The Distribution of the Sample Correlation Coefficient
With One Variable Fixed

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For the usual straight-line model, in which the independent variable takes on a fixed, known set of values, it is shown that the sample correlation coefficient is distributed as \( Q \) with \( (n-2) \) degrees of freedom and noncentrality \( \theta = (\beta / \sigma) \sqrt{\sum (x_i - \bar{x})^2} \). The \( Q \) variate has been defined and studied elsewhere by Hogben et al. It is noted that the square of the correlation coefficient is distributed as a noncentral beta variable.

Key Words: Analysis of variance, calibration, correlation coefficient, degrees of freedom, distribution, fixed variable, noncentral beta variable, noncentrality, \( Q \) variate.

1. Introduction

Consider the straight-line model

\[
Y_i = \alpha + \beta x_i + \epsilon_i, \quad i = 1, 2, \ldots, n
\]

where

(i) the \( \epsilon_i \) are assumed to behave as normally and independently distributed random variables with mean zero and common variance \( \sigma^2 \),

(ii) \( \alpha \) and \( \beta \) are unknown parameters, and

(iii) lowercase italic letters denote fixed, known constants and uppercase italic letters denote random variables, i.e., \( x \) is fixed and \( Y \) is random. This and other straight-line models are discussed in detail by Acton [1959].

The sample correlation coefficient \( r_{xy} \) is defined by

\[
r_{xy} = \frac{\sum_{i=1}^{n} (x_i - \bar{x})(Y_i - \bar{Y})}{\sqrt{\sum_{i=1}^{n} (x_i - \bar{x})^2 \sum_{i=1}^{n} (Y_i - \bar{Y})^2}}.
\]

where

\[
\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n} \text{ and } \bar{Y} = \frac{\sum_{i=1}^{n} Y_i}{n}.
\]

It is of some interest in the calibration problem where a fitted straight-line is used in reverse for estimating an unknown \( x_0 \) corresponding to an observed \( Y_0 \). The distribution of \( r_{xy} \) where \( X \) and \( Y \) follow the bivariate normal is well known; see for example Kendall and Stuart [1961, pp. 383–390]. In the present paper the distribution of \( r_{xy} \) as defined by (2) is derived for \( x \) fixed. This distribution
is well known for the special case with all $Y_i$ identically distributed (i.e., $\beta = 0$), in which case it is the same as the distribution of $r_{XY}$ for $X$, $Y$ independent and normal. See, e.g., Hotelling [1953, p. 196]. J. N. K. Rao and an unidentified person have pointed out that the distribution of $r_{XY}^2$ can be obtained as a special case of the conditional distribution of the multiple correlation coefficient for the multi-variate normal; see, e.g., C. R. Rao [1965, p. 509].

2. Derivation

In an analysis of variance for the model (1) the (corrected) total sum of squares with $(n - 1)$ degrees of freedom may be partitioned into two independent components; the first being the sum of squares due to the slope with 1 degree of freedom and the second being the residual sum of squares with $(n - 2)$ degrees of freedom. This partition can be expressed by

$$
\sum_{i=1}^{n} (Y_i - \bar{Y})^2 = \left[\sum (x_i - \bar{x}) (Y_i - \bar{Y}) \right]^2 + \sum (Y_i - \hat{Y}_i)^2,
$$

where

$$
\hat{Y}_i = \bar{Y} + \hat{\beta} (x_i - \bar{x})
$$

and

$$
\hat{\beta} = \frac{\sum (x_i - \bar{x}) (Y_i - \bar{Y})}{\sum (x_i - \bar{x})^2}
$$

is the usual least squares estimator for $\beta$. Let the random variables $W$ and $X^2$ be defined by

$$
W = \frac{\sum (x_i - \bar{x}) (Y_i - \bar{Y})}{\sigma \sqrt{\sum (x_i - \bar{x})^2}},
$$

and

$$
X^2 = \frac{\sum (Y_i - \hat{Y}_i)^2}{\sigma^2}.
$$

Using (4) and (5) and dividing both sides of eq (3) by $\sigma^2$ we have

$$
\sum_{i=1}^{n} (Y_i - \bar{Y})^2/\sigma^2 = W^2 + X^2.
$$

If both the numerator and denominator of $r_{XY}$ are divided by $\sigma^2$ and the first factor of the denominator is combined with the numerator, the correlation coefficient may be written as

$$
r_{XY} = \frac{W}{\sqrt{W^2 + X^2}}.
$$

Since the $Y_i$ are normally distributed, it is easily shown that $W$ is normally distributed with mean $\theta = (\beta/\sigma) \sqrt{\Sigma (x_i - \bar{x})^2}$ and variance 1. Further, it is well known from the theory of the general linear hypothesis that under model (1) $W$ and $X^2$ are independently distributed and $X^2$ is distributed as chi-squared with $(n - 2)$ degrees of freedom. Therefore, $r_{XY}$ is equal to the random variable $Q$ defined and studied in Hogben et al., [1964a] and [1964b]. Hence, the following theorem is proved.

**Theorem:** The correlation coefficient $r_{XY}$, defined by (2) under model (1), is distributed as $Q$ with $(n - 2)$ degrees of freedom and noncentrality $\theta = (\beta/\sigma) \sqrt{\Sigma (x_i - \bar{x})^2}$.
Various properties of $Q$ are given in the previous two references, including analytic expressions and recurrence relations for the moments about zero, numerical values for the first four central moments and an approximation to the distribution of $Q$ by that of a linearly transformed beta variable. It follows from (7) that $r_{xy}^2$ is distributed as noncentral beta; see for example Seber [1963], where in his notation $n_1 = 1$, $n_2 = n - 2$ and $\lambda = \theta^2/2$. Furthermore, $t = \sqrt{(n - 2)r^2/(1 - r^2)}$ is distributed as noncentral $t$ with noncentrality $\theta$ and $(n - 2)$ degrees of freedom. The distribution of $r_{xy}$ also follows from the interesting and easily derived relation

$$
r_{xy} = \frac{\hat{\beta}}{\sqrt{\beta^2 + (n - 2)s^2}}
$$

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3. References


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