$$(1-\lambda^2)P''_{n-1}-2\lambda P'_{n-1}+(n-1)nP_{n-1}=0 \tag{73}$$

and one easily deduces from (73) that if $g=P'_{n-1}$, then $g$ satisfies (72). Moreover, by method of Frobenius, (72) has a unique (up to constant factor) polynomial solution. Returning to our original $f$, it follows that (ii) holds. For the converse when $H$ has eigenvalues as described in (ii), we have

$$(\lambda-\lambda_1)(\lambda-\lambda_n)f''(\lambda)=n(n-1)f(\lambda)$$

and hence $f''(\lambda_j)=0$ for all $j\neq 1, n$. Then Theorems 4 or 5 show that each $_{j-1}A_j$ covers its full interval of permissible values.

## 6. References

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