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Symmetrizable Generalized Inverses of Symmetrizable Matrices*

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The matrix A is said to be symmetrizable by V when V is positive definite and AV is hermitian. Several lemmas regarding symmetrizability are given. For three classes of generalized inverses it is shown that if A is smmetrizable by V there exists a generalized inverse in each class which is symmetrizable by V. The Moore-Penrose inverse (or pseudo-inverse) of a matrix symmetrizable by V is also symmetrizable by V if and only if the matrix and the pseudo-inverse commute.

Key Words: Generalized inverse, symmetrizable matrix.

1. Introduction

We call a matrix A symmetrizable if there exists a positive definite V such that AV is hermitian. In that case A is said to be symmetrizable by V. Given an A symmetrizable by V we inquire for the existence of generalized inverses of A which are symmetrizable by the same matrix V. For several classes of generalized inverses which have been previously discussed [3, 4, 7 and references therein] [1] it is shown that such symmetrizable generalized inverses exist. In particular it is shown that the C_2-inverse [3, 4] (also called reflexive generalized inverse [7] or semi-inverse [1]) of a symmetrizable matrix which commutes with the matrix is symmetrizable by the same V. Finally it is shown that the Moore-Penrose inverse, B, of a matrix A symmetrizable by V is symmetrizable by V if and only if A and B commute.

2. Preliminaries and Notation

All matrices are considered to have complex entries. For any matrix M, we denote by M^* and $\rho(M)$ the conjugate transpose and rank of M respectively. We write $A\epsilon\mathscr{S}(V)$ when and only when V is positive definite and AV is hermitian. If $A\epsilon\mathscr{S}(V)$, we say that A is symmetrizable to the right by V. We consider only symmetrizability to the right. Since we show, Lemma 1, that A is symmetrizable to the right by V if and only if A is symmetrizable to the left by V^{-1}, an analogous set of results could be derived regarding left symmetrizability. For a given matrix A we define $C_i(A)$ to be the set of all matrices B which satisfy the first i of the relations (i) $ABA=A$, (ii) $BAB=B$, (iii) $AB=(AB)^*$ and (iv) $BA=(BA)^*$. A matrix $B\epsilon C_i(A)$ is called a C_i-inverse of A. The correspondence between this terminology and others which are in use has been noted elsewhere [3, 4]. The set $C_4(A)$ contains a single uniquely determined matrix which is the Moore-Penrose inverse [6] of A. If $B\epsilon C_i(A)$, $i<4$, then B is not uniquely determined by A unless other conditions are imposed. For example, if $B\epsilon C_2(A)$ and commutes with A then B is uniquely determined by A [4].

3. Symmetrizable Matrices

In this section we give several lemmas which are needed in the remainder of the paper.

LEMMA 1. *Let* A *be a given matrix,* V *be positive definite, and* T *be the positive definite square root of* V. *If any one of the matrices* $S_1=AV$, $S_2=V^{-1}A$ *and* $S_3=T^{-1}AT$ *is hermitian, then all are hermitian. There exists a positive definite* H *such that* $H^{-1}AH$ *is hermitian, if and only if* A *is similar to a real diagonal matrix.*

PROOF. From $T^2=V$ and the definitions of the S_i we have $S_1=VS_2V=TS_3T$ from which it follows at once that if any S_i is hermitian then every S_i is hermitian. If $H^{-1}AH$ is hermitian then A is similar to a hermitian matrix and thus has real roots and is diagonable. Conversely, let $P^{-1}AP=\Lambda$ where Λ is real and diagonal. If $P=HQ$ is the polar factorization of P, where H is positive definite and Q is unitary, then we have that $H^{-1}AH=Q\Lambda Q^*$ is hermitian.

For ready reference we have included the above simple proof of Lemma 1, but the content of the lemma is known: That $A\epsilon\mathscr{S}(V)$ is equivalent to $A^*\epsilon\mathscr{S}(V^{-1})$ has been shown [2]. Further if $S_1=S_1^*$,

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[1] Figures in brackets indicate the literature references at the end of this paper.

then A can be written as the product of two hermitian matrices one of which is positive definite. That this is possible if and only if A is similar to a real diagonal matrix is a known theorem [9].

LEMMA 2. *If* $A \epsilon \mathscr{S}(V)$ *then* $A^p \epsilon \mathscr{S}(V)$ *for every integer* $p \geqslant 0$.

PROOF. $A \epsilon \mathscr{S}(V)$, by Lemma 1, implies

$$T^{-1}AT = S = S^*,$$

where $T^2 = V$ and T is positive definite. But then $T^{-1}A^pT = S^p$ is hermitian and Lemma 1 then gives $A^p \epsilon \mathscr{S}(V)$.

Lemma 2 has been proved in a much more general context [8] and a slightly different proof has been given elsewhere [2].

LEMMA 3. *Let* $A \epsilon \mathscr{S}(V)$ *and* $B \epsilon \mathscr{S}(V)$. *Then* $AB \epsilon \mathscr{S}(V)$ *if and only if* $AB = BA$.

PROOF. Let $C = AB$ and $T^{-1}CT = P$. Then

$$(T^{-1}AT)(T^{-1}BT) = P.$$

We choose T to be the positive definite square root of V and then, by Lemma 1, P is the product of two hermitian matrices. Thus $P = P^*$ if and only if A and B commute, But, by Lemma 1, $P = P^*$ if and only if

$$C \epsilon \mathscr{S}(V).$$

LEMMA 4. *Let* $A \epsilon \mathscr{S}(V)$, $B \epsilon \mathscr{S}(V)$ *and define* $C_1 = AB$, $C_2 = BA$. *If* C_1 *is hermitian, then* C_1 *is similar to* C_2. *If* C_1 *and* C_2 *are hermitian, then* $C_1 = C_2$.

PROOF. By Lemma 1 we may write C_1 as the product of two hermitian matrices: $C_1 = (AV)(V^{-1}B)$. If $C_1 = C_1^*$ we have $C_1 = V^{-1}BAV = V^{-1}C_2V$, and the first assertion is proved. If additionally $C_2 = C_2^*$, then $C_1 = V^{-1}C_2V = VC_2V^{-1} = C_1^*$ which implies $C_2V = VC_2$. But then $C_1 = C_2$.

4. Symmetrizable Generalized Inverses

THEOREM 1. *Let* $A \epsilon \mathscr{S}(V)$. *Then there exist matrices* $B_i \epsilon \mathscr{S}(V)$, $i = 1, 2, 3$, *such that* $B_i \epsilon C_i(A)$.

PROOF. Let $AV = S = S^*$. Then there exists [7] an $H = H^*$ such that $H \epsilon C_1(S)$. Given this,

$$SHS = S = AV = AVHAV$$

shows that $B_1 = VH \epsilon C_1(A)$. Further $V^{-1}B_1 = H$ implies, by Lemma 1, that $B_1 \epsilon \mathscr{S}(V)$. By a known theorem [3], if $B_2 = B_1AB_1$ then $B_2 \epsilon C_2(A)$. But

$$V^{-1}B_2 = (V^{-1}B_1)(AV)(V^{-1}B_1) = HSH$$

is hermitian and, by Lemma 1, we have $B_2 \epsilon \mathscr{S}(V)$. Now let $K \epsilon C_4(S)$. Then [6], $K = K^*$. Further,

$$SKS = S = AV = AVKAV$$

and $KSK = K = KAVK$ show that $B_3 = VK \epsilon C_2(A)$. Since SK is hermitian and $SK = (AV)(V^{-1}B_3) = AB_3$, we have $B_3 \epsilon C_3(A)$. Finally $V^{-1}B_3 = K = K^*$ implies, by Lemma 1, that $B_3 \epsilon \mathscr{S}(V)$.

THEOREM 2. *Let* $A \epsilon \mathscr{S}(V)$. *Then there exists a*

$$B \epsilon C_2(A),$$

uniquely determined by A, *such that* $AB = BA$. *Further,* $B \epsilon \mathscr{S}(V)$.

PROOF. From Lemma 1, $A \epsilon \mathscr{S}(V)$ implies that A is diagonable and hence that $\rho(A) = \rho(A^2)$. Given this condition on the rank of A, it follows from a known theorem [4] that there exists a uniquely determined $B \epsilon C_2(A)$ which commutes with A; furthermore this B is a polynomial $g(A)$ in A. From the construction [4] of $B = g(A)$, the coefficients of g are real if the roots of A are real, a condition insured by Lemma 1. This being the case, $A \epsilon \mathscr{S}(V)$ implies, by Lemma 2, that $g(A) \epsilon \mathscr{S}(V)$ and the theorem is proved.

THEOREM 3. *Let* $A \epsilon \mathscr{S}(V)$ *and* $B \epsilon C_2(A)$ *commute. Then* $T^{-1}BT \epsilon C_4(T^{-1}AT)$, *where* T *is the positive definite square root of* V.

PROOF. By Theorem 2, $B \epsilon \mathscr{S}(V)$ and given this we have, from Lemma 3, that the projection $C = AB = BA$ is such that $C \epsilon \mathscr{S}(V)$. By Lemma 1, $T^{-1}CT$ is a hermitian projection. Since $B \epsilon C_2(A)$ is clearly equivalent to $T^{-1}BT \epsilon C_2(T^{-1}AT)$ we have $T^{-1}BT \epsilon C_4(T^{-1}AT)$, for we have shown

$$(T^{-1}AT)(T^{-1}BT) = (T^{-1}BT)(T^{-1}AT) = T^{-1}CT$$

to be hermitian.

THEOREM 4. *Let* $A \epsilon \mathscr{S}(V)$ *and* $B \epsilon C_4(A)$. *Then* $B \epsilon \mathscr{S}(V)$ *if and only if* $AB = BA$.

PROOF. Let $B \epsilon \mathscr{S}(V)$. Then from $B \epsilon C_4(A)$ we have that AB and BA are hermitian and it follows from Lemma 4 that $AB = BA$. Conversely let $AB = BA$. Then it follows from Theorem 2 that $B \epsilon \mathscr{S}(V)$.

It is known [4, 5] that $B \epsilon C_4(A)$ commutes with A if and only if B is a polynomial in A, and that $B \epsilon C_4(A)$ is a polynomial in A if and only if A is an EPr matrix [5]. We combine these results with Theorem 4 to obtain:

THEOREM 5. *Let* $A \epsilon \mathscr{S}(V)$ *and* $B \epsilon C_4(A)$. *Then the following conditions are equivalent.*

(i) $B \epsilon \mathscr{S}(V)$

(ii) $AB = BA$

(iii) B *is a polynomial in* A

(iv) A *is an* EPr *matrix.*

5. References

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