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Analysis of a Market Split Model*

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A mathematical analysis is given for a class of models describing how a "market" (i.e., some subset of the consuming public) might divide its patronage among p competing products ($p > 1$). The analysis is confined to the question of how the respective shares of market change with respect to changes in the variables describing the competing products. The split fractions which define the share of market are assumed to be functions of the choice-influencing attributes of *all* the competing products. The elasticities of the split fractions with respect to these attributes are assumed to be functions only of the split fractions themselves. Some functional forms (including the linear case) leading to self-consistent models are analyzed and their solutions derived.

Key Words: Demand, elasticity, mathematical economics, partial differential equations.

1. Introduction

This paper is concerned with a class of mathematical models of how the "market" (i.e., the consuming public) might divide itself among several competing products

$$P_1, P_2, \ldots, P_p$$

where $p > 1$. Subscripts j, k, m, n, J will be used as "product indices," taking values between 1 and p inclusive.

The notation

$$\mathbf{w} = (w_1, w_2, \ldots, w_p)$$

will be used for the vector of *split fractions*

$$w_j = \text{fraction of market which selects } P_j; \tag{1.1}$$

these necessarily satisfy the conditions

$$w_j \geqslant 0, \tag{1.2}$$

$$\sum_{j=1}^{p} w_j = 1. \tag{1.3}$$

With the further notation

$$M = \text{total market size}, \tag{1.4}$$

$$M_j = \text{size of } P_j\text{'s market share} \tag{1.5}$$

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we then have

$$M_j = w_j M. \tag{1.6}$$

The attributes of P_j which influence the market split are described by a vector

$$\mathbf{x}_j = (x_{1j}, x_{2j}, \ldots, x_{n(j),j})$$

of real parameters. To clarify the role of the first subscript of x_{ij}, we note that x_{11} need not have the same interpretation (e.g., durability) as x_{12}, and might even refer to some quality of P_1 which is meaningless for P_2. The ensemble of *all* the x's (for all products) will be denoted $\mathbf{x}$.

The market share of P_j can depend on the relative attractiveness of the other products, and so we have $w_j(\mathbf{x})$ and $M_j(\mathbf{x})$ rather than $w_j(\mathbf{x}_j)$ and $M_j(\mathbf{x}_j)$. The total market size is not assumed constant, so that in general $M = M(\mathbf{x})$. We make the *smoothness assumption* that the functions $w_j(\mathbf{x})$ and $M(\mathbf{x})$ have first-order partial derivatives, so that the same is true of $M_j(\mathbf{x})$ as well.

This hypothesis is not made explicit merely for the sake of mathematical rigor; it has substantive content. For example, the equality of right-hand and left-hand derivatives can be interpreted as ruling out different degrees of "stickiness" associated with gains and with losses in market share. Also, if P_1 and P_2 are "strictly comparable" in the sense that $\mathbf{x}_1$ and $\mathbf{x}_2$ have the same number of components and corresponding components have identical interpretations, and if P_1 is inferior to P_2 with respect to *each* of these components, the smoothness assumption rules out any automatic conclusion of a zero market share for P_1. A nonzero market share for P_1 might seem at first to represent irrational consumer behavior. However, continuous and nonextreme variation of market shares seems likely to be typical of *real* situations, where different consumers appraise products differently (e.g., may "perceive" different values for the x_{ij}'s), and where all significant choice-influencing factors are unlikely to be fully represented (or even represented at all) in any usable model.

Two apparently innocuous model assumptions will be stated next. They assert that the x_{ij}'s have been redefined (if necessary) so that

$$x_{ij} > 0, \tag{1.7}$$

and so that increasing x_{ij} makes P_j *less* attractive than before (or at least no more attractive), thus tending to decrease w_j and to increase the other w_k's. Formally,

$$\partial w_j / \partial x_{ij} \leqslant 0, \tag{1.8}$$

$$\partial w_k / \partial x_{ij} \geqslant 0 \qquad \text{for } k \neq j. \tag{1.9}$$

The only purpose of (1.7) is to permit transformation to new variables

$$y_{ij} = \log x_{ij} \tag{1.10}$$

for subsequent simplifications.

We recall that the *elasticity* of a function $Q(\mathbf{x})$, with respect to changes in x_{ij}, is defined (when $Q \neq 0$) as

$$E_{ij}(Q) = (\partial Q/\partial x_{ij})/(Q/x_{ij}) = Q^{-1}\partial Q/\partial y_{ij}, \tag{1.11}$$

i.e., as the (limiting) rate of relative change in Q, dQ/Q, per unit relative change in x_{ij}, dx_{ij}/x_{ij}. Q might be associated with one of the products P_k, where either $k = j$ (self-elasticity for P_j) or $k \neq j$ (cross-elasticity) might hold.

It is traditional in economics to consider the elasticities $E_{ij}(M_k)$, and to focus on the M_k's. From (1.6), however, we have

$$E_{ij}(M_k) = E_{ij}(M) + E_{ij}(w_k). \tag{1.12}$$

This suggests—as does (1.6) itself—a decomposition of effort in which one focuses separately on M and on $\mathbf{w}$. Such a suggestion is supported by the fact that variables outside $\mathbf{x}$, e.g., level of institutional advertising by a trade association, or price levels held constant across an industry by regulatory agencies, may affect M in a way largely irrelevant to the competitive aspects represented by $\mathbf{w}$. *In this paper, we focus attention exclusively on* $\mathbf{w}$. It was for this reason that we did not list, together with (1.8), the analogous

$$\partial M/\partial x_{ij} \leqslant 0.$$

If a satisfactory model dealing with $\mathbf{w}$ is arrived at, then at least same information about $M(\mathbf{x})$ could be inferred from the plausible hypothesis that

$$\partial M_j/\partial x_{ij} \leqslant \partial M/\partial x_{ij}$$

which by (1.6) can be written (for $w_j \neq 1$)

$$(1-w_j)^{-1}\partial w_j/\partial x_{ij} \leqslant M^{-1}\partial M/\partial x_{ij} \leqslant 0. \tag{1.13}$$

Since we are dealing with a model for $\mathbf{w}$, the following *irredundancy hypothesis* becomes innocuous: For each pair (i, j), with $1 \leqslant j \leqslant p$ and $1 \leqslant i \leqslant n(j)$, there is at least one product P_k for which

$$\partial w_k/\partial x_{ij} \neq 0. \tag{1.14}$$

(Otherwise the parameter x_{ij} has no influence on the market split, and so can and should be omitted from the model.) It follows that such a P_k can be chosen distinct from P_j, for if

$$\partial w_k/\partial x_{ij} = 0$$

for all k with $k \neq j$, then by (1.3)

$$\partial w_j/\partial x_{ij} = \partial(1 - \sum_{k \neq j} w_k)/\partial x_{ij} = 0$$

would hold as well.

The next model assumptions deal with the question of which points in $(w_1, w_2, \ldots, w_p)$-space, among those satisfying (1.2) and (1.3), are *attainable* in the sense of arising as $\mathbf{w}(\mathbf{x})$ for some $\mathbf{x}$. For any product P_j, consider the point defined by

$$w_j = 1;\ w_k = 0 \qquad \text{for } k \neq j. \tag{1.15}$$

The *first* assumption asserts that each of these p points (i.e., for $j = 1, 2, \ldots, p$) is in the attainable region. While we require only that these points be limit points (rather than members) of the attainable region, it is convenient not to have to repeat this distinction everywhere. Therefore, we will speak of points as "in" the region even when only the weaker condition holds. The intended interpretation is that none of the products has a guaranteed minimum market share, nor is any of them artificially precluded from coming arbitrarily close to gaining the entire market if its superiority would lead to this result. For the *second* assumption (which actually subsumes the first) consider, for any distinct products P_j and P_k and any number w with $0 \leqslant w \leqslant 1$, the point defined by

$$w_j = w,\ w_k = 1 - w,\ w_m = 0 \qquad \text{for } m \neq j, k. \tag{1.16}$$

The assertion now is that each such point is also a limit point of the attainable region, i.e., any two products can come arbitrarily close to capturing the entire market and sharing it in a prescribed ratio. These two assertions will be called the *competitiveness hypotheses*, since they are most naturally interpreted as referring to (i) the degree of direct competitive confrontation among the products, and (ii) the absence of "constraints on competition" which would limit the variety of market splits possible under changes in the relative merits of the products. The *third* competitiveness hypothesis, which applies only when $p > 2$ and which then subsumes the others, makes the analogous assertion for any *triple* of distinct products.

For verifying that the models and solutions determined later actually *do* satisfy the competitiveness hypothesis, we must be more explicit than the limiting (1.7) about the extent of the attainable region in $\mathbf{x}$-space. (This region would of course be altered by rescaling or other admissable transformations of the x's.) Note that such a region need not be a Cartesian product of regions in $\mathbf{x}_j$-space for $j = 1, 2, \ldots, p$, since for example products might compete as customers for one or more scarce resources important to their quality. Even with the parameters for all but a single P_j fixed, the resulting attainable region in $\mathbf{x}_j$-space need not be a Cartesian product; it may have a "curved" boundary representing "tradeoffs" among x_{ij}'s "at the limits" set by available technology and resources. Also, parameters x_{ij} may well have bounds, beyond which one would choose to speak not of P_j but rather of a "different" product perhaps competing in a different market.

There are many assumptions, on the attainable region in $\mathbf{x}$-space, which will permit the desired verifications to be carried out. The particular hypothesis chosen for definiteness, though fairly natural mathematically, is perhaps not fully satisfactory in the light of the preceding paragraph. It's somewhat complicated statement is deferred to the point in the analysis (near the end of sec. 3) at which it is invoked.

The final assumption is the one which actually specifies the form of the model. Consider the elasticities $E_{ij}(w_k)$. They are (initially unknown) functions of the $\sum_{j=1}^{p} n(j)$ components of $\mathbf{x}$, but it would clearly be much more convenient *if* they could be determined by observing only the p split fractions (the components of $\mathbf{w}$). This suggests examining models of the form

$$E_{ij}(w_k) = F_{ijk}(\mathbf{w}),$$

which by (1.11) is equivalent to the system

$$\partial w_k / \partial y_{ij} = w_k F_{ijk}(\mathbf{w}) \tag{1.17}$$

of partial differential equations.

It is natural to begin with the simple case in which each F_{ijk} is *linear*, so that (1.17) becomes

$$\partial w_k / \partial y_{ij} = \sum_{m=1}^{p} b_{ijkm} w_k w_m \tag{1.18}$$

where the b's are constants. This is no less general than the (possibly) *in*homogeneous linear case

$$F_{ijk}(\mathbf{w}) = a_{ijk} + \sum_{m} b'_{ijkm} w_m,$$

since the latter can be brought into the form of (1.18) by setting

$$b_{ijkm} = b'_{ijkm} + a_{ijk}$$

and appealing to (1.3).

We shall deal with a generalization in which the F_{ijk} are *separable*, i.e.,

$$F_{ijk}(\mathbf{w}) = \sum_m b_{ijkm} g_m(w_m),$$

and in fact with the further generalization given by

$$\partial w_k / \partial y_{ij} = \sum_{m=1}^{p} b_{ijkm} f_k(w_k) g_m(w_m) \tag{1.19}$$

where the functions f_k and $g_k (k = 1, 2, \ldots, p)$, defined on $0 \leq w \leq 1$, satisfy

$$f_k(0) = g_k(0) = 0 \tag{1.20}$$

and have continuous derivatives f'_k and g'_k such that

$$f'_k > 0,\ g'_k > 0. \tag{1.21}$$

We call this "the model," and will use this term when referring to eqs (1.19) to (1.21). These last two conditions are of course satisfied for the particular choices

$$f_k(w) = g_k(w) = w$$

which specialize the model eqs (1.19) to (1.18). It will be clear from the proofs to come that finitely many points of exception to (1.21) can be permitted.

Our analysis, of the class of models described by (1.19) and the other assumptions listed above, will be *complete* in the following sense: Those models in the class which are *consistent* (i.e., have at least one solution $\mathbf{w}(\mathbf{x})$) will be identified, and for each of these consistent models the general solution will be given in explicit form. Those parts of the argument common to the cases $p > 2$ and $p = 2$ are presented in section 2, but these cases then require separate treatment; section 3 treats situations with three or more competing products, while section 4 deals with the case of just a pair of competitors.

There are three reasons for passing from the linear model (1.18) to the (possibly) nonlinear (1.19). One is simply intellectual curiosity as to how the generalization might affect the analysis. Second, is the possibility that some special insight into the competitive situation at hand will strongly suggest that linearity is implausible. Third, if it should prove impossible to obtain a satisfactory "fit" to empirical data using the linear model, then perhaps more parameters (which can be adjusted to improve the fit) can be smuggled in *via* the f_k's and g_k's. The conclusions of section 3, however, show that the second and third of these hopes are *in vain* when $p > 2$; the only consistent models are a subclass of the linear ones given by (1.18). For $p = 2$, however, the class of consistent models is shown (in sec. 4) to contain many nonlinear ones.

In the application motivating this work, the "market" in question is to consist of a single "cell" in some stratification of the population of travelers between a particular origin and a particular destination. The "products" are the services offered by the various transport alternatives; the latter might be taken as the traditional transportation "modes" (air, rail, bus, private auto) plus whatever novelties social and technological change may produce, or might reflect a finer classification (e.g., particular auto routes, particular airlines, first-class versus coach service). The components of $\mathbf{x}$ might be measures of trip time, trip cost, variability from published schedules, trip fatigue, frequency and severity of accidents, etc. Validation and subsequent use (for prediction)

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of such a model would of course require operationally meaningful specifications of the transport alternatives (more generally, the products) and of the x's, and also appropriate "calibration" based on empirical data. In the present paper, however, we are solely concerned with the mathematical consequences of the model's assumptions. In particular, the interpretive discussion of the model is concluded at this point, the remaining sections of the paper consisting of mathematical analysis only.

The reader who works through the following derivations is bound to be struck by the very heavy use of the competitiveness hypotheses, especially in showing the strong interconnectedness of all the products early in section 3. As a topic for further investigation, we would suggest the problem of replacing these hypotheses by fruitful but weaker assumptions on what market splits are theoretically attainable, and of determining the resulting classes of consistent models and their solutions. Those consistent models and associated solutions developed in this paper will remain valid, the technical change being that some of those b's which are not forced to be zero by model hypotheses other than the competitiveness hypothesis, might actually be zero. The essential question, however, is that of what *additional* models might prove consistent under the relaxed hypotheses.

2. Preliminaries

We begin this section by showing that

$$\partial(\partial w_k/\partial y_{ij})/\partial y_{IJ} = \partial(\partial w_k/\partial y_{IJ})/\partial y_{ij}. \tag{2.1}$$

As is well known, to prove this it suffices to show that the two second-order partial derivatives exist and are continuous. For the derivatives on the left in (1.19) to exist, $\mathbf{w}(\mathbf{x})$ must be continuous. Since the f's and g's are continuous, it follows from (1.19) that all of the first-order partial derivatives of $\mathbf{w}(\mathbf{x})$ are continuous. We can evaluate the left-hand side of (2.1) by applying the chain rule to (1.19):

$$\partial(\partial w_k/\partial y_{ij})/\partial y_{IJ} = \sum_m b_{ijkm}\{f_k'(w_k)g_m(w_m)\,\partial w_k/\partial y_{IJ} + f_k(w_k)g_m'(w_m)\,\partial w_m/\partial y_{IJ}\}. \tag{2.2}$$

Since the f's and g's and their derivatives are continuous, and the first-order partial derivatives on the right in (2.2) were just proved continuous, it follows that the left-hand side of (2.1) is continuous; similarly for the right-hand side.

The derivatives on the right in (2.2) can be evaluated using (1.19); the resulting expression for the *left* side of (2.1) is

$$\sum_m \sum_n b_{ijkm}b_{IJkn}f_k'(w_k)g_m(w_m)f_k(w_k)g_n(w_n) + \sum_m \sum_n b_{ijkm}b_{IJmn}f_k(w_k)g_m'(w_m)f_m(w_m)g_n(w_n). \tag{2.3}$$

The corresponding expression for the *right* side of (2.1) can be obtained from (2.3) by interchanging (i, j) and (I, J); the first of its two summands, after interchanging the dummy indices m and n, coincides with the first summand of (2.3). Thus (2.1) yields

$$f_k(w_k)\sum_m \sum_n \{b_{ijkm}b_{IJmn} - b_{IJkm}b_{ijmn}\}\; g_m'(w_m)f_m(w_m)g_n(w_n) = 0. \tag{2.4}$$

We next show that a consistent model is *sparse*, i.e., that most of the b's must vanish. Specifically we show that

$$b_{ijkk} = 0 \tag{2.5}$$

and that

$$b_{ijkm} = 0 \qquad (j \neq k,\ m), \tag{2.6}$$

so that only the b_{ijjk}'s and b_{ijkj}'s with $k \neq j$ can possibly be nonzero. For this purpose, first note from (1.3) that

$$\sum_k \partial w_k / \partial y_{ij} = \partial \Big(\sum_k w_k \Big) / \partial y_{ij} = 0.$$

Substituting (1.19) into this, we obtain

$$\sum_k \sum_m b_{ijkm} f_k(w_k) g_m(w_m) = 0. \tag{2.7}$$

Because the point defined by $w_k = 1$ and $w_m = 0$ for $m \neq k$ is a limit point of the attainable region, while f_k and g_m are continuous and $g_m(0) = 0$, it follows from (2.7) that

$$b_{ijkk} f_k^{(1)} g_k^{(1)} = 0.$$

Since (1.20) and (1.21) imply $f_k(1) g_k(1) > 0$, (2.5) is proved.

Suppose now that (j, k, m) are distinct. By (1.9), we have

$$\sum_n b_{ijkn} f_k(w_k) g_n(w_n) = \partial w_k / \partial y_{ij} \geqslant 0. \tag{2.8}$$

For any w with $0 \leqslant w \leqslant 1$, the point defined by

$$w_k = w,\ w_m = 1 - w,\ w_n = 0 \text{ for } n \neq k,\ m$$

is a limit point of the attainable region; using this, (2.5), and the continuity of the f's and g's, we see that (2.8) implies

$$b_{ijkm} f_k(w) g_m(1 - w) \geqslant 0$$

for $0 \leqslant w \leqslant 1$, which in turn implies

$$b_{ijkm} \geqslant 0. \tag{2.9}$$

Similarly (2.7) implies

$$b_{ijkm} f_k(w) g_m(1 - w) + b_{ijmk} f_m(1 - w) g_k(w) = 0. \tag{2.10}$$

But by (2.9) both b's in (2.10) are nonnegative, while by (1.20) and (1.21) the f-values and g-values are strictly positive for $0 < w < 1$; hence (2.6) must hold.

Equations (2.5) and (2.6) have a number of consequences. *First*, we see that for consistent models, (1.19) must have the form

$$\partial w_k / \partial y_{ij} = b_{ijkj} f_k(w_k) g_j(w_j) \qquad (k \neq j), \tag{2.11}$$

$$\partial w_j / \partial y_{ij} = \sum_{m \neq j} b_{ijjm} f_j(w_j) g_m(w_m). \tag{2.12}$$

Since the points defined by $w_j + w_k = 1$ together with (1.2) and (1.3) are limit points of the attainable region, it follows from (2.11) and (1.9) that

$$b_{ijkj} \geqslant 0. \tag{2.13}$$

Second, it follows that the only interesting case of (2.10) is

$$b_{ijkj} f_k(w) g_j(1-w) + b_{ijjk} f_j(1-w) g_k(w) = 0. \tag{2.14}$$

From this and (2.13) it follows that

$$b_{ijjk} \leq 0. \tag{2.15}$$

Another implication of (2.14) is

$$b_{ijkj} = 0 \qquad \text{iff } b_{ijjk} = 0, \tag{2.16}$$

a result useful in studying the pattern of nonzero b's. From (2.11) we can infer that constant nonzero elasticities $E_{ij}(w_k)\,(k \neq j)$ are impossible, while from (2.12) and (2.15) it follows that constant nonzero self-elasticities $E_{ij}(w_j)$ are also impossible. *Third*, we observe that (2.7) becomes

$$\sum_{m \neq j} b_{ijjm} f_j(w_j) g_m(w_m) + \sum_{k \neq j} b_{ijkj} f_k(w_k) g_j(w_j) = 0;$$

a neater form is

$$\sum_{k \neq j} \{ b_{ijjk} f_j(w_j) g_k(w_k) + b_{ijkj} f_k(w_k) g_j(w_j) \} = 0. \tag{2.16a}$$

We can also use (2.5) and (2.6) to simplify (2.4). For (j, k, J) distinct, which implies $p > 2$, (2.4) becomes

$$f_k(w_k) \{ b_{ijkj} b_{IJjJ} g_j'(w_j) f_j(w_j) g_J(w_J) \} = f_k(w_k) \{ b_{IJkJ} b_{ijJj} g_J'(w_J) f_J(w_J) g_j(w_j) \}. \tag{2.17}$$

This has of course only been proved for the case in which (w_j, w_k, w_J) are the indicated components of some **w** which lies in (or . . . by continuity . . . is a limit point of) the attainable region. But by the third competiveness hypothesis, for any nonnegative w_j, w_J with $w_j + w_J < 1$, such a **W** is obtained by setting $w_k = 1 - w_j - w_J$; this implies $f_k(w_k) > 0$, so that (2.17) yields (for *distinct* j, k, J)

$$b_{ijkj} b_{IJjJ} g_j'(w_j) f_j(w_j) g_J(w_J) = b_{IJkJ} b_{ijJj} g_J'(w_J) f_J(w_J) g_j(w_j) \tag{2.18}$$

as valid for any nonnegative (w_j, w_J) with $w_j + w_J < 1$, and hence (by continuity) for $w_j + w_J = 1$ as well.

Next take $j = k \neq J$ in (2.4). Then application of (2.5) and (2.6) yields

$$f_j(w_j) \{ g_J'(w_J) f_J(w_J) \left[\sum_{n \neq J} b_{ijjJ} b_{IJJn} g_n(w_n) - b_{IJjJ} b_{ijJj} g_j(w_j) \right] + \sum_{m \neq J,\, j} b_{ijjm} b_{IJmJ} g_m'(w_m) f_m(w_m) g_J(w_J) \} = 0,$$

which can be rearranged as

$$f_j(w_j) \{ (b_{ijjJ} b_{IJJj} - b_{IJjJ} b_{ijJj}) g_J'(w_J) f_J(w_J) g_j(w_j)$$
$$+ \sum_{m \neq J,\, j} [b_{ijjJ} b_{IJJm} g_J'(w_J) f_J(w_J) g_m(w_m) + b_{ijjm} b_{IJmJ} g_m'(w_m) f_m(w_m) g_J(w_J)] \} = 0. \tag{2.19}$$

This has of course only been proved for those cases in which $j \neq J$ and the w's are the components of some **w** which lies in or is a limit point of the attainable region.

Now consider any two products P_j and P_J, $j \neq J$. We will call P_j *weakly disconnected* (*strongly disconnected*) from P_J if $\partial w_J / \partial y_{ij} = 0$ holds for some i (for *all* i) with $1 \leqslant i \leqslant n(j)$. Clearly strong disconnectedness implies weak disconnectedness. It will now be shown that, conversely, weak disconnectedness implies strong disconnectedness, so that we can speak simply of "disconnectedness" and its opposite, "connectedness." It will also be shown that disconnectedness (and hence connectedness) is a *symmetric* relation, i.e., if P_j is disconnected from P_J, then P_J is disconnected from P_j.

For the proof, assume $\partial w_J / \partial y_{ij} = 0$ for some i. Then $b_{ijJj} = 0$, by (2.11). By the irredundancy hypothesis (used only here!) there is a k, with $k \neq j$, such that $\partial w_k / \partial y_{ij} \neq 0$ and hence by (2.11) such that $b_{ijkj} \neq 0$; thus $k \neq J$. Since (j, k, J) are distinct, we can apply (2.18) to infer that $b_{iJjJ} = 0$ for *all* I with $1 \leqslant I \leqslant n(J)$, i.e., that P_J is strongly disconnected from P_j. Applying the same argument with j and J interchanged, we have P_j strongly disconnected from P_J.

3. Analysis for More Than Two Products

We begin the analysis for $p > 2$ by showing that the relation of connectivity, proved symmetric at the end of the previous section, is also *transitive* in the sense that for distinct (j, k, J), if P_j is connected to P_k and P_k is connected to P_J, then P_j is connected to P_J. For the proof, note that for any w with $0 \leqslant w \leqslant 1$, the point defined by

$$w_k = w,\ w_J = 1 - w,\ w_m = 0 \text{ for } m \neq k, J$$

is a limit point of the attainable region. From this and (2.19), we have

$$b_{ijjk} b_{IJkJ} g_k'(w) f_k(w) g_J(1-w) + b_{ijjJ} b_{iJJk}\, g_J'(1-w) f_J(1-w) g_k(w) = 0.$$

If P_j were *dis*connected from P_J, the second summand would vanish, leaving

$$b_{ijjk} b_{IJkJ} g_k'(w) f_k(w) g_J(1-w) = 0$$

for $0 \leqslant w \leqslant 1$, and thus $b_{ijjk} b_{IJkJ} = 0$. This however is impossible because P_j is connected to P_k, and P_k to P_J (note the use of (2.16)); the proof of transitivity is complete.

Next it will be shown that *total connectivity* holds, i.e., for *any* distinct j and J, P_j and P_J are connected. If not, then since "connectedness" is both symmetric and transitive, the set of p products would decompose into two or more subsets such that

(i) any two products in the same subset are connected, but

(ii) no two products in different subsets are connected. Suppose for example that P_1 and P_2 lie in different subsets S_1 and S_2. Then $\partial w_1 / \partial x_{ij} = 0$ unless P_j is in S_1, i.e., w_1 depends only on the parameters of the products in S_1, and similarly for w_2 and S_2. By the first competitiveness hypothesis there exist choices of $\{\mathbf{x}_j : j \epsilon S_1\}$ for which $\mathbf{w}(\mathbf{x})$ has w_1 arbitrarily close to 1, which requires that w_2 be arbitrarily close to 0. This is impossible because the parameters of products in S_1 cannot influence w_2. It follows from this contradiction that, for $p > 2$, total connectivity must hold, i.e., all of the *possibly* nonzero b's (the b_{ijkj}'s and b_{ijjk}'s for $k \neq j$) *are* in fact nonzero. (From this and the argument below (2.16), it follows that for $p > 2$ there can be no *constant* elasticities $E_{ij}(w_k)$ in a consistent model.)

Thus for distinct (j, k, J), the b's in (2.18) are nonzero. For $w_j > 0$ and $w_J > 0$, (2.18) can be written

$$b_{ijkj} b_{IJjJ} g_j'(w_j) f_j(w_j) / g_j(w_j) = b_{IJkJ} b_{ijJj} g_J'(w_J) f_J(w_J) / g_J(w_J).$$

The left side is a function of w_j only, the right one a function of w_J only. Since $p > 2$, it follows that both are *constant*, and since the b's are nonzero, there exist (necessarily positive) constants d_j such that for $w_j > 0$

$$f_j(w_j)g_j'(w_j)/g_j(w_j) = d_j,$$

i.e. (by continuity for $w_j = 0$ also)

$$f_j = d_j g_j / g_j'. \tag{3.1}$$

Thus the f's are uniquely determined by the g's (in a consistent model).

At this point it is convenient to introduce the normalization

$$g_m(1) = 1 \qquad (m = 1, 2, \ldots, p), \tag{3.2}$$

which is possible since $g_m(1) > 0$, i.e., we can replace b_{ijkm} with $b_{ijkm}g_m(1)$ and g_m with $g_m/g_m(1)$.

Now we return to (2.14), with (j, k) distinct, and apply (3.1) to obtain

$$(b_{ijkj}d_k g_k(w)g_j(1-w)/g_k'(w) + (b_{ijjk}d_j g_j(1-w)g_k(w))/g_j'(1-w) = 0.$$

Thus for $0 < w < 1$, and hence by continuity for $0 \leqslant w \leqslant 1$,

$$b_{ijkj}d_k g_j'(1-w) + b_{ijjk}d_j g_k'(w) = 0. \tag{3.3}$$

Indefinite integration with lower limit zero gives

$$b_{ijjk}d_j g_k(w) - b_{ijkj}d_k g_j(1-w) = -b_{ijkj}d_k, \tag{3.4}$$

where (3.2) has been used to evaluate the right-hand side. Setting $w = 1$, we have

$$b_{ijjk}d_j = -b_{ijkj}d_k. \tag{3.5}$$

Substitution of this into (3.4) yields

$$g_k(w) + g_j(1-w) = 1 \qquad (k \neq j). \tag{3.6}$$

Since $p > 2$, this implies the existence of a *single* function $g(w)$ such that

$$g_k(w) = g(w) \qquad \text{for } k = 1, 2, \ldots, p; \tag{3.7}$$

i.e., the g's coincide.

Now consider any distinct (j, k, J). The third competitiveness hypothesis, applied to (2.19), shows that

$$f_j(w)\{(b_{ijjJ}b_{IJJj} - b_{IJjJ}b_{ijJj})g_J' f_J g_j(w)$$

$$+ b_{ijjJ}b_{IJJk}g_J'(w_J)f_J(w_J)g_k(w_k) + b_{ijjk}b_{IJkJ}g_k'(w_k)f_k(w_k)g_J(w_J)\} = 0$$

holds if $w_j + w_k + w_J = 1$. But by (3.1) $g_j' f_J = d_J g_J$, so the last equation can be written as

$$f_j(w_j)g_J(w_J)\{(b_{ijjJ}b_{IJJj} - b_{IJjJ})b_{ijJj}d_J g_j(w_j) + b_{ijjJ}b_{IJJk}d_J g_k(w_k) + b_{ijjk}b_{IJkJ}d_k g_k(w_k)\} = 0.$$

Thus if $w_j+w_k+w_J=1$, then

$$(b_{jjJ}b_{IJJj}-b_{IJjJ}b_{ijJj})d_J g_j(w_j)+b_{ijjJ}b_{IJJk}d_J g_k(w_k)+b_{ijjk}b_{IJkJ}d_k g_k(w_k)=0$$

holds if $w_j>0$ and $w_J>0$, and hence (by continuity) even without this extra hypothesis. Taking $w_k=1$ leads to

$$b_{ijjJ}b_{IJJk}d_J+b_{ijjk}b_{IJkJ}d_k=0.$$

But by (3.5),

$$b_{IJkJ}d_k=-b_{IJJk}d_J,$$

so the preceding equation becomes

$$b_{IJJk}d_J(b_{ijjJ}-b_{ijjk})=0 \qquad (j,\ k,\ J \text{ distinct}).$$

Since $d_J\neq 0$, this implies the existence of constants b_{ij} (necessarily negative, by (2.15) and total connectivity) such that

$$\mathrm{b}_{ijjk}=b_{ij} \qquad \text{for all } k\neq j. \tag{3.8}$$

We turn now to (2.16a). By (3.1), it becomes

$$g_j(w_j)\sum_{k\neq j} g_k(w_k)\{b_{ijjk}d_j/g_j'(w_j)+b_{ijkj}d_k/g_k'(w_k)\}=0,$$

which by (3.5) can be written

$$g_j(w_j)d_j\sum_{k\neq j} b_{ijjk}g_k(w_k)\{1/g_j'(w_j)-1/g_k'(w_k)\}=0,$$

and then by (3.8) can be written

$$g_j(w_j)d_j b_{ij}\sum_{k\neq j} g_k(w_k)\{1/g_j'(w_j)-1/g_k'(w_k)\}=0. \tag{3.9}$$

Choose any distinct $(j,\ k,\ J)$. Then the third competitiveness hypothesis, applied to (3.9) shows that for $w_j+w_k+w_J=1$,

$$g_k(w_k)\ \{1/g_j'(w_j)-1/g_k'(w_k)\}+g_J(w_J)\ \{1/g_j'(w_j)-1/g_J'(w_J)\}=0$$

if $w_j>0$, and hence (by continuity) also if $w_j=0$. Choosing

$$w_k=w,\ w_J=1-w,\ w_j=0,$$

and applying (3.7), we have

$$g(w)\ \{1/g'(0)-1/g'(w)\}+g(1-w)\ \{1/g'(0)-1/g'(1-w)\}=0,$$

which by (3.6) and the result of differentiating it becomes

$$\{g(w)+g(1-w)\}\ \{1/g'(0)-1/g'(w)\}=1/g'(0)-1/g'(w)=0.$$

This implies $g'(w) = g'(0)$ for $w > 0$, i.e., g' is *constant*. Thus g is *linear*, and since $g(0) = 0$ and $g(1) = 1$ we have

$$g(w) = w. \tag{3.10}$$

By (3.1),

$$f_j(w) = d_j w. \tag{3.11}$$

Thus every consistent model is a *linear* one in the sense of (1.17)! By (2.11), (3.5), (3.8), and (3.10–11) we have

$$\begin{aligned}\partial w_k / \partial y_{ij} &= b_{ijkj} d_k w_k w_j \\ &= - b_{ijjk} d_j w_k w_j = - b_{ij} d_j w_k w_j\end{aligned}$$

for $k \neq j$, while use of (2.12) leads to

$$\begin{aligned}\partial w_j / \partial y_{ij} &= \sum_{m \neq j} b_{ijjm} d_j w_j w_m \\ &= b_{ij} d_j w_j \sum_{m \neq j} w_m = b_{ij} d_j w_j (1 - w_j).\end{aligned}$$

Both forms can be combined, with the aid of the Kronecker delta, in

$$\partial w_k / \partial y_{ij} = b_{ij} d_j w_k (\delta_{jk} - w_j). \tag{3.12}$$

We now proceed to the explicit solution of (3.12). We have

$$\begin{aligned}(1/w_k)\, dw_k &= (1/w_k) \sum_{i,j} (\partial w_k / \partial y_{ij})(dy_{ij}). \\ &= \sum_{i,j} b_{ij} d_j (\delta_{jk} - w_j)(dy_{ij}).\end{aligned}$$

Therefore

$$\begin{aligned}dw_k / w_k - dw_1 / w_1 &= \sum_{i,j} b_{ij} d_j (\delta_{jk} - \delta_{j1})(dy_{ij}). \\ &= \sum_i b_{ik} d_k (dy_{ik}) - \sum_i b_{i1} d_1 (dy_{i1}).\end{aligned}$$

There is therefore a constant c_k such that

$$\log\,(w_k / w_1) = \sum_i b_{ik} d_k y_{ik} - \sum_i b_{i1} d_1 y_{i1} + c_k,$$

and hence such that

$$w_k = C_k w_1 W_k / W_1 \tag{3.13}$$

where

$$C_k = \exp\,(c_k) > 0 \qquad (C_1 = 1),$$

$$W_k = \exp\left(\sum_i b_{ik} d_k y_{ik}\right).$$

From the definition (1.10) of y_{ik}, we have

$$W_k = \left(\prod_i x_{ik}^{b_{ik}}\right)^{d_k}. \tag{3.14}$$

By summing (3.13) over $1 \leqslant k \leqslant p$ and applying (1.3), we obtain

$$1 = (w_1/W_1) \sum_k C_k W_k,$$

$$w_k = C_k W_k \Big/ \sum_j C_j W_j. \tag{3.15}$$

We have shown that *if* the model (1.19) is to be consistent for $p > 2$, *then* it has the special form (3.12), and its solutions $\mathbf{w}(\mathbf{x})$ have the form

$$w_k = C_k \left(\prod_i x_{ik}^{b_{ik}}\right)^{d_k} \Big/ \sum_n C_n \left(\prod_i x_{in}^{b_{in}}\right)^{d_n} \tag{3.16}$$

involving parameters

$$d_k > 0,\ b_{ik} < 0,\ C_k > 0 \tag{3.17}$$

where the C's are determined only up to a common positive multiplicative factor. Although (3.12) admits the singular solution $w_k(\mathbf{x}) \equiv 0$ corresponding to $C_k = 0$, this is ruled out by the competitiveness hypothesis.

Conversely, consider *any* sets of b's, C's and d's satisfying (3.17), and *define* $\mathbf{w}(\mathbf{x})$ by (3.16). It is readily verified that (1.2) and (1.3), as well as (3.12), are satisfied. The irredundancy hypothesis is also clearly satisfied. So it only remains to check the third competitiveness hypothesis, which subsumes the others. Consider, then, any distinct (j, k, J) and any nonnegative $(w_j^\circ, w_k^\circ, w_J^\circ)$ summing to 1.

To show (as desired) that the point defined by

$$w_j = w_j^\circ,\ w_k = w_k^\circ,\ w_J = w_J^\circ,\ w_m = 0 \qquad \text{for } m \neq j, k, J \tag{3.18}$$

lies in or at least is a limit point of the attainable region in $\mathbf{w}$-space, it is necessary to be more explicit than the delimiting (1.7) about the extent of the feasible region in $\mathbf{x}$-space. To be definite, we will adopt the following somewhat complicated hypothesis: For each distinct (j, k, J), there exists a triple of parameters $(x_{i(1)j}, x_{i(2),k}, x_{i(3),J})$ with

$$1 \leqslant i(1) \leqslant n(j),\ 1 \leqslant i(2) \leqslant n(k),\ 1 \leqslant i(3) \leqslant n(J)$$

. . . we immediately renumber them x_{1j}, x_{1k}, x_{1J} . . . such that for certain fixed settings $x_{im} = x_{im}^\circ > 0$ of all *other* parameters

$$\{x_{im}\colon m \neq j, k, J \qquad \textit{or} \qquad i > 1\},$$

there exist constants

$$x_{1j}^{+} > 0,\ x_{1k}^{+} > 0,\ x_{1J}^{+} > 0$$

with the property that *any* numerical triple (x_{1j}, x_{1k}, x_{1J}) obeying

$$0 < x_{1j} < x_{1j}^{+},\ 0 < x_{1k} < x_{1k}^{+},\ 0 < x_{1J} < x_{1J}^{+} \tag{3.19}$$

forms, together with the fixed settings $x_{im} = x_{im}^{\circ}$, a point $\mathbf{x}^{\circ}$ in the attainable region in $\mathbf{x}$-space.

The proof will involve a limiting process in which

$$x_{1j} \to 0+,\ x_{1k} \to 0+,\ x_{1J} \to 0+ \tag{3.20}$$

while all other x_{im} are maintained at the fixed settings x_{im}°. From this and (3.16–17), we have

$$w_m(\mathbf{x}) \to 0 \qquad \text{for } m \neq j, k, J,$$

as desired. In view of the fixed settings, we have from (3.16) that

$$w_j = C_j^*(x_{1j})^{b_{1j}d_j} / (C_j^*(x_{1j})^{b_{1j}d_j} + C_k^*(x_{1k})^{b_{1k}d_k} + C_J^*(x_{1J})^{b_{1J}d_J} + C^*),$$

$$w_k = C_k^*(x_{1k})^{b_{1k}d_k} / (C_j^*(x_{1j})^{b_{1j}d_j} + C_k^*(x_{1k})^{b_{1k}d_k} + C_J^*(x_{1J})^{b_{1J}d_J} + C^*),$$

$$w_J = C_J^*(x_{1J})^{b_{1J}d_J} / (C_j^*(x_{1j})^{b_{1j}d_j} + C_k^*(x_{1k})^{b_{1k}d_k} + C_J^*(x_{1J})^{b_{1J}d_J} + C^*),$$

where C_j^*, C_k^*, C_J^* are positive constants and C^* a nonnegative constant (zero only when $p=3$).

One of $(w_j^{\circ}, w_k^{\circ}, w_J^{\circ})$, say the last, must be strictly positive. The limiting process can be chosen so that (3.19) holds, while x_{1j} and x_{1k} are defined in terms of x_{1J} in such a way that

$$w_j(\mathbf{x})/w_J(\mathbf{x}) = C_j^*(x_{1j})^{b_{1j}d_j}/C_J^*(x_{1J})^{b_{1J}d_J} = w_j^{\circ}/w_J^{\circ},$$

$$w_k(\mathbf{x})/w_J(\mathbf{x}) = C_k^*(x_{1k})^{b_{1k}d_k}/C_J^*(x_{1J})^{b_{1J}d_J} = w_k^{\circ}/w_J^{\circ}.$$

Since all points $\mathbf{w}(\mathbf{x})$ associated with the process lie in the closed bounded set defined by

$$\mathbf{w} \geqslant \mathbf{0},\ \sum_m w_m = w_j + w_k + w_J = 1,$$

these points will have a limit point w^* obeying the same conditions, and such that

$$w_j^*/w_J^* = w_j^{\circ}/w_J^{\circ},\ w_k^*/w_J^* = w_k^{\circ}/w_J^{\circ}.$$

Thus

$$w_j^* = w_j^{\circ},\ w_k^* = w_k^{\circ},\ w_J^* = w_J^{\circ},$$

i.e., w^*, defined as a limit point of the attainable region in $\mathbf{w}$-space, coincides with the point (3.18).

We have thus shown that for $p > 2$, (3.12), (3.16), and (3.17) give precisely the class of consistent models and their solutions.

4. The Two-Product Case

In this section we assume $p=2$. The situation will be shown to be quite different from that with $p>2$, in that there is an abundance of consistent nonlinear models.

It is convenient to introduce the continuous functions

$$h_1(w)=f_1(w)g_2(1-w), \tag{4.1}$$

$$h_2(w)=f_2(w)g_1(1-w), \tag{4.2}$$

so that for $j=1, 2$

$$h_j(0)=h_j(1)=0,\ h_j(w)>0 \text{ for } 0<w<1. \tag{4.3}$$

By (2.11) and (2.12), the model takes the form

$$\partial w_1/\partial y_{i1}=b_{i112}h_1(w_1), \tag{4.4}$$

$$\partial w_1/\partial y_{i2}=b_{i212}h_1(w_1), \tag{4.5}$$

$$\partial w_2/\partial y_{i1}=b_{i121}h_2(w_2), \tag{4.6}$$

$$\partial w_2/\partial y_{i2}=b_{i221}h_2(w_2), \tag{4.7}$$

while (2.14) yields

$$b_{i212}h_1(w)+b_{i221}h_2(1-w)=0, \tag{4.8}$$

$$b_{i121}h_2(w)+b_{i112}h_1(1-w)=0. \tag{4.9}$$

The same argument used (in the second paragraph of section 3) to prove total connectivity when $p>2$, can be readily adapted to show that P_1 and P_2 must be connected. Thus (here (2.16) is used) all the b's in (4.4) through (4.7) are nonzero. From (4.8) and (4.9) we have, for $0<w<1$,

$$b_{i112}/b_{i121}=-h_2(w)/h_1(1-w)=b_{I212}/b_{I221}.$$

Thus there is a constant

$$R<0 \tag{4.10}$$

such that

$$Rh_1(1-w)+h_2(w)=0, \tag{4.11}$$

$$b_{i112}=Rb_{i121},\ b_{I212}=Rb_{I221}. \tag{4.12}$$

The sign of R follows from (4.12) together with (2.13) and (2.15). The simplifying substitutions

$$z_{ij}=b_{ij21}y_{ij} \tag{4.13}$$

then convert the model into

$$\partial w_1/\partial z_{i1}=\partial w_1/\partial z_{i2}=Rh_1(w_1), \tag{4.14}$$

$$\partial w_2/\partial z_{i1}=\partial w_2/\partial z_{i2}=h_2(w_2). \tag{4.15}$$

Furthermore, the positive factor $(-R)$ can be absorbed into h_1, i.e., we can assume $R=-1$ so that (4.14) becomes

$$\partial w_1/\partial z_{i1}=\partial w_1/\partial z_{i2}=-h_1(w_1) \tag{4.16}$$

and (4.11) simplifies to

$$h_2(w)=h_1(1-w). \tag{4.17}$$

Substitution of (4.15) or (4.16) into the differential identity

$$dw_j=\sum_1^{n(1)}(\partial w_j/\partial z_{i1})(dz_{i1})+\sum_1^{n(2)}(\partial w_j/\partial z_{i2})(dz_{i2})$$

yields

$$(dw_j)/h_j(w_j)=(-1)^j\left[\sum_1^{n(1)}(dz_{i1})+\sum_1^{n(2)}(dz_{i2})\right]. \tag{4.18}$$

For $j=1, 2$, choose points w_{0j} with

$$0<w_{0j}<1,\ w_{01}+w_{02}=1. \tag{4.19}$$

For $0<w<1$, let

$$H_j(w)=\int_{w_{0j}}^{w}(1/h_j(w))\,dw. \tag{4.20}$$

Then in terms of

$$u=\sum_1^{n(1)}z_{i1}+\sum_1^{n(2)}z_{i2}, \tag{4.21}$$

(4.18) yields

$$H_j(w_j)=(-1)^j(u+c_j) \tag{4.22}$$

where c_j is a constant of integration. However, (4.17) and (4.19) lead to

$$H_2(w)=-H_1(1-w), \tag{4.23}$$

which together with (4.22) and (1.3) yields $c_1=c_2$. Thus

$$H_j(w_j)=(-1)^j(u+c) \tag{4.24}$$

where c is an arbitrary constant.

It follows from (4.3) that the differentiable functions H_j are strictly increasing, with

$$H_j(0)=-\infty,\ H_j(1)=\infty.$$

They therefore possess differentiable strictly increasing *inverse* functions, denoted $\overline{H}_j$, with

$$\overline{H}_j(-\infty)=0,\ \overline{H}_j(\infty)=1.$$

It follows from (4.23) that these functions satisfy the identity

$$\overline{H}_1(-v)+\overline{H}_2(v)=1. \tag{4.25}$$

Finally, from (4.24) we see that the general solution is

$$w_1=\overline{H}_1(-(u+c)), \tag{4.26}$$

$$w_2=\overline{H}_2(u+c), \tag{4.27}$$

where u is defined by (4.21) and c is an arbitrary constant.

To relate this to the general solution (3.16) for $p>2$, we set

$$C=\exp(c)>0,$$

follow (3.8) in defining

$$b_{i112}=b_{i1},\ b_{i221}=b_{i2}, \tag{4.28}$$

and introduce the functions

$$K_j(v)=\overline{H}_j(\log v). \tag{4.29}$$

Then by (4.12),

$$b_{i121}=b_{i1}/R=-b_{i1},$$

and by (4.13) and (1.10) we have

$$u+c=\log\left[C\prod_1^{n(2)}(x_{i2})^{b_{i2}}\Big/\prod_1^{n(1)}(x_{i1})^{b_{i1}}\right].$$

Thus (4.26) and (4.27) take the form

$$w_1=K_1\left(C^{-1}\prod_1^{n(1)}(x_{i1})^{b_{i1}}\Big/\prod_1^{n(2)}(x_{i2})^{b_{i2}}\right), \tag{4.30}$$

$$w_2=K_2\left(C\prod_1^{n(2)}(x_{i2})^{b_{i2}}\Big/\prod_1^{n(1)}(x_{i1})^{b_{i1}}\right), \tag{4.31}$$

involving the same products as in (3.16). Note that the competitiveness hypothesis has been used to rule out the singular solutions $w_j=0$ and $w_j=1$ of (4.15) and (4.16).

Conversely, assume h_1 and h_2 are continuous functions satisfying (4.3) and (4.17). Using some w_{01} and w_{02} as in (4.19), and any constant c, *define* functions w_1 and w_2 by (4.21), (4.26), and (4.27). They satisfy (1.2), and by (4.25) they also obey (1.3). They are readily shown to be solutions of (4.15) and (4.16). Next set $R=-1$, and for any set of

$$b_{i121}>0,\ b_{l221}<0, \tag{4.32}$$

define the remaining b's by (4.12), define the y's from the z's by (4.13), and the x's from the y's by (1.10). Then w_1 and w_2 satisfy (4.4) through (4.7), and satisfy the irredundancy hypothesis because the b's are nonzero. It remains to show that they obey the competitiveness hypothesis.

Consider any nonnegative pair (w_1^0, w_2^0) summing to 1. It must be shown that $\mathbf{w}^0 = (w_1^0, w_2^0)$ is in or at least a limit point of the attainable region in $\mathbf{w}$-space. The situation regarding the attainable region in $\mathbf{x}$-space is like that discussed for $p > 2$ near the end of section 3, and we adopt for definiteness the hypothesis about that region analogous to the one chosen earlier for $p > 2$: There are variables $x_{i(1)1}$ and $x_{i(2)2}$, which we immediately make x_{11} and x_{12} by renumbering, and numerical settings $x_{i1} = x_{i1}^0 > 0$ and $x_{i2} = x_{i2}^0 > 0$ for all the *other* x's, and numbers $x_{11}^+ > 0$ and $x_{12}^+ > 0$, such that *any* numerical pair (x_{11}, x_{12}) satisfying

$$0 < x_{11} < x_{11}^+, \; 0 < x_{12} < x_{12}^+$$

forms, together with the fixed settings x_{i1}^0 and x_{i2}^0, a point $\mathbf{x}$ in the attainable region.

Let the fixed settings z_{i1}^0 and z_{i2}^0 correspond via (1.10) and (4.13) to the x_{i1}^0 and x_{i2}^0. Taking into account the signs in (4.32), we find that there exist intervals

$$(-\infty, z_{11}^+) \text{ and } (z_{12}^+, \infty)$$

such that any z_{11} and z_{12} lying respectively in these intervals form, together with the fixed settings, a point in the attainable region in $\mathbf{z}$-space. If now $0 < w_2^0 < 1$, then we can choose such z_{11} and z_{12} so that

$$z_{11} + \sum_{2}^{n(1)} z_{i1}^0 + z_{12} + \sum_{2}^{n(2)} z_{i2}^0 + c = H_2(w_2^0),$$

and so $w_2 = w_2^0$ and hence $w_1 = w_1^0$. If $w_2^0 = 0$ we use a limiting process with z_{12} fixed and $z_{11} \to (-\infty)$, while if $w_2^0 = 1$ we use a limiting process with z_{11} fixed and $z_{12} \to \infty$.

This completes the proof that all consistent models and their general solutions have been found. We note (omitting details) that for the linear case

$$f_j(w) = d_j w, \quad g_j(w) = w,$$

the solution derived above has exactly the same form as that obtained in section 3 for $p > 2$.

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