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Means and the Minimization of Errors*

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Let $0 < a < b$. How should a number p be chosen so that the maximum 'relative error' obtained, by replacing a number x varying in the closed interval $[a, b]$, by p, is a minimum? For a large number of 'relative errors,' p must be chosen as the geometric mean of a and b.

Key Words: Arithmetic mean, geometric mean, harmonic mean, means, relative error.

Let $0 < a < b$. If x is a number contained in the closed interval $[a, b]$, there are various measures of the "error" committed in replacing x by an approximation p. For example the "absolute error" $|p-x|$, or the "relative error" $\frac{|p-x|}{x}$. In [1],[1] Huntington suggests more general relative errors of the form $\frac{|p-x|}{\phi(p, x)}$ where $\phi(p, x)$ is a mean of p and x (that is for all p and x, $\phi(p, x)$ lies between (not necessarily strictly) p and x). We shall consider errors of this type subject to a few other conditions. If $E(p, x)=\frac{|p-x|}{\phi(p, x)}$ is given, for each p in $[a, b]$, let $\lambda(p)=\max_x E(p, x)$ and let μ in $[a, b]$ satisfy $\lambda(\mu)=\min_p \lambda(p)$. The conditions we impose on ϕ besides being a mean are that $\lambda(p)$ exist and that μ exist and be unique. If ϕ^* denotes the 'transpose' of $\phi(\phi^*(p, x)=\phi(x, p))$, we will also require that ϕ^* satisfy the same conditions as ϕ. In [2], Pólya showed that for $\phi(p, x)=x$, $\mu=\frac{2ab}{a+b}$. In this note we compute μ for a variety of other means. As a final condition imposed on ϕ and hence on E, we demand that if p' is strictly between p and x then $E(p', x)$ be strictly smaller than $E(p, x)$ and that $E(x, x)=0$. A set of sufficient conditions on ϕ to ensure all these requirements are

(1) ϕ is a mean.
(2) ϕ is continuous on the boundary of the square, $[a, b] \times [a, b]$.
(3) $\phi(t, u)$ and $\phi(u, t)$ as functions of u are monotone in each of the intervals $[a, t]$ and $[t, b]$, for each $t \in [a, b]$. (A)

Under the conditions assumed in paragraph 1 preceding (A),

$$\lambda(p)=\max\,(E(p, a), E(p, b)). \tag{1}$$

and μ can be characterized as the unique zero in $[a, b]$ of the equation

$$E(\mu, a)=E(\mu, b) \tag{2}$$

or

$$(\mu-a)\phi(\mu, b)+(\mu-b)\phi(\mu, a)=0 \tag{3}$$

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[1] Figures in brackets indicate the literature references at the end of this paper.

For many choices of ϕ the solution of (3) is completely routine and we merely record the results for some of them. By $A(u, v)$, $G(u, v)$, and $H(u, v)$ we mean the arithmetic mean, geometric mean, and harmonic mean of u and v, respectively.

TABLE 1.

$\phi(p, x)$	$\mu(a, b)$
x	$H(a, b)$
p	$A(a, b)$
max (p, x)	$G(a, b)$
min (p, x)	$G(a, b)$
$A(p, x)$	$G(a, b)$
$G(p, x)$	$G(a, b)$
$H(p, x)$	$G(a, b)$

We discuss another example in some detail. Let

$$\phi_\tau(p, x) = A(p, x) + \frac{\tau}{2}(p-x). \tag{4}$$

For $-1 \leqq \tau \leqq 1$, ϕ_τ is admissible for our discussion. If $\mu_\tau(a, b)$ is the corresponding value of μ, (3) becomes

$$Q_\tau(\mu_\tau) = 0 \tag{5}$$

where

$$Q_\tau(u) = (\tau+1)u^2 - \tau(a+b)u + (\tau-1)ab, \tag{6}$$

or equivalently

$$Q_\tau(u) = \tau(u-a)(u-b) + u^2 - ab. \tag{7}$$

From (7), $Q_\tau(a) = a(a-b) < 0$ and $Q_\tau(b) = b(b-a) > 0$. Since Q_τ is of degree at most 2, there is a unique zero in the interval $[a, b]$. For $-1 \leqq \tau \leqq 1$, this would follow from the general theory, but the argument just presented does not depend on ϕ_τ being a mean. For all real τ we define $\mu_\tau(a, b)$ as the unique zero of (5) in the interval $[a, b]$.

From (7) it follows that

$$Q_{\tau+\delta}(u) - Q_\tau(u) = \delta(u-a)(u-b).$$

Setting $u = \mu_\tau$, for $\delta > 0$, we obtain $Q_{\tau+\delta}(\mu_\tau) < 0$. Since $Q_{\tau+\delta}(b) > 0$, we obtain $a < \mu_\tau < \mu_{\tau+\delta} < b$. Hence as a function of τ, μ_τ is strictly increasing. As $\tau \to +\infty$, one of the zeros of (5) approaches b and the other becomes unbounded. By the monotonicity we then have $\mu_\tau(a, b) \to b$ as $\tau \to +\infty$. Similarly $\mu_\tau(a, b) \to a$ as $\tau \to -\infty$. This motivates the definitions $\mu_\infty(a, b) = +b$, $\mu_{-\infty}(a, b) = a$.

We may uniquely extend μ_τ for all pairs of positive numbers by defining $\mu_\tau(x, x) = x$ and insisting that $\mu_\tau(x, y) = \mu_\tau(y, x)$. With these extensions μ is continuous in all variables $-\infty \leqq \tau \leqq \infty$, $0 < x < \infty$, $0 < y < \infty$.

It is interesting to compare this one parameter family of means with the standard means $M_\tau(a, b)$ [3]. For $\tau = -\infty, -1, 0, 1, \infty$; $\mu_\tau = M_\tau$, each is homogeneous and symmetric. For $\tau = 2$, $\mu_2(a, b) \leqq M_2(a, b)$ for all (a, b). We conjecture that for each τ, μ_τ and M_τ are comparable.

From (6) or (7) we find that if $x \neq 0$

$$x^2 Q_{-\tau}\left(\frac{ab}{x}\right) = -abQ_\tau(x). \tag{8}$$

Also

$$a \leqslant x \leqslant b \rightarrow a \leqslant \frac{ab}{x} \leqslant b. \tag{9}$$

Hence

$$\mu_{-\tau}(a, b) = \frac{ab}{\mu_\tau(a, b)}. \tag{10}$$

For $\tau = 1$, this is the elementary property

$$G(A(a, b), H(a, b)) = G(a, b). \tag{11}$$

From (4) it follows that $\phi_{-\tau} = \phi_\tau^*$. This suggests the following generalization of (10). We recall that a mean ϕ is homogeneous if for positive x, y, and k,

$$\phi(kx, ky) = k\phi(x, y).$$

THEOREM: *Let ϕ be a homogeneous mean satisfying the conditions (A). Let ϕ^* be the transposed mean. If μ and μ^* are the corresponding means of* a *and* b, *then*

$$\mu\mu^* = \mathrm{ab}.$$

PROOF: μ is characterized by

(i) $a < \mu < b$

(ii) $(\mu - a)\phi(\mu, b) + (\mu - b)\phi(\mu, a) = 0.$

μ^* is characterized by

(iii) $a < \mu^* < b$

(iv) $(\mu^* - a)\phi^*(\mu^*, b) + (\mu^* - b)\phi^*(\mu^*, a) = 0.$

Let

$$P(x) = (x - a)\phi^*(x, b) + (x - b)\phi^*(x, a)$$

$$P\left(\frac{ab}{\mu}\right) = \left(\frac{ab}{\mu} - a\right)\phi^*\left(\frac{ab}{\mu}, b\right) + \left(\frac{ab}{\mu} - b\right)\phi^*\left(\frac{ab}{\mu}, a\right).$$

Since ϕ is homogeneous, $\phi^*\left(\frac{ab}{\mu}, b\right) = \phi\left(b, \frac{ab}{\mu}\right) = \frac{b}{\mu}\phi(\mu, a)$ and $\phi^*\left(\frac{ab}{\mu}, a\right) = \frac{a}{\mu}\phi(\mu, b)$.

Hence

$$P\left(\frac{ab}{\mu}\right) = \frac{b}{\mu}\left(\frac{ab}{\mu} - a\right)\phi(\mu, a) + \frac{a}{\mu}\left(\frac{ab}{\mu} - b\right)\phi(\mu, b)$$

$$= -\frac{1}{\mu^2}ab\,\{(\mu - b)\phi(\mu, a) + (\mu - a)\phi(\mu, b)\} = 0.$$

Since $a < \mu < b$, $a < \frac{ab}{\mu} < b$.

Hence $\mu^* = \frac{ab}{\mu}$ or $\mu\mu^* = ab$.

COROLLARY: *If in addition to the hypotheses of the theorem, ϕ is symmetric, ($\phi = \phi^*$), then* $\mu = \mathrm{G(a, b)}$.

The corollary explains the frequent occurrence of $G(a, b)$ in table 1. Symmetry without homogeneity is not sufficient for $\mu(a, b) = G(a, b)$. For an example let $\phi(p, x) = \frac{p^2 + x^2}{p^2 + x^2 + 1}$ max $(p, x) + \frac{1}{p^2 + x^2 + 1}$ min (p, x). By direct substitution in (3) it can be shown that $G(a, b)$ is not $\mu(a, b)$.

References

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