JOURNAL OF RESEARCH of the National Bureau of Standards — B. Mathematical Sciences
Vol. 72B, No. 1, January–March 1968

A Note on the G-Transformation*

H. L. Gray** and T. A. Atchison**

(December 12, 1967)

Recent literature concerning the use of nonlinear transformations to evaluate numerically certain improper integrals of the first kind has shown that difficulties are encountered if the integrand f is such that

$$\lim_{t\to\infty}\frac{f(t+k)}{f(t)}=1.$$

This note introduces a new nonlinear transformation which is in some cases quite useful when the above limit is one. A simple example is given to illustrate the use of this transformation.

Key Words: Improper integrals, nonlinear transformations.

1. Introduction

In a recent paper [1],[1] H. L. Gray and T. A. Atchison have introduced a nonlinear transformation for the purpose of evaluating improper integrals of the first kind. This transformation is most useful on integrals of the type

$$\int_a^\infty f(x)dx, \tag{1.1}$$

where

$$\lim_{t\to\infty}\frac{f(t+k)}{f(t)}=R\neq 0 \text{ or } 1. \tag{1.2}$$

In this note, a new transformation is introduced which will be more suitable when $R=1$ and which reduces to the transformation defined in [1] when $R\neq 1$.

2. The Transformation

Let

$$F(t)=\int_a^t f(x)dx\to S \text{ as } t\to\infty. \tag{2.1}$$

When the following limit exits, let

$$\alpha=\lim_{t\to\infty}\frac{1-R(t;\,k)}{R(t;\,k)}\,\frac{E(t+k)}{F(t+k)-F(t)} \tag{2.2}$$

where $R(t;\,k)=f(t+k)/f(t)$ and $E(t+k)=S-F(t+k)$.

*An invited paper.

**Present address: Texas Technological College, Department of Mathematics, P.O. Box 4319, Lubbock, Texas 79409.

[1] H. L. Gray and T. A. Atchison. Nonlinear transformations related to the evaluation of certain improper integrals, SIAM Journal on Numerical Analysis **4**, No. 3, 363–371 (1967).

DEFINITION 2.1. *Let*

$$L[F;\ t,\ k]=F(t+k)+\alpha R(t;\ k)\,\frac{F(t+k)-F(t)}{1-R(t;\ k)}, \tag{2.3}$$

where we assume α exists and $R(t;\ k)\neq 1$.

In order to determine α from (2.2), the value of the integral S appears to be necessary. The following considerations show that this is not always true. Note that

$$\frac{1-R(t;\ k)}{R(t;\ k)}\,\frac{E(t+k)}{F(t+k)-F(t)}=\frac{E(t+k)/[1/f(t+k)-1/f(t)]^{-1}}{[F(t+k)-F(t)]/f(t)}. \tag{2.4}$$

If $1/f(t+k)-1/f(t)\rightarrow\infty$ as $t\rightarrow\infty$, then L'Hospital's rule can be applied to the numerator and denominator separately to assist in determining α. That is, if the limits exist and the denominator limit is not zero,

$$\alpha=\frac{\lim\limits_{t\rightarrow\infty}-f(t+k)/\dfrac{d}{dt}[1/f(t+k)-1/f(t)]^{-1}}{\lim\limits_{t\rightarrow\infty}\ [f(t+k)-f(t)]/f'(t)}. \tag{2.5}$$

Note that (2.5) does not involve any integration.

THEOREM 2.1. $L[F;\ t,\ k]\rightarrow S$ *as* $t\rightarrow\infty$ *if, and only if,*

$$\alpha\ \lim_{t\rightarrow\infty} R(t;\ k)\,\frac{F(t+k)-F(t)}{1-R(t;\ k)}=0. \tag{2.6}$$

PROOF. This follows immediately from Definition 2.1.

THEOREM 2.2. $L[F;\ t,\ k]\rightarrow S$ *as* $t\rightarrow\infty$.

PROOF. If $\alpha=0$, the result is immediate from Theorem 2.1.

If $\alpha\neq 0$, then

$$\alpha\lim_{t\rightarrow\infty} R(t;\ k)\,\frac{F(t+k)-F(t)}{1-R(t;\ k)}=\lim_{t\rightarrow\infty} E(t+k)=0 \tag{2.7}$$

and the result follows from Theorem 2.1.

The fact that $L[F;\ t,\ k]$ converges to S is of importance. However, the purpose of this transformation is to obtain a function which converges to S more rapidly than the original integral.

THEOREM 2.3. *If* $\alpha\neq 0$, *then* $L[F;\ t,\ k]\rightarrow S$ *more rapidly than* $F(t+k)$ *as* $t\rightarrow\infty$.

PROOF. Since

$$\frac{S-L[F;\ t,\ k]}{S-F(t+k)}=\frac{S-F(t+k)-\alpha R(t;\ k)\,\dfrac{F(t+k)-F(t)}{1-R(t;\ k)}}{S-F(t+k)} \tag{2.8}$$

$$=1-\alpha\,\frac{R(t;\ k)}{1-R(t;\ k)}\,\frac{F(t+k)-F(t)}{E(t+k)}$$

and $\alpha\neq 0$, then

$$\lim_{t\rightarrow\infty}\frac{S-L[F;\ t,\ k]}{S-F(t+k)}=1-\alpha\,\frac{1}{\alpha}=0. \tag{2.9}$$

Note that if $\alpha=0$, then $L[F;\ t,\ k]=F(t+k)$ and more rapid convergence is not achieved. Another interesting relation occurs if $\alpha=1$. In this case $L[F;\ t,\ k]=G[F;\ t,\ k]$, the nonlinear transformation introduced in [1]. The circumstances under which L will reduce to G are considered in the next theorem.

THEOREM 2.4. *If* $\lim_{t\to\infty} R(t; k) = R(k) \neq 0, 1$, *then* $L[F, t, k] = G[F; t, k]$.

PROOF. Since

$$\lim_{t\to\infty} \frac{E(t+k)}{F(t+k)-F(t)} = \lim_{t\to\infty} \frac{-f(t+k)}{f(t+k)-f(t)}$$

$$= \lim_{t\to\infty} \frac{R(t; k)}{1-R(t; k)} = \frac{R(k)}{1-R(k)}, \tag{2.10}$$

then $\alpha = 1$ and the theorem follows.

The importance of the transformation L lies in the fact that regardless of the limit of $R(t; k)$, if α exists and is different from zero, then more rapid convergence to S is still achieved. This is best illustrated by considering the following simple example in which $G[F; t, k]$ fails to converge more rapidly than $F(t+k)$ but $L[F; t, k]$ converges more rapidly than $G[F; t, k]$ or $F(t+k)$.

Let $f(x) = 1/(1+x^2)$. Then

$$R(t; k) = \frac{1+t^2}{1+(t+k)^2} \to 1 \text{ as } t \to \infty. \tag{2.11}$$

However,

$$\alpha = \lim_{t\to\infty} \frac{(2kt+k^2)\int_{t+k}^{\infty} \frac{1}{1+x^2}\,dx}{(1+t^2)\int_t^{t+k} \frac{1}{1+x^2}\,dx} = \frac{2k}{k} = 2 \tag{2.12}$$

as may be determined by using eq (2.5). The transformation described in this paper becomes

$$L[F; t, k] = \int_0^{t+k} \frac{1}{1+x^2}\,dx + 2\,\frac{1+t^2}{2kt+k^2}\int_t^{t+k} \frac{1}{1+x^2}\,dx. \tag{2.13}$$

Taking $t = 20$ and $k = 0.1$,

$$L[F; 20, 0.1] \simeq 1.571213756 \tag{2.14}$$

which is in error by about 0.0004. It should be noted that arc tan 20.1 is in error by about 0.05 while $G[F; 20, 0.1]$ is in error by approximately 0.02.

Clearly the integral above, being very simple, could be integrated quite satisfactorily by a number of other numerical methods. However it adequately illustrates the comparison between L and G.

(Paper 72B1–256)

289-930 OL-68—3