JOURNAL OF RESEARCH of the National Bureau of Standards — C. Engineering and Instrumentation
Vol. 71C, No. 2, April–June 1967

Sensitivity of a Correlation Radiometer*

John J. Faris**

Institute for Basic Standards, National Bureau of Standards, Boulder, Colo. 80302

(November 14, 1966)

The correlation radiometer is analyzed to determine the sensitivity that can be obtained under various operating conditions.

The radiometer using a sine wave comparison signal is analyzed and compared with the usual radiometer that employs a random noise for the comparison signal. It is found that the radiometer employing the sine wave comparison signal is the more sensitive of the two circuits, particularly in the case that the effective temperature of the input noise signal is greater than the effective input temperature of the amplifiers.

It is shown that if nonidentical amplifiers are used in the correlation circuit, the properties of the radiometer are determined by the portion of the amplifier response functions in the frequency interval that the two response functions overlap. The effect of amplifier gain fluctuations are considered, and although the correlation scheme reduces the effect of gain fluctuations, it is shown that they still do contribute to the output fluctuations of the radiometer.

Calculations are included showing that the effect of a differential phase shift between the two channels is a reduction in radiometer sensitivity. The same conclusion is reached concerning the effect of a differential time delay.

Finally, it is shown that if the comparison signal and the input signal have the same statistical properties, the requirements on the multiplier are less stringent than if the two signals have different statistical properties.

Key Words: Correlation radiometer, differential time delay, gain fluctuations, imperfect multiplier, noise comparison, nonidentical amplifiers, sine-wave comparison signal.

1. Introduction

The use of correlation techniques in radiometry has been suggested recently by several workers [Strum, 1958; Blum, 1959; Colvin, 1961; Allred, 1962]. One of the major reasons for this interest is that by using correlation techniques, it is possible to build a radiometer in which amplifier gain fluctuations contribute less to the fluctuations present in the output of the instrument than they do in the conventional radiometer; thus, the correlation radiometer might be expected to be superior to the conventional radiometer under conditions in which gain fluctuations are important. Strum [1958] has pointed out that as the noise level in a conventional radiometer system is reduced, the fluctuations due to changes in amplifier gain become more serious; therefore, at cryogenic temperatures it is very important to reduce this contribution to the total radiometer fluctuation. Thus, it was felt to be desirable to calculate the sensitivity obtainable from instruments of this type.

The analysis was carried out on the correlation receiver shown in figure 1. Although several different types of correlation receivers have been suggested in the literature, the operation of all of them is quite similar to the one chosen here. Therefore, the results obtained here should be applicable with appropriate modifications to many of the correlation radiometer circuits in the literature.

The first problem to be considered (see sec. 2) is the calculation of the sensitivity of a radiometer with amplifiers whose gain and passband are identical but with different effective tempera-

*This research was supported by the Advanced Research Projects Agency under Project DEFENDER and was monitored by M. I. Witow under ARPA Order No. 515.

**Radio Standards Laboratory, NBS Laboratories, Boulder, Colo. 80302. Permanent address, Colo. State University, Ft. Collins, Colo. 80521.

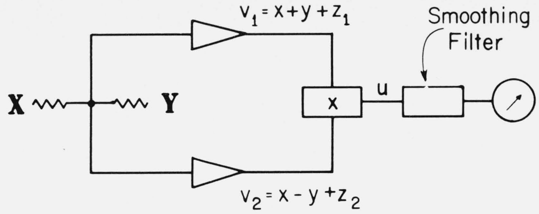

FIGURE 1. *Radiometer circuit.*

tures at the input circuit. These calculations are carried out for two types of input signals: (1) both input signals consist of white noise and, (2) one input signal is white noise and the other is sinusoidal. The first case was selected because it corresponds to the problem considered by Colvin [1961], and the second was chosen because it was the problem considered by Allred [1962]. Throughout this calculation, all filters are assumed to have a square passband. This assumption is made because it allows the results of the calculation to be written rather explicitly and thus it is easy to see the physical significance of the results. When it is desired to apply these results to a real radiometer with filters that are not square, it is only necessary to use the formalism of Colvin [1959] that expresses the effective bandwidth of a filter in terms of a convolution integral. Thus, this generalization can be carried out rather simply.

Section 3 is an analysis of the correlation radiometer in which it is assumed that the two amplifiers have different gains and different response curves.

Section 4 deals with the same circuit, but here the effect of variation in phase and in time delay of the signals in the two amplifiers is considered.

Finally, section 5 deals with some effects produced by an imperfect multiplier.

2. Sensitivity of an Ideal Correlation Radiometer

The circuit to be analyzed is shown in figure 1. Two noise sources, designated X and Y, are connected to opposite arms of a matched hybrid junction. The signals from the other pair of arms are applied to amplifiers whose transfer functions are $R_1(\omega)$ and $R_2(\omega)$ respectively. The outputs of the amplifiers are multiplied together and the product is filtered with a smoothing filter (low pass filter). The output of the smoothing filter is displayed on a d-c instrument.

As the following analysis will show, the average value of the deflection of the output instrument is proportional to the difference in noise temperature of the two sources, X and Y. Thus, when the two noise sources have the same effective temperature, the average value of the output deflection is zero and the radiometer is said to be balanced. The instantaneous deflection is a stochastic function of the time; thus the random fluctuations of the output deflection produce an uncertainty in the experimental conditions that correspond to a balance.

The sensitivity of a radiometer is usually defined as the change in temperature of one of the noise sources that will produce a deflection whose magnitude is equal to the root mean square of the output fluctuation. Therefore, in order to calculate the sensitivity of a radiometer it is necessary to obtain expressions for both the rms value of the output fluctuation and the change in average deflection per degree change in effective temperature of one of the noise sources.

In this section the assumption will be made that the two amplifiers have the same power transfer function, and for simplicity, a "square passband" will be assumed.

If $x(t)$ is the signal voltage due to source X that leaves each amplifier, and $y(t)$ is the signal voltage due to source Y that leaves amplifier 1, the signal voltage $-y(t)$ due to source Y must leave amplifier 2. It will be further assumed that the amplifiers introduce noise voltages z_1 and z_2 respectively.

Since the random functions $x(t)$, $y(t)$, $z_1(t)$, and $z_2(t)$ represent noise from physically separate noise generators, it will be assumed that they are mutually independent. It is also assumed that each is a second order, stationary Gaussian process possessing a continuous spectral density function. These assumptions imply that the processes are ergodic so that time averages and ensemble averages coincide. It is also assumed that each of these functions has zero mean.

If the output of amplifier 1 is designated by v_1 and the output of amplifier 2 by v_2, then

$$v_1 = x + y + z_1 \tag{1}$$

and

$$v_2 = x - y + z_2. \tag{1a}$$

The multiplier will be assumed to have a response law such that

$$u = a v_1 v_2 \tag{2}$$

where u is the voltage output and a is a constant of proportionality.

For any random quantity ξ, let $\overline{\xi}$ be the expectation or ensemble average of ξ. (This is the equivalent of the notation $E(\xi)$ used in the literature of mathematical statistics.) Using the independence and normality of the random processes and the fact that each process has zero mean, the autocorrelation function of u is

$$\psi_u(\tau) = \overline{u(t)u(t+\tau)} = a^2\ \overline{v_1(t)v_2(t)v_1(t+\tau)v_2(t+\tau)}$$

$$= a^2\ \overline{[x^2(t) - y^2(t) + z_1(t)(x(t) - y(t)) + z_2(t)\,(x(t) + y(t)) + z_1(t)z_2(t)][x^2(t+\tau) - y^2(t+\tau) + \ldots]}.$$

$$\psi_u(\tau) = a^2[\psi_{x^2}(\tau) + \psi_{y^2}(\tau) - 2\psi_x(0)\psi_y(0) + \psi_{z_1}(\tau)\,(\psi_x(\tau) + \psi_y(\tau)) + \psi_{z_2}(\tau)(\psi_x(\tau) + \psi_y(\tau)) + \psi_{z_1}(\tau)\psi_{z_2}(\tau)]. \tag{3}$$

If ξ_i, $i = 1, 2, 3, 4$ are Gaussian variables, each with zero mean and

$$\overline{\xi_i \xi_j} = \sigma_{ij}$$

then

$$\overline{\xi_1\xi_2\xi_3\xi_4} = \sigma_{12}\sigma_{34} + \sigma_{13}\sigma_{24} + \sigma_{14}\sigma_{23}. \tag{4}$$

Setting $\xi_1 = \xi_2 = x(t)$ and $\xi_3 = \xi_4 = x(t+\tau)$, there results

$$\psi_{x^2}(\tau) \equiv \overline{x^2(t)x^2(t+\tau)} = 2\psi_x^2(\tau) + \psi_x^2(0). \tag{5}$$

When (4) is applied to (3), the result is

$$a^{-2}\psi_u(\tau) = 2[\psi_x^2(\tau) + \psi_y^2(\tau)] + [\psi_x(0) - \Psi_y(0)]^2 + \psi_{z_1}(\tau)[\psi_x(\tau) + \psi_y(\tau)]$$
$$+ \psi_{z_2}(\tau)[\psi_x(\tau) + \psi_y(\tau)] + \psi_{z_1}(\tau)\psi_{z_2}(\tau). \tag{6}$$

The Fourier transform of the autocorrelation function of a process is the spectral density of the process. Further, the Fourier transform of a product of functions is the convolution of the Fourier transforms. Thus, designating the spectral density by Q, and the convolution operation by $*$,

$$a^{-2}Q_u(f) = [\psi_x(0) - \psi_y(0)]^2\delta(f) + 2[Q_x * Q_x(f) + Q_y * Q_y(f)]$$
$$+ [Q_{z_1} + Q_{z_2}] * [Q_x + Q_y](f) + Q_{z_1} * Q_{z_2}(f). \tag{7}$$

Case I. Consider both X and Y to be white noise generators at temperature T_x and T_y respectively. When the spectral densities are evaluated, it is necessary to consider the power division that takes place at the hybrid junction. These spectral densities are (see appendix)

$$Q_x * Q_x \approx \frac{1}{8}(kT_x)^2 B, \tag{8}$$

$$Q_x * Q_{z_1} \approx \frac{1}{4} k^2 T_x T_{z_1} B, \tag{8a}$$

$$Q_{z_1} * Q_{z_2} \approx \frac{1}{2} k^2 T_{z_1} T_{z_2} B. \tag{8b}$$

Thus, using Q' to designate the portion of the spectral density that contributes to fluctuations, this portion of (7) becomes

$$Q'_u = \frac{a^2k^2B}{4}\left[T_x^2 + T_y^2 + T_x(T_{z_1} + T_{z_2}) + T_y(T_{z_1} + T_{z_2}) + 2T_{z_1}T_{z_2}\right]. \tag{9}$$

With the assumption that the two amplifiers have the same effective noise temperature, this becomes

$$Q'_u = \frac{a^2k^2B}{4}\left[T_x^2 + T_y^2 + 2T_xT_z + 2T_yT_z + 2T_z^2\right]. \tag{10}$$

If the smoothing filter has a power response $G(f)$, the output power spectrum from the smoothing filter is

$$W(f) = G(f)Q'_u(f). \tag{11}$$

It is well known that integrating a power spectrum over all frequencies results in the mean square of the output voltage or current; so if the output voltage from the smoothing filter is $w(t)$, and $G(f)$ represents a square pass filter of width b, then

$$\overline{w(t)^2} = \int_{-\infty}^{\infty} W(f)df = \frac{1}{2} a^2k^2G_0bB\left[T_x^2 + T_y^2 + 2T_z^2 + 2(T_xT_z + T_yT_z)\right] \tag{12}$$

where G_0 is the power response of the filter at zero frequency. The fluctuations at balance are required, so the balance condition is substituted into $\overline{w^2}$. The balance occurs when $T_x = T_y$, so for balance

$$\overline{w^2} = a^2k^2G_0bB(T_x + T_z)^2. \tag{13}$$

Therefore, in terms of an rms voltage

$$w_{\text{rms}} = (\overline{w^2})^{1/2} = ak(G_0bB)^{1/2}(T_x + T_z). \tag{14}$$

Next, we calculate the deflection arising from a certain imbalance of the input signals. From (2)

$$\begin{aligned} u = av_1v_2 &= a(x + y + z_1)(x - y + z_2), \\ &= a[x^2 - y^2 + z_1z_2 + z_1(x - y) + z_2(x + y)]. \end{aligned} \tag{15}$$

Since $x(t)$, $y(t)$, $z_1(t)$ and $z_2(t)$ are all uncorrelated, when (15) is averaged over time, the result is

$$\bar{u} = a(\overline{x^2} - \overline{y^2}). \tag{16}$$

Again applying the theorem that the integral of the power spectrum over all frequencies is the average square of a function, (16) becomes

$$\begin{aligned}\bar{u} &= a\left[\int Q_x df - \int Q_y df\right], \\ &= \frac{akB}{2}[T_x - T_y].\end{aligned} \tag{17}$$

In order to examine the effect of a small imbalance, set

$$T_y = T_x + \Delta T. \tag{18}$$

Then

$$u + \Delta u = \frac{akB}{2}[T_x - (T_x + \Delta T)].$$

Since $u=0$ when $T_x = T_y$,

$$|\Delta u| = \frac{akB}{2}\Delta T. \tag{19}$$

The resulting signal at the output of the smoothing filter is

$$|\Delta w| = G_0^{1/2}|\Delta u| = \frac{a}{2}G_0^{1/2}kB\Delta T. \tag{20}$$

As is usual in radiometer calculations, the assumption is made that the minimum detectable signal occurs when the deflection is equal to w_{rms}. Thus, equating the right-hand sides of (14) and (20), the condition corresponding to minimum detectable signal results; i.e.,

$$ak(G_0 bB)^{1/2}(T_x + T_z) = \frac{a}{2}G_0^{1/2}kB\Delta T. \tag{21}$$

Thus, the minimum detectable temperature difference is

$$\Delta T = 2(b/B)^{1/2}(T_x + T_z). \tag{22}$$

Case II. Let Y produce a sinusoidal signal, so that

$$Q_y(f) = \frac{v^2}{4}[\delta(f - f_0) + \delta(f + f_0)]. \tag{23}$$

The expression for $\psi_{x^2-y^2}$ is needed, so it will be calculated first. From the previous calculation, it is known that

$$\psi_{x^2-y^2} = \psi_{x^2} + \psi_{y^2} - 2\overline{x^2 y^2}.$$

Thus, it is necessary to compute ψ_{y^2}:

$$\begin{aligned}\psi_{y^2} &= \overline{y^2(t)y^2(t+\tau)}, \\ &= \overline{\frac{v^4}{4}\cos^2\omega_0 t\cos^2\omega_0(t+\tau)},\end{aligned} \tag{24}$$

$$\psi_{y^2} = \frac{v^4}{4}\left[\frac{3}{8}\cos^2\omega_0\tau + \frac{1}{8}\sin^2\omega_0\tau\right]. \tag{25}$$

Making use of the above and (5), the result is

$$\psi_{x^2-y^2}=\overline{x^2}^2+2(\psi_x)^2+\frac{v^4}{4}\left(\frac{3}{8}\cos^2\omega_0\tau+\frac{1}{8}\sin^2\omega_0\tau\right)-2\overline{x^2}\,\frac{v^2}{4}. \tag{26}$$

From this, the power spectral density is

$$Q_{x^2-y^2}=\int \psi_{x^2-y^2}e^{-i\omega\tau}d\tau,$$

$$=\left(\overline{x^2}^2-\overline{x^2}\,\frac{v^2}{2}\right)\delta(f)+2(Q_x*Q_x)+\frac{v^4}{4}\left[\frac{1}{4}\,\delta(f)+\frac{1}{16}\,\delta(f-2f_0)+\frac{1}{16}\,\delta(f+2f_0)\right]. \tag{27}$$

Therefore

$$Q_u=a^2Q_{v_1v_2}=a^2\Big[Q_{x^2-y^2}+Q_{z_1(x-y)}+Q_{z_2(x+y)}+Q_{z_1z_2}\Big] \tag{28}$$

$$=a^2\left\{\left(\overline{x^2}-\frac{v^2}{4}\right)^2\delta(f)+\frac{v^4}{64}\,[\delta(f-2f_0)+\delta(f+2f_0)]+2(Q_x*Q_x)\right.$$

$$\left.+Q_{z_1}*(Q_x+Q_y)+Q_{z_2}*(Q_x+Q_y)+Q_{z_1}*Q_{z_2}\right\}. \tag{29}$$

Since the present calculation is to compute the fluctuations occurring in this radiometer, only the terms that contribute to the fluctuation need be considered; therefore they will be expressed as

$$Q_u'=a^2\{2(Q_x*Q_x)+(Q_{z_1}+Q_{z_2})*(Q_x+Q_y)+Q_{z_1}*Q_{z_2}\}, \tag{30}$$

where $Q_u'(f)$ denotes the fluctuating portion of Q_u.

All of the evaluations of the convolutions that occur in (30) can be taken from (8), except $Q_y*Q_{z_1}$; which is,

$$Q_y*Q_{z_1}=Q_{z_1}*Q_y=\int Q_{z_1}(f')Q_y(f-f')df',$$

where Q_y is given in (23). Thus,

$$Q_y*Q_{z_1}=\int Q_{z_1}(f')\,\frac{v^2}{4}\,[\delta(f-f'+f_0)+\delta(f-f'-f_0)]df',$$

$$=\frac{v^2}{4}\,[Q_{z_1}(f+f_0)+Q_{z_1}(f-f_0)]. \tag{31}$$

Since Q_{z_1} is assumed to be constant through the passband, when f_0 is in the passband and f is small, this convolution becomes

$$Q_y*Q_{z_1}=\frac{v^2}{4}\,kT_{z_1}. \tag{32}$$

Therefore

$$Q_u'=a^2\left\{\frac{1}{4}\,(kT_x)^2+\frac{1}{4}\,k^2BT_x(T_{z_1}+T_{z_2})+\frac{1}{4}\,v^2k(T_{z_1}+T_{z_2})+\frac{1}{2}\,k^2T_{z_1}T_{z_2}B\right\}. \tag{33}$$

By analogy with the previous calculation, the spectral density of the output of the filter can be written as

$$W(f) = G(f)Q_u(f),$$

and

$$\overline{w^2} = \int_{-\infty}^{\infty} W(f)df = a^2G_0(2b)\left\{\frac{1}{4}(kT_x)^2B + \frac{1}{4}k^2BT_x(T_{z_1}+T_{z_2}) + \frac{1}{4}v^2k(T_{z_1}+T_{z_2}) + \frac{1}{2}k^2T_{z_1}T_{z_2}B\right\}. \quad (34)$$

The d-c term of (29) indicates that the system is balanced when $kT_xB = v^2/2$. Also, to simplify the expression, it will be assumed that $T_{z_1} = T_{z_2}$. Then, at balance

$$\overline{w^2} = a^2G_0bk^2B\left(\frac{1}{2}T_x^2 + 2T_xT_z + T_z^2\right). \quad (35)$$

By analogy with the previous calculation, a root mean square fluctuation amplitude may be defined by

$$w_{\text{rms}} = ak(G_0bB)^{1/2}\left(\frac{1}{2}T_x^2 + 2T_xT_z + T_z^2\right)^{1/2}. \quad (36)$$

As mentioned previously, in order to arrive at the radiometer sensitivity, two parameters must be computed. The first is w_{rms}, which is given in (36). The second is the change in deflection due to a small change in temperature of the thermal source. The deflection sensitivity will be the same as it was in the previous calculation, so it is

$$|\Delta w| = G_0^{1/2}\frac{akB}{2}\Delta T.$$

Again, the minimum change in temperature that is observable will be taken to be a change that produces a deflection equal to the w_{rms}. Therefore

$$G_0^{1/2}\frac{akB}{2}\Delta T = ak(G_0B)^{1/2}\left(\frac{1}{2}T_x^2 + 2T_xT_z + T_z^2\right)^{1/2},$$

and the final result is

$$\Delta T = 2\left(\frac{b}{B}\right)^{1/2}\left(\frac{1}{2}T_x^2 + 2T_xT_z + T_z^2\right)^{1/2}. \quad (37)$$

When this result is compared with (22), the sensitivity of a radiometer with both signals assumed to be white noise, it is evident that this instrument has a somewhat higher sensitivity. This is particularly true in the case that the input signal is the dominant noise in the system and thus sets the fluctuation level. In this case the improvement in sensitivity approaches $\sqrt{2}$.

3. Radiometer With Dissimilar Amplifiers

In section 2 the assumption was made that the two amplifiers were identical, except that they were allowed to have different effective temperatures. Here the more general problem, in which the two amplifiers may differ with respect to other parameters, will be considered.

The considerations are limited at first to a case in which the two amplifiers have the same shape of gain function but with differing gain amplitudes. If $H_1(\omega)$ is the complex voltage gain of the first amplifier, then the complex voltage gain of the second will be taken to have the form

$$H_2(\omega, t) = [1+\alpha(t)]H_1(\omega). \tag{38}$$

where $\alpha(t)$ is a stochastic function.

If $x_1(t)$ is the signal resulting from amplifying the signal from the source X (this signal is $l(t)$) with the first amplifier and $x_2(t)$ is the result of amplifying $l(t)$ with the second amplifier, then a straightforward Fourier analysis shows that

$$x_2(t) = [1+\alpha(t)]x_1(t), \tag{39}$$

and a similar relationship occurs for the signals $y_1(t)$ and $y_2(t)$ that arise from the signal $m(t)$ from the source Y. This notation is summarized in figure 2.

The most general form of the autocorrelation function of the sum of two random functions is

$$\begin{aligned}\psi_{a\pm b} &= \overline{[a(t)\pm b(t)][a(t+\tau)\pm b(t+\tau)]},\\ &= \psi_a + \psi_b \pm \overline{a(t)b(t+\tau)+b(t)a(t+\tau)}.\end{aligned} \tag{40}$$

From this it follows that

$$\begin{aligned}\psi_{(x_1+y_1)(x_2-y_2)} &= \psi_{x_1x_2}+\psi_{x_1y_2}+\psi_{y_1x_2}+\psi_{y_1y_2}-2\overline{x_1x_2y_1y_2}\\ &\quad -\overline{x_1(t)y_1(t+\tau)y_2(t)x_2(t+\tau)}+\overline{y_1(t)x_1(t+\tau)x_2(t)y_2(t+\tau)}.\end{aligned} \tag{41}$$

From (39) it follows that

$$\begin{aligned}\psi_{x_1x_2} &= \overline{x_1(t)x_2(t)x_1(t+\tau)x_2(t+\tau)},\\ &= \overline{x_1^2(t)x_1^2(t+\tau)[1+\alpha(t)][1+\alpha(t+\tau)]},\\ &= \psi_{x_1^2}(1+\psi_\alpha+2\overline{\alpha}).\end{aligned} \tag{42}$$

By means of a similar line of reasoning, every term in (41) can be put into a form similar to that in (42). Thus,

$$\psi_{(x_1+y_1)(x_2-y_2)} = \psi_{(x_1+y_1)(x_1-y_1)}(1+\psi_\alpha+2\overline{\alpha}), \tag{43}$$

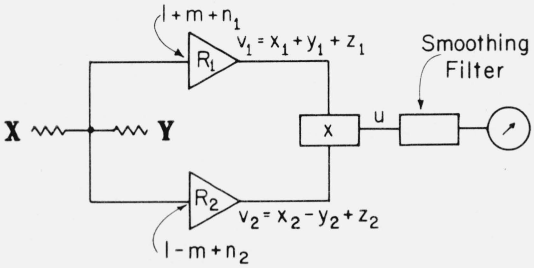

FIGURE 2. *Radiometer circuit with terminology for section 3.*

again

$$\psi_{z_1(x_2-y_2)}=(\psi_{z_1x_1}+\psi_{z_1y_1})(1+\psi_\alpha+2\overline{\alpha}), \tag{44}$$

and in the same way

$$\psi_{z_2(x_2-y_2)}=(\psi_{z_1^1x_1}+\psi_{z_1^1y_1})(1+\psi_\alpha+2\overline{\alpha}), \tag{45}$$

and

$$\psi_{z_1z_2}=\psi_{z_1z_1^1}(1+\psi_\alpha+2\overline{\alpha}), \tag{46}$$

where z_1^1 is the signal that would result if the noise signal $n_2(t)$ (the equivalent noise generated in the input of the second amplifier) were amplified by the first amplifier.

When these terms are all collected together, the result is

$$\psi_{v_1v_2}=(1+\psi_\alpha+2\overline{\alpha})\{(x_1^2-y_1^2)^2+\psi_{x_1}^2+\psi_{y_1}^2+\psi_{z_1x_1}+\psi_{z_1y_1}+\psi_{z_1^1x_1}+\psi_{z_1y_1}+\psi_{z_1^1z_1}\}\cdot \tag{47}$$

Notice that in this case the second and third terms of (41) are cancelled by the sixth term.

In order to simplify the analysis, we will consider two cases: (1) α a constant, independent of time, and (2) $\alpha(t)$ a random variable with $\overline{\alpha}=0$.

Case I. When α is a constant independent of time, it is elementary to show that

$$\left.\begin{aligned}\psi_\alpha&=\alpha^2\\ \overline{\alpha}&=\alpha\end{aligned}\right\}. \tag{48}$$

Thus

$$\psi_{v_1v_2}=(1+\alpha)^2\{(x_1^2-y_1^2)^2+2(\psi_{x_1}^2+\psi_{y_1}^2)+\psi_{z_1x_1}+\psi_{z_1y_1}+\psi_{z_1^1x_1}+\psi_{z_1^1y_1}+\psi_{z_1z_1^1}\}. \tag{49}$$

Therefore, both the d-c and the fluctuation terms are multiplied by $(1+\alpha)^2$. Since this affects both the fluctuations and the deflection by the same amount, the sensitivity is not changed.

Case II. Assume that $\alpha(t)$ is a random function of time, such that $\overline{\alpha}=0$. At balance the term $\overline{(x_1^2-y_1^2)}=0$, so using a prime to denote quantities evaluated at the balance point, the result is

$$\psi'_{v_1v_2}=(1+\psi_\alpha)\{2(\psi_{x_1}^2+\psi_{y_1}^2)+\psi_{z_1x_1}+\psi_{z_1y_1}+\psi_{z_1^1x_1}+\psi_{z_1^1y_1}+\psi_{z_1^1z_1}\}. \tag{50}$$

From this the spectral density is

$$Q'_{v_1v_2}=(1+Q_\alpha)*\{2(Q_{x_1}*Q_{x_1}+Q_{y_1}*Q_{y_1})+Q_{z_1}*(Q_{x_1}+Q_{y_1})+Q_{z_1^1}*(Q_{x_1}+Q_{y_1})+Q_{z_1^1}*Q_{z_1}\}. \tag{51}$$

Notice that (51) is similar to the result that is obtained in the case of identical amplifiers; the only difference is that the present result is the convolution of the original result with $(1+Q_\alpha)$.

The spectral density of the multiplier output is

$$Q_u=a^2Q_{v_1v_2},$$

where $Q_{v_1 v_2}$ is given by (51). The spectral density at the output of the low pass filter is

$$W(f) = G(f) Q_u(f),$$

$$= a^2 G(f)(1+Q_\alpha) * \{2(Q_{x_1} * Q_{y_1} + Q_{y_1} * Q_{y_1}) + Q_{z_1} * (Q_{x_1} + Q_{y_1}) + Q^1_{z_1} * (Q_{x_1} + Q_{y_1}) + Q_{z1_1} * Q_{z_1}\} \cdot \tag{52}$$

Following the same line of reasoning that was used from (11) to (14), the root mean square fluctuation on the output can be seen to be

$$w_{\mathrm{rms}} = \left[\int_{-\infty}^{\infty} W(f)\,df\right]^{1/2},$$

$$= \left[\int_{-\infty}^{\infty} a^2 G(f)(1+Q_\alpha) * \{2(Q_{x_1} * Q_{x_1} + Q_{y_1} * Q_{y_1}) + Q_{z_1} * (Q_{x_1} + Q_{y_1}) + Q^1_{z_1} * (Q_{x_1} + Q_{y_1}) + Q_{z^1_1} * Q_{z_1}\}\,df\right]^{1/2}. \tag{53}$$

To see the effect of the gain fluctuations, consider that the amplifiers have a square bandpass of bandwidth B. Then,

$$Q_{x_1} * Q_{x_1} = 2Q_l^2 R_1^2 B,$$

where $$R_1(\omega) = H_1(\omega)\, H_1^*(\omega),$$

is the power gain of the amplifier, and

$$Q_\alpha * Q_{x_1} * Q_{x_1} = \int_{-\infty}^{\infty} 2Q_l^2 R_1^2 B Q_\alpha(f-f')\,df' = 2Q_l^2 R_1^2 B \overline{\alpha^2}. \tag{54}$$

Thus the term $(1+Q_\alpha)*(Q_{x_1}*Q_{x_1})$ can be written as $2Q_l^2 R_1^2 B(1+\overline{\alpha^2})$. Each term in (53) will produce the same effect, so the result is

$$w_{\mathrm{rms}} = (1+\overline{\alpha^2})^{1/2} w_{\mathrm{rms}}(0), \tag{55}$$

where $w_{\mathrm{rms}}(0)$ is the fluctuation level that occurs when there are no gain fluctuations, i.e., when $\alpha = 0$.

Since gain fluctuations increase the fluctuation level of the output but have no effect on the average deflection, it is evident that this effect results in a reduction in radiometer sensitivity by the factor $(1+\overline{\alpha^2})^{1/2}$.

While the above expression for the reduction in radiometer sensitivity has been derived only in the case of white noise and amplifiers with square bandpass, examination of the integral that leads to (54) shows that the result will be qualitatively similar even in a less idealized case.

The next consideration will be to calculate the sensitivity of a radiometer with differing amplifier gain functions (transfer functions). In general these amplifiers could have differing center frequencies, and also the shape of the gain-frequency functions could differ.

The average square output can be expressed as

$$\overline{w(t)^2} = \int_{-\infty}^{\infty} G(f) Q_u\,df \tag{56}$$

where $G(f)$ is the power transfer function of the low pass filter. The input to the filter is given by

$$u = av_1v_2 = a(x_1 + y_1 + z_1)(x_2 - y_2 + z_2). \tag{2}$$

By means of the theorem expressed in (4), the autocorrelation function of u can be put into the form

$$a^{-2}\psi_u = (\overline{x_1x_2} - \overline{y_1y_2})^2 + (\psi_{x_1} + \psi_{y_1} + \psi_{z_1})(\psi_{x_2} + \psi_{y_2} + \psi_{z_2})$$
$$+ \overline{x_1(t)x_2(t+\tau)}\ \overline{x_2(t)x_1(t+\tau)} + \overline{y_1(t)y_2(t+\tau)}\ \overline{y_2(t)y_1(t+\tau)}$$
$$- \overline{x_2(t)x_1(t+\tau)}\ \overline{y_1(t)y_2(t+\tau)} - \overline{x_1(t)x_2(t+\tau)}\ \overline{y_2(t)y_1(t+\tau)}. \tag{57}$$

Recalling that

$$Q_{x_1} = R_1Q_l = H_1H_1^*Q_l,$$

and similar relationships exist for the other variables, the spectral density of u becomes

$$a^{-2}Q_u = (\overline{x_1x_2} - \overline{y_1y_2})^2\delta(f) + (R_1Q_l + R_1Q_m + R_1Q_{n_1})*(R_2Q_l + R_2Q_m + R_2Q_{n_2})$$
$$+ \mathrm{Re}\,\{[(H_1H_2^*Q_l)*(H_2H_1^*Q_l)] + [(H_1H_2^*Q_m)*(H_2H_1^*Q_m)]$$
$$- [(H_2H_1^*Q_l)*(H_1H_2^*Q_m)] - [(H_1H_2^*Q_l)*(H_2H_1^*Q_m)]\}. \tag{58}$$

The form of the last term arises from the theory of cross-spectra [Goodman, 1957; Korn and Korn, 1961].

The complete expression for the spectral density of the multiplier output is obtained by evaluating (58). From the resulting expression the fluctuation amplitude of the radiometer can be obtained by following the same procedure used previously.

Next it is necessary to calculate the average deflection resulting from a particular combination of input signals. This can be obtained by evaluating (2) of the simple theory in this more general case.

First, consider

$$\bar{u} = a\overline{v_1v_2} = a(\overline{x_1 + y_1 + z_1)(x_2 - y_2 + z_2}). \tag{2}$$

Of the six random variables that appear in the right-hand term of (2), the only correlations that are not zero are the partial correlations of x_1 with x_2 and of y_1 with y_2. Therefore this expression reduces to

$$\bar{u} = a(\overline{x_1x_2} - \overline{y_1y_2}). \tag{59}$$

Equation (59) must be evaluated in terms of the transfer functions of the two amplifiers, $H_1(f) + H_2(f)$, and the spectral densities of the noise outputs of the two noise sources. The Fourier transform of the random function l is

$$S_l(f) = \int l(t)e^{-2\pi ift}dt. \tag{60}$$

In the theory of cross-correlations, it is shown that [Korn and Korn, 1961] the time average of the cross correlation of x_1 and x_2 can be expressed as

$$\langle x_1x_2\rangle = \int \mathrm{Re}\left[\lim_{T\to\infty} \frac{1}{T}(S_{x_1}S_{x_2}^*)\right]df = \lim_{T\to\infty} \frac{1}{2T}\int (S_{x_1}S_{x_2}^* + S_{x_1}^*S_{x_2})df = \frac{1}{2}\int (H_1H_2^* + H_1^*H_2)Q_l df, \tag{61}$$

where the symbol $\langle\ \rangle$ is used to indicate a time average. The final form of (61) results from the fact that $S_{x_1} = H_1 S_l$, etc. Also, due to the fact that the random processes are ergodic, the time average in (61) may be replaced by an ensemble average.

In the same way

$$\overline{y_1 y_2} = \frac{1}{2} \int (H_1 H_2^* + H_2 H_1^*) Q_m df. \tag{62}$$

Therefore,

$$\bar{u} = a[\overline{x_1 x_2} - \overline{y_1 y_2}] = \frac{a}{2} \int (H_1 H_2^* + H_2 H_1^*)(Q_l - Q_m) df. \tag{63}$$

The above expression leads to the following conclusions: (1) In case the two sources have the same form of spectral density over the amplifier bandwidth, balance occurs independent of the amplifier response curves. Thus, for white noise, balance occurs when $T_l = T_m$. (2) In case Q_l and Q_m have different forms, balance occurs where

$$\int (H_1 H_2^* + H_2 H_1^*) Q_l df = \int (H_1 H_2^* + H_2 H_1^*) Q_m df. \tag{64}$$

An expression for $\Delta\bar{u}$, the change in $\bar{u}$ due to a deviation from balance, is required. In the case where both signals consist of white noise,

$$Q_l - Q_m = \frac{1}{4} k(T_l - T_m). \tag{65}$$

Next, the assumption is made that

$$T_m = T_l - \Delta T,$$

then,

$$Q_l - Q_m = \frac{1}{4} k[T_l - (T_l - \Delta T)] = \frac{1}{4} k \Delta T. \tag{66}$$

In this case

$$\Delta\bar{u} = \frac{a}{8} k \Delta T \int (H_1 H_2^* + H_2 H_1^*) df. \tag{67}$$

If the smoothing filter has a response function at zero frequency given by G_0, then

$$\Delta\bar{w} = G_0^{1/2} \Delta\bar{u} = \frac{a}{8} G_0^{1/2} k \Delta T \int (H_1 H_2^* + H_2 H_1^*) df. \tag{68}$$

Again the assumption is made that the minimum detectable change in deflection is a change equal to the rms value of the fluctuations. The minimum detectable value of ΔT is evaluated by equating the right-hand side of (68) and the square root of $\overline{w^2}$. Using the fact that

$$w_{\mathrm{rms}} = (\overline{w^2})^{1/2} = \left[\int_{-\infty}^{\infty} W(f) df \right]^{1/2},$$

the result is

$$\left[\int_{-\infty}^{\infty} G(f) Q_u'(f) df \right]^{1/2} = \frac{a}{8} G_0^{1/2} k \Delta T \int (H_1 H_2^* + H_2 H_1^*) df. \tag{69}$$

When this solved for ΔT, the result is

$$\Delta T = \frac{\left[\int_{-\infty}^{\infty} G(f) Q_u'(f) df\right]^{1/2}}{\frac{a}{8} G_0^{1/2} k \int (H_1 H_2^* + H_2 H_1^*) df}. \tag{70}$$

Equation (70) is the general expression for the sensitivity of a correlation radiometer with amplifiers with differing complex gain functions. In order to interpret this result and obtain a feeling for the importance of the various parameters, it is convenient to look at two examples. First the case of two amplifiers with square gain functions centered on the same frequency f_0; the first amplifier with bandwidth B_1 and gain R_1^0 and the second with bandwidth B_2 and gain R_2^0 will be considered. To be specific, it will be assumed that $B_1 > B_2$. It will be assumed that no phase shifts occur in these amplifiers.

In order to evaluate (70) it is necessary to evaluate the various terms of (58), recalling that the numerator of (70) contains only fluctuation terms; i.e., any d-c contributions must be ignored. It is reasonable to evaluate the convolution integrals that occur with the restriction that f is small enough that B_2 is enclosed within B_1. When these evaluations are carried out with the assumptions that the radiometer is balanced and also that the low pass filter has a square passband of width b, the result is

$$[\text{numerator of (70)}] = a G_0^{1/2} b^{1/2} (B_2 R_1^0 R_2^0)^{1/2} k (T_l + T_n).$$

When this is compared to the similar result from the simple theory, for example (14), it is evident that the bandwidth that sets the fluctuation level is B_2, the narrower of the two amplifier bandwidths. Also, the amplifier gains appear as the product $R_1^0 R_2^0$.

The denominator of (70) must also be evaluated under the same conditions. This involves evaluating the integral

$$\int (H_1 H_2^* + H_2 H_1^*) df, \tag{71}$$

where H_1 and H_2 are the discontinuous real functions as defined above. When this is carried out, the result is

$$[\text{denominator of (70)}] = \frac{a}{8} G_0^{1/2} k (4 R_1^0 R_2^0 B_2).$$

These results lead to

$$\Delta T = 2\left(\frac{b}{B_2}\right)^{1/2} (T_l + T_n). \tag{72}$$

Thus, in this case the same sort of expression is obtained as in the case of the simple theory except that the bandwidth of the narrowest amplifier sets the sensitivity.

Finally the case of two amplifiers with square bandpass with identical bandwidth but tuned to slightly different center frequencies will be considered. Again phase shifts will be neglected, the complex gains will be treated as real positive functions, and the radiometer will be assumed to be balanced. The resulting sensitivity is

$$\Delta T = 2\left(\frac{b}{B_0}\right)^{1/2} (T_l + T_n) \tag{73}$$

where B_0 is the "overlap bandwidth" or the frequency interval common to both amplifiers.

247-602 O-67—6

4. Effect of Variation in Time Delay and Phase in the Amplifiers of a Correlation Type Radiometer

Here it will be assumed that a differential variation in time delay can occur to the signals in the two amplifiers. Thus, at any instant the multiplier is comparing voltages corresponding to two different instants of time. Therefore, the multiplier output can be expressed as

$$u(t) = v_1(t) v_2(t+\tau), \tag{74}$$

where τ is the time delay difference in the two signal channels. In order to simplify this analysis, the assumption will be made that the two radiometer channels are identical except for the difference in time delay. Thus the voltages that are multiplied together are

$$\begin{aligned} v_1(t) &= x(t) + y(t) + z_1(t), \\ v_2(t+\tau) &= x(t+\tau) - y(t+\tau) + z_2(t+\tau). \end{aligned} \tag{75}$$

Equation (7) shows that the fluctuation level of the radiometer depends only on convolutions of power spectra. Since these are not a function of τ, the fluctuation level does not vary as a result of a differential time delay. Therefore, the change in sensitivity will be related to the change in average deflection caused by the time delay.

The change in average deflection can be obtained from

$$\begin{aligned} \overline{u(t,\tau)} &= \overline{[x(t) + y(t) + z_1(t)]\,[x(t+\tau) - y(t+\tau) + z_2(t+\tau)]} \\ &= \overline{x(t)x(t+\tau)} - \overline{y(t)y(t+\tau)} = \psi_x(\tau) - \psi_y(\tau). \end{aligned} \tag{76}$$

The Wiener-Khintchine theorem allows this to be put into the form

$$\overline{u(t,\tau)} = \int (Q_x - Q_y) e^{i\omega\tau} df. \tag{77}$$

In order to be able to evaluate the above integral, it is convenient to consider that both sources generate white noise and the amplifiers have square bandpass of width B centered on f_0. Thus, in this case,

$$\begin{aligned} \overline{u}(\tau) &= (Q_x - Q_y) \left[\int_{-f_0-B/2}^{-f_0+B/2} e^{i\omega\tau} df + \int_{f_0-B/2}^{f_0+B/2} e^{i\omega\tau} df \right], \\ &= 2B(Q_x - Q_y) \cos 2\pi\tau f_0 \frac{\sin \pi\tau B}{\pi\tau B}. \end{aligned} \tag{78}$$

This can be put in more convenient form by noting that

$$\overline{u}(0) = 2B(Q_x - Q_y).$$

Thus

$$\overline{u}(\tau) = \overline{u}(0) \cos 2\pi\tau f_0 \frac{\sin \pi\tau B}{\pi\tau B}. \tag{79}$$

For the usual case in which $B \ll f_0$ and τ is on the order of $1/f_0$, this can be approximated by

$$\overline{u}(\tau) \approx \overline{u}(0) \cos 2\pi\tau f_0. \tag{80}$$

Thus, in the case of white noise sources and square bandpass amplifiers, time delay variations will not influence the balance point. They will affect sensitivity, however. For maximum sensitivity, it is important to keep the difference in time delay in the two amplifiers small enough that $\cos 2\pi\tau f_0$ is approximately unity.

A somewhat related question is the effect of a differential phase change in the two amplifiers. Again assuming white noise sources and amplifiers that are identical except for the phase shift, the average deflection can be computed by means of (63). With the assumption that

$$R_2 = R_1 e^{i\phi}, \tag{81}$$

this equation becomes

$$\bar{u}(\phi) = \frac{a}{2}(Q_l - Q_m)R_1(2\cos\phi).$$

This can again be expressed in the form

$$\bar{u}(\phi) = \bar{u}(0)\cos\phi. \tag{82}$$

5. Some Effects of Using an Imperfect Multiplier

The simplest form of multiplier to multiply two microwave signals together is probably that shown in figure 3. The input signals (v_1 and v_2) are applied to two opposite arms of a hybrid junction. The other pair of opposite arms are terminated by square law detectors. The outputs from these detectors are applied to a difference amplifier. In case the hybrid T is matched, the detectors have the same sensitivity and their reflection coefficient is zero; and, if they are assumed to have a perfect square law envelope response, the output voltage z will be

$$z = t_1 - t_2 = \alpha\left(\frac{v_1 + v_2}{\sqrt{2}}\right)^2 - \alpha\left(\frac{v_1 - v_2}{\sqrt{2}}\right)^2, \tag{83}$$

where α is the coefficient of proportionality in the multiplier law. Thus,

$$z = 2\alpha v_1 v_2. \tag{84}$$

In an actual multiplier constructed on this principle, the ideal conditions suggested above will be only approximated. Thus, the output will not be an accurate multiplication of the input signals. A scattering matrix analysis of the multiplier junction indicates that if the junction is matched and perfect, the wave arriving at the detectors is equal to that assumed in the ideal case as long as either pair of leads are terminated in matched loads. Since the detectors (if bolometers) can be quite accurately matched, this will probably not be the phenomenon that sets the multiplier accuracy. This accuracy can depend on the accuracy of the "square law" of the detectors, however.

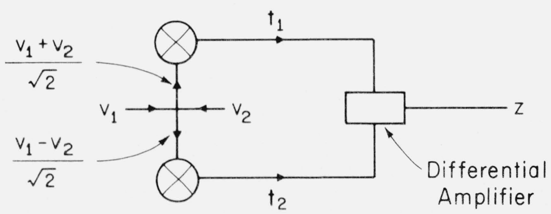

FIGURE 3. *Microwave multiplier.*

In order to investigate the effect of the detector law on the multiplier accuracy, assume that both detectors follow the same law. If one detector is more sensitive than the other, the incoming signal can be attenuated, so that, referred to the input of the attenuator, the two detectors are assumed to have the same law with the same numerical coefficients. Thus, the voltages from the detectors can be expressed as power series

$$t_1=\sum_n a_n\left(\frac{v_1+v_2}{\sqrt{2}}\right)^n, \tag{85a}$$

$$t_2=\sum_n a_n\left(\frac{v_1-v_2}{\sqrt{2}}\right)^n. \tag{85b}$$

Then the multiplier output is

$$z=t_1-t_2=\sum_n a_n\left[\left(\frac{v_1+v_2}{\sqrt{2}}\right)^n-\left(\frac{v_1-v_2}{\sqrt{2}}\right)^n\right]. \tag{86}$$

These terms may be expanded with the binomial theorem; the result is

$$z=2\left[\frac{a_1}{2^{1/2}}v_2+\frac{a_2}{2^{2/2}}(2v_1v_2)+\frac{a_3}{2^{3/2}}(3v_1^2v_2+v_2^3)+\frac{a_4}{2^{4/2}}(4v_1^3v_2+4v_1v_2^3)+\ldots\right]. \tag{87}$$

The smoothing filter provides an estimate of the average of the multiplier output; thus it is necessary to compute $\bar{z}$. This is,

$$\bar{z}=2[a_2\overline{(v_1v_2)}+a_4\overline{(v_1^3v_2+v_1v_2^3)}+\ldots],$$

$$=2[a_2\overline{(x+y+z_1)(x-y+z_2)}+a_4\{\overline{(x+y+z_1)^3(x-y+z_2)}+\overline{(x+y+z_1)(x-y+z_2)^3}\}+\ldots],$$

$$=2[a_2(\overline{x^2}-\overline{y^2})+2a_4(\overline{x^4}-\overline{y^4})+3a_4(\overline{x^2}-\overline{y^2})(\overline{z_1^2}+\overline{z_2^2})+\ldots]. \tag{88}$$

The result is that if $\overline{x^4}-\overline{y^4}=0$ at the same conditions that $\overline{x^2}-\overline{y^2}=0$, the balance condition is not affected by any term up to the fourth in the crystal law expansion. This would be the case if both x and y are signals with the same statistical properties; for example, if both are thermal noise sources, they both possess Gaussian statistics.

However, in the case that the two signals have different statistical properties, $(\overline{x^4}-\overline{y^4})$ would not necessarily equal zero when $(\overline{x^2}-\overline{y^2})$ does. Thus, in this case, it is important to select detectors that are accurately square law.

6. Conclusions

The sensitivity of a correlation radiometer has been computed under a variety of conditions. In section 2 a very simplified calculation is used to derive the usual expression for the sensitivity of the radiometer. This is followed by a calculation of the sensitivity of a radiometer of the type suggested by Allred (1962) in which the unknown noise signal is balanced against a sinusoidal reference signal. By performing these two calculations in an analogous manner, it is particularly easy to compare the sensitivities that result. In the case that the output fluctuations are predominately due to the input noise signals, Allred's radiometer is more sensitive than the conventional circuit that compares two noise signals.

The first part of section 3 demonstrates that it is not necessary for the two amplifiers to have the same gain. The calculation shows that the product of the gains is the parameter that determines the output amplitude, instead of the individual amplifier gains. Next, it is shown that gain fluctuations produce a decrease in sensitivity. The last part of this section is concerned with the effect of using two dissimilar amplifiers. A general expression is obtained for the sensitivity of a radiometer with amplifiers with arbitrary gain functions. This expression is evaluated for the case of amplifiers with square bandpass and no phase shifts. It is shown that under these conditions, whether the two amplifiers have the same bandpass and different center frequencies or whether they have the same center frequency and different bandwidths, the sensitivity is determined by the "overlap" bandwidth.

Section 4 deals with the effect of a differential time delay or differential phase shift in the two radiometer channels. It is shown that both of these effects result in a decrease of sensitivity.

Finally in section 5 it is shown that, if the multiplier carries out the multiplication operation by forming the difference of the squares of the sum and difference of the two input signals, errors can result if the "square law" elements do not have a perfect square law response and if the two input voltages have differing statistical distributions. Thus, the radiometer proposed by Allred, in which the comparison signal is sinusoidal, places a more stringent requirement on the multiplier than does the more usual correlation radiometer, in which both input signals are Gaussian noise.

The author gratefully acknowledges many helpful discussions with D. F. Wait and G. F. Engen. M. M. Siddiqui's comments on the manuscript proved to be very helpful.

7. References

Allred, C. M. (1962), A precision noise spectral density comparator, J. Res. NBS **66C** (Engr. and Instr.), No. 4, 323–330.

Blum, E. J. (1959), Sensibilité des radioteléscopes et recepteurs a correlation, Annales d' Astrophysique **22,** No. 2, 140–163.

Colvin, R. S. (1959), Faint signal limitations of radiometers, Inst. of Radio Engr., WESCON Convention Record **3,** Pt. 8, 52–58.

Colvin, R. S. (1961), A study of radio-astronomy receivers; Stanford Radio Astronomy Institute Publication No. 18A (Stanford University, Stanford, Calif.).

Goodman, N. R. (Mar. 1957), On the joint estimation of the spectra, cospectrum and quadrature spectrum of a two-dimensional stationary Gaussian process, Engineering Statistics Laboratory Scientific paper No. 10, New York University, p. 23.

Korn, G. A. and T. M. Korn (1961), Mathematical Handbook for Scientists and Engineers (McGraw-Hill Book Co., Inc., New York, N.Y.).

8. Appendix—Evaluation of Convolution Integrals

Assume a random signal with white spectral density amplified by an amplifier with square passband as shown in figure A–1.

If the spectral density of the input signal is Q_0 as measured, the amplitude of Q_x is $RQ_0/2$, where R is the power gain of the amplifier. The division by 2 occurs because the spectral density is assumed to be split equally between the positive and negative frequency regions. Then,

$$Q_x * Q_x = \int Q_x(f-f')Q_x(f')df'.$$

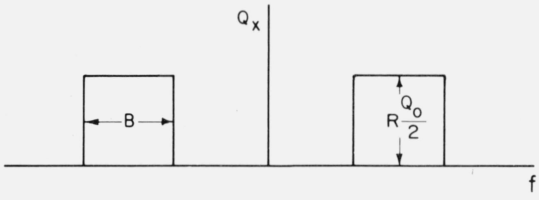

FIGURE A–1. *Square spectral density.*

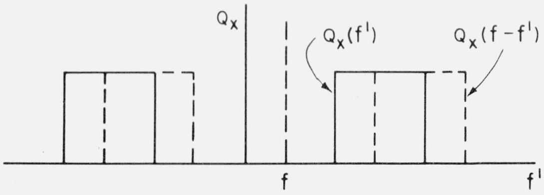

FIGURE A–2. *Functions appearing in integrand of convolution integral.*

The value of this integral for small values of the parameter f is required. The term $Q_x(f-f')$ has the same shape as $Q_x(f')$ except that it is displaced an amount f and inverted on the frequency axis, as shown in figure A–2. Since the integrand is the product of these two functions, the value of the integral is the product of the amplitudes of these functions times the frequency interval over which the product is nonzero. Thus,

$$Q_x * Q_x = \left(R\frac{Q_0}{2}\right)^2 2(B-f) \qquad \text{(A–1)}$$

for $f < B$.

In this paper, the value of this integral is required under the condition that $f \ll B$. To this approximation, the result is

$$Q_x * Q_x = 2B\left(R\frac{Q_0}{2}\right)^2. \qquad \text{(A–2)}$$

In general, the convolutions of the other spectral densities appearing in this paper are evaluated in the same way. The reason that the various convolutions of power spectra that are used have differing numerical coefficients is that the hybrid junction at the input of the radiometer divides the input powers; however, the noise powers appearing due to the effective input temperatures of the amplifiers do not undergo this power division.

(Paper 71C1–253)