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On Taylor's Theorem*

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A simple way of looking at and proving Taylor's theorem.

Key Words: Iterated integrals, remainders, Taylor series.

Let $-\infty < a < b < \infty$, n an integer > 1, and f a real function with $f^{(n)}$ continuous on the closed interval $[a, b]$. Consider the number

$$I = \int_a^b \int_a^{t_n} \cdots \int_a^{t_3} \int_a^{t_2} f^{(n)}(t_1)\, dt_1 dt_2 \ldots dt_n. \tag{1}$$

Performing the integrations, one obtains

$$I = f(b) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} (b-a)^k, \tag{2}$$

a Taylor formula with "remainder" I. To obtain the Lagrange form of the remainder, observe that taking $f(x) \equiv (x-a)^n/n!$, we have by (1) and (2),

$$I_0 = \int_a^b \int_a^{t_n} \cdots \int_a^{t_2} dt_1 dt_2 \ldots dt_n = \frac{(b-a)^n}{n!}.$$

Returning to our original f, let $M = \max_{a \leq x \leq b} f^{(n)}(x)$, $m = \min_{a \leq x \leq b} f^{(n)}(x)$. Then clearly $mI_0 \leq I \leq MI_0$, and unless $f^{(n)}$ is constant on $[a, b]$, strict inequalities hold. Therefore, there exists a c, $a < c < b$, such that $I/I_0 = f^{(n)}(c)$, and so

$$f(b) = \left[\sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} (b-a)^k \right] + \frac{f^{(n)}(c)}{n!} (b-a)^n.$$

REMARK: The usual proof of Taylor's theorem using integration by parts gives as the value of the right-hand side of (2) the number $\int_a^b f^{(n)}(t) \frac{(b-t)^{n-1}}{(n-1)!} dt$, which must therefore equal the right-hand side of (1). The equality is also evident from the known representation of an iterated indefinite integral as a single integral.

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