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Matrices of Class $\mathscr{J}_2$*

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Let J_2 be the set of $n \times n$ complex matrices $A = (a_{ij})$ such that $a_{j_1j_2}a_{j_2j_3} \ . \ . \ . \ a_{j_rj_1} = 0$ for all r such that $3 \leq r \leq n$ and all distinct $j_1, j_2, \ . \ . \ ., j_r$. Then many properties of this set are given, which may be regarded as generalizations of the properties of the set of triple diagonal matrices.

Key words: Chains, cycles, Jacobi matrices, triple diagonal matrices.

1. Introduction

In a previous paper, [1],[1] we introduced the class of matrices $\mathscr{J}_2$ as a generalization of the class $\mathscr{J}$ of Jacobi matrices. We also developed a few of the spectral properties of certain types of matrices in $\mathscr{J}_2$. In [2] we have shown why the members of $\mathscr{J}_2$ enjoy many special properties not shared by matrices with more complex structure.

Several years prior to the work done in [1] and [2] S. Parter in [3] obtained a spectral theorem for certain elements of $\mathscr{J}_2$ by an ingenious argument based upon the theory of linear graphs.

Finally, in a recent paper [10], Quirk and Ruppert obtained very deep stability results for a subset of $\mathscr{J}_2$ quite different from that considered by Parter. The theorems in [1] overlap with the work of Quirk and Ruppert as well as with the work of Parter.

The various results of the papers [1], [2], [3], and [10] point to the desirability of constructing a comprehensive theory of the properties of elements of $\mathscr{J}_2$. The purpose of the present paper is to initiate such a study. We will show that a remarkable number of the properties of Jacobi matrices can be generalized to the elements of $\mathscr{J}_2$.

It turns out that the class $\mathscr{J}_2$ contains two subclasses of particular importance. We shall refer to the elements of these classes as matrices of semi-positive type and semi-negative type respectively. The semi-positive matrices occur in a variety of problems in the physical sciences, particularly in the small vibrations of mechanical systems. The semi-negative matrices occur in the theory of qualitative stability and find their primary application in the social sciences, particularly in economics.

With these applications in mind we have concentrated mainly upon the semi-positive and semi-negative elements of $\mathscr{J}_2$ although these two classes certainly do not exhaust the possibilities.

The paper is organized into several parts as follows: In section 2 we set forth the basic definitions and develop what may be called the combinational properties of the elements of $\mathscr{J}_2$. This section is not restricted by any considerations of positivity or negativity. The same is true of section 3 in which we give a variety of determinant formulas. Some of these results generalize known formulas for Jacobi matrices, but the problem for elements of $\mathscr{J}_2$ is more difficult and some problems remain open. In particular, our formulas for the exterior p-th power of elements of $\mathscr{J}_2$ are not sufficiently well worked out to enable us to decide which matrices have exterior p-th powers of positive cyclic type. On the other hand many of these formulas hold for quite general matrices.

Section 4 consists of a study of semi-positive elements of $\mathscr{J}_2$. Here we concentrate mainly upon the questions of interlacing and spectral multiplicity. Other, more special, aspects of the semi-positive matrices are more properly reserved for a general treatment of certain mechanical problems where such questions naturally arise. We intend to treat these matters shortly in another paper.

Finally, section 5 consists of a few miscellaneous properties of semi-negative matrices in $\mathscr{J}_2$. The problems here are more difficult than those associated with the semi-positive matrices because they center around the problem of stability and this in turn seems to be an inherently difficult problem. We have been unable to give an essentially simpler proof of the fundamental stability theorem of Quirk and Ruppert [10]. This being so, we have confined ourselves to stating the theorem and to deriving those few results which follow readily from the general determinant formulas of section 3.

A basic problem remaining unsolved is that of determining for the elements of $\mathscr{J}_2$ how the eigenvalues of A, or at least the real parts of the eigenvalues, vary with the elements along the principal diagonal of A.

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[1] Figures in brackets indicate the literature references at the end of this paper.

A solution to this problem in combination with lemma 5.2 might yield an elegant proof of half of the Quirk-Ruppert theorem. Another problem not treated here is the structure of the eigenvectors for the semi-positive and semi-negative matrices. For certain important subclasses of semi-positive matrices this question is most strongly motivated by problems in mechanics and we intend to deal with it later. Parter [3] has some results about eigenvectors, but a glance at the extensive results available for the Jacobi matrices makes it clear that much more extensive theorems should be available for $\mathscr{J}_2$. As of the present writing nothing is known about the eigenvectors of semi-negative elements of $\mathscr{J}_2$.

2. Notation and Fundamental Preliminaries

We deal with $n\times n$ matrices over the real or complex field, $n>2$. $A=(a_{jk})_1^n$ is a Jacobi matrix, $A\in\mathscr{J}$, if $a_{jk}=0$ whenever $|j-k|>1$. To define the class $\mathscr{J}_2$ we recall that an r-cycle in A is an object of the form

$$\hat{a}(j_1\ .\ .\ .\ j_r)=a_{j_1j_2}\ .\ .\ .\ a_{j_{r-1}j_r}a_{j_rj_1} \qquad (2.1)$$

where the indices in $J=(j_1\ .\ .\ .\ j_r)$ are distinct elements of the set $S_n=(1\ .\ .\ .\ n)$. The elements a_{jj} on the principal diagonal of A are the 1-cycles of A.

DEFINITION 2.1: The matrix $A=(a_{jk})_1^n$ is an element of $\mathscr{J}_2$ if every r-cycle of A of length $r>2$ is zero.

Definition 2.1 leads at once to a formula of fundamental importance in any study of the properties of the elements of $\mathscr{J}_2$. Let $J=(j_1\ .\ .\ .\ j_r)$ be any set of distinct indices from S_n; then we denote by $A[J]$ (or $A[j_1\ .\ .\ .\ j_r]$) the principal submatrix of A in rows and columns J. We use both the symbol d to denote the determinant function and the Bourbaki notation. Thus,

$$d(A[J])=A_J=d(A[j_1\ .\ .\ .\ j_r]). \qquad (2.2)$$

We also write $A(\lambda)$ for $A-\lambda I$, $A_J(\lambda)=d(A[J;\ \lambda])$. For each $r=2, 3, \ldots, n$, each sequence $J=(j_1\ .\ .\ .\ j_r)$ of distinct indices, and each $1\leq p\leq r$, we have

$$A_J(\lambda)=(\hat{a}(j_p)-\lambda)A_{J-j_p}(\lambda)-\sum_{\substack{q=1\\ q\neq p}}^{r}\hat{a}(j_pj_q)A_{J-(j_pj_q)}(\lambda). \qquad (2.3)$$

In this formula and subsequently we denote by $J-H$ (or $J-(h_1\ .\ .\ .\ h_s)$) the set of indices remaining in J after deleting the indices in H.

In [1] we defined the class $\mathscr{J}_2$ directly by a formula having the form (2.3). We were led naturally to it by the corresponding formula connecting the principal minors of Jacobi matrices.

The key to our method consists in first showing that practically every determinental formula normally associated with the Matrix A can be expressed in terms of determinents of principal submatrices and certain special products of elements of A which are easily studied. The new formulas obtained in this way are then systematically exploited. We have already done this with the basic determinant formula introduced in [2] and the results obtained there are freely used below. Formula (2.3) is at the same time a special case of the classical formula of Cauchy and a special case of the general formula of [2] for an arbitrary square matrix.

To expedite the study of various products in a matrix A we introduce the following concepts.

Definition 2.2: The product

$$a(J)=a(j_1\ .\ .\ .\ j_p)=a_{j_1j_2}\ .\ .\ .\ a_{j_{p-1}j_p}, \qquad (2.4)$$

$(j_1\ .\ .\ .\ j_p)$ distinct, is called a chain of length $p-1$ in A.

In dealing with chains we shall often refer to $a(\alpha J\beta)=a(\alpha j_1\ .\ .\ .\ j_p\beta)$ as a chain from α to β. The number of distinct chains in A of length $p-1$ is $p!\binom{n}{p}$. A complete theory of chains and cycles of A is not essential for the study of $\mathscr{J}_2$ so we limit ourselves here to the few basic concepts and elementary results needed.

Definition 2.3: The product

$$\bar{a}(J)=a(J)a_{j_1j_p} \qquad (2.5)$$

is called the closure of $a(J)$. We also refer to $\bar{a}(J)$ as a closed chain of length p in A. The cycle

$$\hat{a}(J)=a_{j_pj_1}$$

will be referred to as the completion of $a(J)$.

Note that a chain of length 1 is closed. All other closed chains have length ≥ 3. In the sequel when we refer to closed chains we shall exclude the closed chains of length 1.

Clearly there is a unique closure and a unique completion to each chain in A. Conversely each closed chain is the closure of a unique chain so that there is a 1–1 correspondence between chains of length $p-1$ and closed chains of length p. On the other hand a cycle of length p is the completion of each of the p chains of length $p-1$ obtained by striking out any term in $\hat{a}(J)$.

A chain $a(J)\neq 0$ will be called maximal in A if both its closure and its completion are zero. Clearly we may also characterize the class $\mathscr{J}_2$ by the statement that for $A\in\mathscr{J}_2$ no chain of length greater than one has a nonzero completion in A. Recalling (see [1], [2], or [3]) that A is combinatorially symmetric if $a_{jk}\neq 0$ implies $a_{kj}\neq 0$, we see that the existence of nonzero closed chains in $A\in\mathscr{J}_2$ is a measure of the extent to which A fails to be combinatorially symmetric. On the other hand we shall see that $A\in\mathscr{J}_2$ is irreducible (indecomposable) only in the presence of combinatorial symmetry. Hence the existence of one nonzero closed chain in $A\in\mathscr{J}_2$ implies A is reducible. This furnishes us with an occasionally useful criterion for recognizing reducible elements of $\mathscr{J}_2$.

Definition 2.4: The set Q consists of all $A\in\mathscr{J}_2$ which are also irreducible (indecomposable).

In [1] we called the elements of Q quasi-Jacobi matrices. The paper [3] of Parter deals with a proper subset of Q. Some of our results will be valid only for elements of Q; others will be valid for all elements of $\mathscr{J}_2$.

THEOREM 2.1: *Let* $A \in \mathscr{J}_2$. $A \in Q$ *if and only if* A *is combinatorially symmetric and has exactly* $n-1$ *nonzero elements above the principal diagonal.*

PROOF: We confine ourselves here to proving the if portion of the theorem since the only if portion was proved in [1] (see also [10]). Suppose, for contradiction, that the set S_n can be divided into disjoint subsets $J=(j_1 \ . \ . \ . \ j_p)$, $J'=(j_{p+1} \ . \ . \ . \ j_n)$ such that every element having the form a_{jk} with $j \in J$, $k \in J'$ is zero. By combinatorial symmetry the elements a_{kj} are also zero. The principal submatrices $A[J]$ and $A[J']$ are also combinatorially symmetric. Either $A[J]$ contains more than $p-1$ nonzero elements above the principal diagonal or $A[J']$ contains more than $n-p-1$ nonzero elements above the principal diagonal. For in the contrary case A can have no more than $n-2$ nonzero elements above the principal diagonal. By theorem 1 of [1] it follows that $A \notin \mathscr{J}_2$ yielding the desired contradiction. Thus the conditions imply A is irreducible.

We may observe now on the basis of this theorem and the previous remarks that $A \in Q$ if and only if every chain in A of length $\geqslant 2$ is maximal.

For many purposes it is desirable to know the relative positions of the elements of $A \in \mathscr{J}_2$ which are zero and the nonzero elements of A. Of course, each nonzero off diagonal element of A is a chain of length one. The existence of nonzero chains of greater length in A does imply information on the location of zeroes. The following result is almost obvious.

LEMMA 2.2: *Let* $a(i_1 \ . \ . \ . \ i_p)$ *be a nonzero chain in* $A \in \mathscr{J}_2$ *of length* $p \geqslant 2$. *Then every element* $a_{\alpha\beta}$ *of* A *such that* $\alpha = i_q$, $\beta = i_r$ *with* $1 \leqslant r \leqslant q-1 \leqslant p$ *is zero.*

The lemma implies that each nonzero chain of length $p->2$ in $A \in \mathscr{J}_2$ forces $(p-1)(p-2)/2$ elements of A to be zero. If $A \in Q$ the number of zeroes becomes $(p-1)(p-2)$. It is instructive to look at the two extreme cases below (the x's denote the location of nonzero elements),

$$A_1 = \begin{pmatrix} x & x & 0 & 0 & 0 & 0 \\ x & x & 0 & x & 0 & 0 \\ 0 & 0 & x & 0 & x & 0 \\ 0 & x & 0 & x & 0 & x \\ 0 & 0 & x & 0 & x & x \\ 0 & 0 & 0 & x & x & x \end{pmatrix},$$

$$A_2 = \begin{pmatrix} x & x & x & x & x & x \\ x & x & 0 & 0 & 0 & 0 \\ x & 0 & x & 0 & 0 & 0 \\ x & 0 & 0 & x & 0 & 0 \\ x & 0 & 0 & 0 & x & 0 \\ x & 0 & 0 & 0 & 0 & x \end{pmatrix}$$

$A_1, A_2 \in Q$. A_1 has the chain $a(124653) \neq 0$ of length 5. By lemma 2.2 this nonzero chain locates all of the zero elements of A_1. On the other hand A_2 clearly has no nonzero chain of length greater than 2.

Here is a result which is complimentary to lemma 2.2 and which we again give without proof.

LEMMA 2.3: *Let* $A \in Q$ *and suppose that for fixed* $1 \leqslant i \leqslant n$ *the elements* $a_{ij_1}, \ . \ . \ ., a_{ij_q}$, $j_1 \neq i, \ . \ . \ .,$ $j_q \neq i$, *are all different from zero. Then the*

$$(q-1)(q-2)/2$$

elements $a_{j_p j_r}$ *with* $1 \leqslant r \leqslant p-1 \leqslant q$ *of* A *are zero, and the symmetrically placed elements are zero.*

These simple ideas may be put to work to further classify the elements of $\mathscr{J}_2$ with aid of

Definition 2.5: The matrix $A \in Q$ belongs to the set Q^{n-1} if there exists a nonzero chain of length $n-1$ in A.

THEOREM 2.4: *The matrix* $A \in Q^{n-1}$ *if and only if there exists a sequence* $A^0 = 1, A^1, \ . \ . \ ., A^n = A$ *of principal submatrices of* A *with* A^j *a matrix of order* j, $j=0, \ . \ . \ ., n$, *and* A^{j-1} *a principal submatrix of* A^j, *satisfying the recurrence formula*

$$d(A^q) = \hat{a}(j_q)d(A^{q-1}) - \hat{\mathbf{a}}(j_q j_{q-1})d(A^{q-2}), \qquad (2.7)$$

$q=2, \ . \ . \ ., n$, *where the products* $\hat{\mathbf{a}}(j_q j_{q-1}) \neq 0$.

PROOF: Suppose first that $A \in Q^{n-1}$ and let

$$a(j_1 \ . \ . \ . \ j_n)$$

be a nonzero chain of length $n-1$ in A. Set $A^0=1$, $J_q = [j_1 \ . \ . \ . \ j_q]$ for each $q=1, \ . \ . \ ., n$, and set $A^q = A[J_q]$. Expand $d(A^q)$ relative to the qth column. By virtue of lemma 2.2 the elements $a_{j_{q-r}j_q} = 0$ for $r \geqslant 2$. On the other hand, we are given that $a_{j_{q-1}j_q} \neq 0$. By combinatorial symmetry the only nonzero off-diagonal element in the qth row of A_q is $a_{j_q j_{q-1}}$. Formula (2.7) follows immediately for $q \geqslant 2$.

To prove the converse, suppose the sequence $(A^0, \ . \ . \ ., A^n)$ exists satisfying (2.7). The chain $a(j, \ . \ . \ ., j_n) \neq 0$ and also the transposed chain $a'(j_1 \ . \ . \ . j_n) = a(j_n \ . \ . \ . j_1) \neq 0$. Thus A has at least $n-1$ nonzero off-diagonal elements above the principal diagonal and at least $n-1$ below. Since $A \in Q$, $A \in Q^{n-1}$.

As an application of this result consider the matrix A above. A sequence of principal submatrices satisfying the formula (2.7) is the following:

$$A^1 = A[1], A^2 = A[12], A^3 = A[124],$$

$$A^4 = A[1246], A^5 = A[12456], A^6 = A[123456].$$

The sequence $A^1 = A[3]$, $A^2 = A[35]$, $A^3 = A[356]$, $A^4 = A[3456]$, $A^5 = A[23456]$, $A^6 = A$ also satisfies such a recurrence formula. It is clear that in general there are exactly two sequences in $A \in Q^{n-1}$ satisfying the formula (2.7), one sequence running in the opposite direction from the other.

Theorem 2.4 can be put a little differently by saying that $A \epsilon Q^{n-1}$ if and only if there is a nested sequence of principal minors of A satisfying a second order difference equation.

Here is an alternative important characterization of Q^{n-1}.

THEOREM 2.5: *Let* $A \epsilon Q$. *Then* $A \epsilon Q^{n-1}$ *if and only if no row (or column) of* A *contains more than two nonzero off-diagonal elements.*

PROOF: Assume first that $A \epsilon Q^{n-1}$ but that row i contains the nonzero elements a_{ip}, a_{iq}, a_{ir},

$$(p \neq i, q \neq i, r \neq i).$$

A contains the nonzero chains $a(i_1 \ . \ . \ . \ i_n)$ and $a'(i_1 \ . \ . \ . \ i_n)$. The product

$$a(i_1 \ . \ . \ . \ i_n) a'(i_1 \ . \ . \ . \ i_n) \neq 0$$

contains all of the nonzero off-diagonal entries of A. Each subscript appears at most twice in $a(i_1 \ . \ . \ . \ i_n)$, once as a row subscript and once as a column subscript, and the same is true of $a'(i_1 \ . \ . \ . \ i_n)$. But if a_{ip}, a_{iq}, a_{ir} are nonzero so are a_{pi}, a_{qi}, a_{ri} and the product $a(i_1 \ . \ . \ . \ i_n)$ must contain all six of these elements so that the index i appears six times in the product. This contradiction establishes the only if portion of the theorem.

Assume next that $A \epsilon Q$ and the condition is satisfied. Denote by $a(i_1 \ . \ . \ . \ i_n i_n \ . \ . \ . \ i_1) \neq 0$ the product of the n - 1 nonzero 2-cycles of A. Each index 1 through n appears in this product at most twice as a row subscript and at most twice as a column subscript, and some pair of indices appear just twice. This follows from the irreducibility of A. That $A \epsilon Q^{n-1}$ is now clear.

Let us now complete our remarks on Q^{n-1} by the following result:

THEOREM 2.6: *Let* $A \epsilon Q^{n-1}$, *then there exists a permutation matrix* P *such that the similarity* P′AP *transforms* A *into a Jacobi matrix.*

PROOF: We have merely to observe that the permutation such that $T(j_q) = q$ will convert the recurrence relation (2.7) into the usual relation for a Jacobi matrix.

For the example A, above the permutation

$$\sigma(123456) = (356421)$$

will do the job.

Theorem 2.6 shows that the elements of Q^{n-1} are essentially Jacobi matrices. We shall exploit this fact below.

3. Various Determinant Formulas

In this section we collect several determinant formulas together with a few of their more immediate applications. Some of these formulas have interesting connections with the linear graph of A, $G(A)$, as introduced by Parter in his study of the spectral properties of the elements of Q. Therefore we shall briefly remark on these connections first.

Definition 3.1.: Let $A = (a_{jk})_1^n$ be combinatorially symmetric. By the graph of A, $G(A)$, we mean a set $\{p_1, \ . \ . \ ., p_n\}$ together with certain distinguished sets of pairs $\{p_i, p_j\}$ corresponding to the elements $a_{ij} \neq 0$ of A. The points $p_1, \ . \ . \ ., p_n$ are called the vertices of the graph and the pairs $\{p_i, p_j\}$ the arcs of the graph.

Note that this graph $G(A)$ is only defined here for combinatorially symmetric matrices, hence it is not the directed linear graph used, for example, by Varga in [13].

$G(A)$ is called a tree if it has Betti number zero and is connected. The next result is a direct consequence of the concepts of section 2.

THEOREM 3.1: *Each of the following two conditions is both necessary and sufficient in order that* G(A) *be a tree.*

(1) $A \epsilon Q$,

(2) *For each pair of indices* $i \neq j$ *there is exactly one nonzero chain* $a(i \ J \ k)$, $J = (j, \ . \ . \ ., j_p)$.

The condition (2) of the theorem turns out to play a crucial role in our examination of the exterior p-th power of A.

Clearly there is a 1–1 correspondence between trees and the elements of Q. Parter exploits this fact systematically in obtaining his results.

We turn next to an examination of the matrices $\Lambda^p A$, $p = 2, \ . \ . \ ., n-1$, $\Lambda^p A$ being the exterior p-th power of A. We remind the reader that if the row vectors of A are $a^1, \ . \ . \ ., a^n$, then the row vectors of $\Lambda^p A$ are just the exterior products

$$a^{j_1} \Lambda a^{j_2} \Lambda \ . \ . \ . \ \Lambda a^{j_p}$$

arranged in lexicographic order. Thus $\Lambda^1 A = A$, $\Lambda^n A = d(A)$, and $\Lambda^{n-1} A$ is, except for certain signs, the cofactor matrix of A. (The matrix $\Lambda^p A$ is usually called the p-th compound matrix of A in the older literature.) One may pass from the matrix $\Lambda^{n-1} A$ to the cofactor matrix, cof A, by multiplying the j, k-th element $\Lambda^{n-1} A$ by $(-1)^{j+k}$. We start with cof A, since it is the simplest case.

For fixed α, β we shall denote by $A_{\alpha\beta}$ the algebraic cofactor of $a_{\alpha\beta}$ so that cof A has the elements $A_{\alpha\beta}$, $\alpha, \beta = 1, \ . \ . \ ., n$.

LEMMA 3.2: *If* $\alpha \neq \beta$, *we have*

$$A_{\alpha\beta} = - \begin{vmatrix} a_{\beta\alpha} & a_{\beta 1} \ . \ . \ . & a_{\beta n} \\ a_{2\alpha} & & \\ & A_{(\alpha\beta)'} & \\ a_{n\alpha} & & \end{vmatrix}, \tag{3.1}$$

where $A_{(\alpha\beta)'}$ *is the principal submatrix of* A *of order* $n-2$ *consisting of the elements in all rows and columns except* α *and* β.

PROOF: By definition

$$A_{\alpha\beta}=(-1)^{\alpha+\beta}\begin{vmatrix} a_{11} & \ldots & a_{1,\beta-1} & \ldots & a_{1,\beta+1} & \ldots & a_{1n} \\ a_{\alpha-1,1} & & & & & & a_{\alpha-1,n} \\ a_{\alpha+1,1} & & & & & & a_{\alpha+1,n} \\ a_{n1} & \ldots & a_{n,\beta-1} & \ldots & a_{n,\beta+1} & \ldots & a_{nn} \end{vmatrix} \tag{3.2}$$

$\alpha+\beta-3$ interchanges of rows and columns converts (3.2) into (3.1).

We now expand the determinant (3.1) relative to the element $a_{\beta\alpha}$ using the fundamental determinant formula of [2]. This yields

$$A_{\alpha\beta}=-\left[a_{\beta\alpha}d(A_{(\alpha\beta)'}) + \sum_{r=0}^{n-3}(-1)^{n-r}\sum_{J\epsilon\Omega r,\,(\alpha\beta)'} A_J A(\alpha|\beta)_{(J')}\right]. \tag{3.3}$$

The notation of formula (3.3) is as follows:

(a) Ω_r, $(\alpha\beta)'$ is the set of all increasing sets of r distinct indices from S_n minus α and β;

(b) J' is a compliment of J relative to either one of the sets $S_n-\alpha$ or $S_n-\beta$; (thus J' has $n-1-r$ distinct elements);

(c) $A(\alpha|\beta)_{(J')}$ is the sum over all complementary cycles to A_J in the matrix of $A_{\alpha\beta}$, (each cycle in the sum is an $n-1-r$ cycle).

A typical element of $A(\alpha|\beta)_{(J')}$ is the sum of chains having the form

$$a(\beta K\alpha)$$

where $K=(k_1\ldots k_q)\subset J'$. In particular, every chain in A from β to α appears as a coefficient in some term in (3.2). Thus we have

LEMMA 3.3: $A_{\alpha\beta}=0$ *if every chain from* β *to* α *is zero.*

LEMMA 3.4: *Let* $A\epsilon Q$, *then for each fixed* $\alpha\neq\beta$ *the expansion* (3.2) *for* $A_{\alpha\beta}$ *has at most one nonzero term.*

By lemma 3.4 formula (3.3) can be put into the form:

$$A_{\alpha\beta}=\pm a(\beta J\alpha)A_{J'} \tag{3.3'}$$

for $A\epsilon Q$ and some suitable index set J. Therefore a cycle of cof A (or of $\Lambda^{n-1}A$ which has equal cycles) has the form:

$$A_{i_1i_2}\ldots A_{i_pi_1} =\pm a(i_1J_pi_p)\ldots a(i_2J_1i_1)A_{J_1'}\ldots A_{J_p'}. \tag{3.4}$$

Now it is clear that $a(i_1J_pi_p)\ldots a(i_2J_1i_1)$ is a product of cycles of A on the index i_1 of length $\geqslant 2$. Hence the product can only be different from zero if each cycle is of length 2. It follows that an even number of the chains in (3.4) must be of odd length. Referring back to formula (3.3) we see that the sign on the right side of (3.3) is therefore positive. Let us formalize this result.

LEMMA 3.5: *Let* $A\epsilon Q$, *then every nonzero cycle of length* $\geqslant 2$ *of* $\Lambda^{n-1}A$ (*or of cof* A) *is a product of 2-cycles of* A *and principal minors of* A *of the form* (3.4) *with a positive sign.*

As one might expect, the structure of the matrix Λ^pA for $p=2,\ldots n-2$ is more difficult to analyze. Here is how the formula for an arbitrary minor of A can be related to the principal minors of A. Let R, S be two increasing sets of indices of length p and suppose the number of indices in $R\cap S$ is α. Let $A_{R,S}$ be the determinant of the submatrix of A in rows R and columns S and let $\beta=p-\alpha$. After suitable interchanges of rows and columns we arrive at the submatrix $A[\tilde{R},\tilde{S}]$ having the form,

$$A[\tilde{R},\tilde{S}]=\begin{pmatrix} a_{h_1k_1} & \ldots & a_{h_1k_\alpha} & a_{h_1j_1} & \ldots & a_{h_1j_\beta} \\ a_{h_\alpha k_1} & \ldots & a_{h_\alpha k_\alpha} & a_{h_\alpha j_1} & \ldots & a_{h_\alpha j_\beta} \\ a_{j_1k_1} & \ldots & a_{j_1k_\alpha} & & A[J] & \\ a_{j_\beta k_1} & \ldots & a_{j_\beta k_\alpha} & & & \end{pmatrix}$$

in which the index sets

$$H=(h_1\ \ldots\ h_\alpha),\ K=(k_1\ \ldots\ k_\alpha),\ J=(j_1\ \ldots\ j_\beta)$$

are pairwise disjoint and $A[J]$ is a principal submatrix. Clearly

$$A_{\tilde{R},\tilde{S}}=\pm A_{R,S}.$$

On the other hand, repeated application of the determinant formula of [2] to $A_{\tilde{R},\tilde{S}}$ shows that the terms in the expansion of this determinant are obtained by forming products of α chains in A connecting an index from the set H to an index from the set K and passing through the set J. (We remark that β may be equal to zero. If $\alpha=0$ we have a principal minor and the discussion does not apply. The term passing through the set J as used here includes the possibility that no element of the set J appears in a given chain.) These products are multiplied by α principal minors of A_J and a suitable sign attached. Thus we have

$$A_{R,S}=\Sigma\pm a(h_1J_1\sigma_1)\ \ldots\ a(h_\alpha j_\alpha\sigma_\alpha)A_{J_1'}\ \ldots\ A_{J_\alpha'}, \tag{3.5}$$

where σ_i is an element from the set K, $A_{J_i'}$ is a minor of A, and the sum is taken over permutations of the set K and partitions of J.

With the aid of this result we may generalize lemma 3.4 to obtain:

LEMMA 3.6: *Let* $A\epsilon Q$, *then the formula* (3.5) *for* $A_{R,S}$ *has at most one nonzero term.*

PROOF: Suppose for contradiction, that there is more than one nonzero term in (3.5). To be specific, suppose the factor, $a(h_1J_1k_1)a(h_2J_2k_2)$ appears in one term and the factor $a(h_1\tilde{J}_1k_2)a(h_2\tilde{J}_2k_1)$ appears in another term. Consider the principal submatrix $A[L]$ in rows and columns h_1, h_2, k_1, k_2, and the q distinct elements from the sets J_1, J_2, $\tilde{J}_1$, and $\tilde{J}_2$. This matrix is of order $p+4$ and contains $p+4$ nonzero elements above the main diagonal. This contradicts the fact that $A\epsilon Q$.

We may also analyze the cycles of $\Lambda^p A$. Let $J_1, \ldots, J_r$ be increasing multiindices of length p, then

$$\Lambda^p(J_1 \ldots J_r) = A_{J_1J_2} \ldots A_{J_rJ_1} \tag{3.6}$$

is an r-cycle of $\Lambda^p A$. It is clear that the interchanges of rows and columns used above in passing from $A_{R,S}$ to $A_{\tilde{R},\tilde{S}}$ will, when applied throughout the cycle (3.6) of $\Lambda^p A$, not result in any change in sign. On the other hand, it is also clear from (3.5) that the product (3.6) consists of cycles of A on the elements of J_1 with principal minors of A. The principal minors of A are, of course, just the one cycles of the several matrices $\Lambda^p A$.

Turning now to the case of $A\epsilon Q$ we have the appropriate generalization of lemma 3.5.

LEMMA 3.7: *Let* $A\epsilon Q$, *then every nonzero cycle of length* $\geqslant 2$ *of* $\Lambda^p A$ *is a product of 2-cycles of* A *and principal minors of* A.

Lemmas 3.4 and 3.6 generalize the known formulas for Jacobi matrices to all elements of Q. Lemmas 3.5 and 3.7 apparently give new results even for the elements of $\mathscr{J}_2$. The cycles of $\Lambda^p A$ for $A\epsilon\mathscr{J}_2$ do not seem to have been studied.

4. Semipositive Elements of $\mathscr{J}_2$

The semipositive elements of Q seem to have been first studied by Parter in [3]. He was aware of a slightly earlier paper of Givens [11], which treats Jacobi matrices of semipositive type. But a fundamental study of the Jacobi case contained in chapter 2 of the beautiful book of Gantmacher and Krein [7] seems to be almost unknown. (The only noteworthy exception to this statement is in the extensive work of Karlin on total positivity. In fact, his forthcoming book [8] will contain many results which supersede to a considerable extent the work in [7] as well as a wealth of new applications.) Our work in this section has been influenced primarily by Gantmacher and Krein and by Parter.

Definition 4.1: $A\epsilon\mathscr{J}_2$ is of semipositive type if every nonzero 2-cycle of A is positive and a_{jj} is real for $j=1, \ldots, n$. A is principally positive if A is of semipositive type and $a_{jj} \geqslant 0$, $j=1, \ldots, n$.

In this section we shall restrict our consideration to the set Q. The reason for this is simply that the author is unaware of any interesting applications of the semipositive elements of $\mathscr{J}_2 - Q$. We start with:

THEOREM 4.1: *Let* $A\epsilon Q$ *be of semipositive type. Then there exists a diagonal matrix* D *uniquely determined by* A *up to the condition* $d(D) \neq 0$ *and the signs of the elements such that*

$$\tilde{A} = D^{-1}AD$$

is a real symmetric matrix.

This theorem has an interesting history. Parter pointed out in [3] that $A\epsilon Q$ of semipositive type is symmetrizable, basing his result upon previous work by Hearon [6] and Goldberg [4, 5]. Independently the present author pointed out in [1] that A is symmetrizable and, apparently, first noted that this could be accomplished with a diagonal matrix. (In fact, the paper [1] gives a more or less explicit formula for D when A is real and a similar formula is not difficult to obtain for A complex.) It is also possible to show that the signs of D can be so chosen that $\tilde{A}$ has nonnegative elements except possibly on the principal diagonal. This leads one to the following theorem which is a transcription for the semipositive case of a theorem proved in [1]. We do not include a proof.

THEOREM 4.2: *Let* $A\epsilon Q$ *be of semipositive type, then:*

(*a*) *for each* $b \geqslant \min(a_{jj}, 0)$ *the matrix* $A+bI$ *has a unique positive eigenvalue,* $\rho(A+bI)$, *such that for every* λ *in the spectrum,* $\sigma(A+bI)$, $\rho(A+bI) > |\lambda|$.

(*b*) *If* x *is an eigenvector of* $A+bI$ *corresponding to* $\rho(A+bI)$ *then there exists a nonsingular diagonal matrix* D *such that* $x = Dy$ *where* $y > 0$.

(*c*) *The spectrum of* A *and the spectrum of every principal submatrix of* A *is real.*

(*d*) *If any element of* A *is increased, then* $\rho(A)$ *is strictly increased.*

In order to explore the spectral properties of elements of Q which are not so near the surface, it turns out to be useful to single out the class Q^{n-1} for special study. For this purpose, let us consider the nested sequence $A^0=1, A^1, \ldots, A^n=A$ of principal submatrices associated with $A\epsilon Q^{n-1}$ in theorem 2.4. It will be convenient to set

$$\left.\begin{aligned} &d_q = d(A^q),\ q=0, 1, \ldots, n, \\ \text{and}\quad &d_q(\lambda) = d(A^q - \lambda I),\ q=0, 1, \ldots, n. \end{aligned}\right\} \tag{4.1}$$

Thus $d_0(\lambda) \equiv 1$.

The theory begins with:

LEMMA 4.3: *If* $A\epsilon Q^{n-1}$ *is of semipositive type, the sequence of polynomials* $(d_0(\lambda), d(\lambda), \ldots, d_n(\lambda))$, *is a Sturm sequence for every real value of* λ.

PROOF: Since the polynomial $d_q(\lambda)$ is of degree q for $q=0, \ldots n$, we need only show that, if $d_p(\lambda_0)=0$,

$$d_p(\lambda_0)d_{p-1}(\lambda_0) < 0. \tag{4.2}$$

In the present notation the formula 2.7 applied to $A-\lambda I$ in place of A reads

$$d_q(\lambda) = (a_{j_qj_q} - \lambda)d_{q-1}(\lambda) - a_{j_{q-1}j_p}a_{j_pj_{q-1}}d_{q-2}(\lambda). \tag{4.3}$$

If we apply (4.3) for $q=p+1$, multiply by $d_{p-1}(\lambda)$ and set $\lambda=\lambda_0$, we obtain

$$d_{p+1}(\lambda_0)d_{p-1}(\lambda_0) = -a_{j_{p-1}j_p}a_{j_pj_{p-1}}d_{p-1}^2(\lambda_0) \leqslant 0,$$

since A is of semipositive type. The lemma is thereby proved unless $d_{p-1}(\lambda_0)=0$. Suppose, in fact, this is so. Applying (4.3) for $q=p$ and setting $\lambda=\lambda_0$ shows $d_{p-2}(\lambda_0)=0$ also. Thus, by the same reasoning,

$$d_{p-3}(\lambda_0)=\ldots=d_1(\lambda_0)=d_0(\lambda_0)=0,$$

contradicting the fact that $d_0(\lambda_0)=1$. It follows that $d_{p-1}(\lambda_0)\neq 0$ and the lemma is proved.

THEOREM 4.4: *Let* $A\epsilon Q^{n-1}$ *be of semipositive type. Then the eigenvalues of* A^q, $q=1,\ldots,n-1$, *strictly interlace with those of* A^{q+1}. *Hence, in particular, the eigenvalues of* A *are distinct.*

PROOF: By Sturm's theorem the Cauchy index $I_a^b d_{q-1}(\lambda)/d_q(\lambda)$ (see Gantmacher [12], Vol. 2) equals $V_q(a)-V_q(b)$, where $V_q(\lambda)$ is the number of changes in sign in the Sturm sequence $(d_0(\lambda),\ldots,d_q(\lambda))$ and where $-\infty\leqslant a<b\leqslant+\infty$. In the present case we have $d_p(\lambda)=(-1)^p\lambda^p+\ldots$ for each value of p. Thus $V_q(-\infty)=0$ and $V_q(+\infty)=q$. The theorem now follows immediately from the meaning of the Cauchy index.

If $A\notin Q$ is of semipositive type and $A\notin Q^{n-1}$, then A may or may not have a multiple eigenvalue. For example

$$A_1=\begin{pmatrix} 3 & -1 & -1 & -1\\ -1 & 3 & 0 & 0\\ -1 & 0 & 2 & 0\\ -1 & 0 & 0 & 1\end{pmatrix}$$

is of semipositive type and not in Q^{n-1} and has no multiple eigenvalue. But

$$A_2=\begin{pmatrix} 3 & -1 & -1 & -1\\ -1 & 2 & 0 & 0\\ -1 & 0 & 2 & 0\\ -1 & 0 & 0 & 2\end{pmatrix}$$

has the double eigenvalue $\lambda=2$. Thus the fact that $A\epsilon Q^{n-1}$ has no multiple eigenvalues if it is of semipositive type does not distinguish this class. We shall show now that it is the interlacing of the eigenvalues in the sequence $A^1,\ldots,A^n=A$ which does distinguish the class.

The following theorem is quite general in its application.

THEOREM 4.5: *The matrix* A *posses a sequence of principal submatrices with strictly interlacing real roots if and only if there is a sequence of principal submatrices whose determinants form a Sturm sequence for all real* λ.

PROOF: The "if" part is the content of theorem 4.4, the proof of which did not make explicit use of the hypothesis $A\epsilon Q^{n-1}$. Hence we may confine ourselves to the only if statement.

To be specific suppose $J_1\subset J_2\subset\ldots\subset J_n$ is a sequence of increasing multiindices with J_q containing q distinct elements from the set $J_n=S_n=(1\ \ldots\ n)$. Let J_0 be the empty set of indices and assume the sequence of polynomials.

$$(A_{J_0}(\lambda),\quad A_{J_1}(\lambda),\ldots,A_{J_n}(\lambda)),\tag{4.4}$$

where $A_{J_0}(\lambda)\equiv1$, is such that $A_{J_p}(\lambda)$ has p distinct real roots which strictly interlace with those of $A_{J_{p+1}}(\lambda)$, $p=1,\ldots,n-1$. Denote by $\lambda_{p,1},\ldots,\lambda_{p,p}$ the zeroes of $A_{J_p}(\lambda)$ in increasing order. For each p we require that

$$\lambda_{p+1,1}<\lambda_{p,1}<\lambda_{p+1,2}<\ldots<\lambda_{p,p}<\lambda_{p+1,p+1}.$$

In view of the fact that the leading coefficient of $A_{J_p}(\lambda)$ is $(-1)^p$, we may write

$$A_{J_p}(\lambda)=\prod_{q=1}^{p}(\lambda_{p,q}-\lambda),\qquad p=1,\ldots,n.$$

For fixed r we have

$$A_{J_{p+1}}(\lambda_{p,r})A_{J_{p-1}}(\lambda_{p,r})$$
$$=\prod_{q=1}^{p+1}(\lambda_{p,r}-\lambda_{p+1,q})\prod_{q=1}^{p-1}(\lambda_{p,r}-\lambda_{p-1,q})\tag{7.4}$$

In the first product there are r positive factors and $p+1-r$ negative factors. In the second product there are $r-1$ positive factors and $p-r$ negative factors. Thus there are $2(p-r)+1$ negative factors in the product and

$$A_{J_{p+1}}(\lambda_{p,r})A_{J_{p-1}}(\lambda_{p,r})<0,$$

for $r=1,\ldots,p$. It follows that the sequence (4.4) is a Sturm sequence. This proves the theorem.

On the basis of theorem 4.5 we can now investigate the consequence of $A\epsilon\mathscr{J}_2$ possessing a Sturm sequence of principal submatrices. Let us assume that the index j_p is adjoined to J_{p-1} to obtain J_p. Then for $A\epsilon\mathscr{J}_2$ the formula (2.3) applied to the sequence (4.4) becomes

$$A_{J_p}(\lambda)=(\hat{a}(j_p)-\lambda)A_{J-j_p}(\lambda)-\sum_{q=1}^{p-1}\hat{a}(j_pj_q)A_{J-(j_pj_q)}(\lambda).\tag{4.5}$$

It follows that

$$A_{J_p}(\lambda_{p-1,r})A_{J_{p-2}}(\lambda_{p-1,r})$$
$$=-\sum_{q=1}^{p-1}\hat{a}(j_pj_q)A_{J-(j_pj_q)}(\lambda_{p-1,r})A_{J_{p-2}}(\lambda_{p-1,r}).$$

Clearly we shall have a Sturm sequence only if each of the numbers

$$\sum_{q=1}^{p-1}\hat{a}(j_pj_q)A_{J-(j_pj_q)}(\lambda_{p-1,r})A_{J_{p-2}}(\lambda_{p-1,r})>0,\tag{4.6}$$

$r=1, \ldots, p-1$. Since the inequality is strict, at least one term in (4.6) must be different from zero. On the other hand, we have conditions of the form (4.6) for the $n-1$ different values of p, $p=2, \ldots, n$, and A has at most $n-1$ nonzero 2-cycles. The same 2-cycle cannot appear in the inequality (4.6) for two different values of p. It follows that the left side of (4.6) contains exactly 1-term for each value of

$$p=2, \ldots, n,$$

and A has exactly $n-1$ nonzero 2-cycles. But this implies that (4.6) becomes

$$\hat{a}(j_p j_{p-1}) A^2_{J_{p-2}}(\lambda_{p-1,\, r}) > 0 \tag{4.7}$$

$p=2, \ldots, n$. Therefore each of the $n-1$ nonzero 2-cycles of A is positive. This result is summarized in the following theorem:

THEOREM 4.6: *Let* $A \epsilon \mathscr{J}_2$ *possess a sequence of principal submatrices with strictly interlacing roots. Then* $A \epsilon Q^{n-1}$ *and* A *is of semipositive type.*

Theorems 4.4 and 4.6 nail down the most important spectral property of the semipositive elements of Q^{n-1}. Since the elements of Q^{n-1} are essentially Jacobi matrices by virtue of theorem 2.6, we can hardly argue that theorem 4.4 is a new result. On the other hand, theorem 4.6 appears to be a new result and, as it complements theorem 4.4, we felt it appropriate to include both.

The foregoing analysis can be used conveniently to study the spectral multiplicity of the elements Q not in Q^{n-1}. Our intention is to indicate briefly the results of Parter, but by a method somewhat more algebraic in character than his. We only sketch the results here, referring the reader to [3] for a more detailed argument.

We use theorem 2.6 to derive a kind of canonical form for an element of $Q-Q^{n-1}$. By theorem 2.5 $A \epsilon Q-Q^{n-1}$ has at least one row with more than two nonzero off-diagonal elements. Suppose the pth row is one such with the nonzero elements in columns $q_1, \ldots, q_\alpha$, $\alpha > 2$. By a modification of the argument used to prove theorem 2.6 there exists a permutation matrix P such that the matrix $P\ A\ P'$ has the following form:

$$P\ A\ P' = \begin{pmatrix} A_{11} & c_1 & 0 & \ldots & 0 \\ r_1 & a_{pp} & r_2 & \ldots & r_\alpha \\ 0 & c_2 & A_{22} & \ldots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & c_\alpha & 0 & \ldots & A_{\alpha\alpha} \end{pmatrix}, \tag{4.8}$$

in which A_{ii} is a square matrix of order n_i, $i=1, \ldots, \alpha$; c_1 is a column vector of zeroes except for the n_1th entry which is $a_{q_1 p}$; r_1 is a row vector of zeroes except for the nonzero element a_{p,q_1} in the n_1th entry; c_i, $i=2, \ldots, \alpha$, is a column of zeroes except for the nonzero element $a_{q_i p}$ in the first position;

$$r_i,\ i=2, \ldots, \alpha$$

is a row of zeroes except for $a_{pq_i} \neq 0$ in the first position; and each block A_{ii} is either a Jacobi matrix or a matrix of the form (4.8). Thus $P\ A\ P'$ is recursively defined in the general case.

Now we apply the basic recurrence formula (2.3) to $P\ A\ P' - \lambda I$ to obtain

$$d(P\ A\ P' - \lambda I) = (a_{pp} - \lambda)\, d(A_{11}(\lambda)\ \ldots\ d(A_{\alpha\alpha}(\lambda))$$

$$-\sum_{i=1}^{\alpha} \hat{a}(p, q_i)\, d(\overline{A}_{ii}(\lambda)) \prod_{\substack{j=1 \\ j \neq i}}^{\alpha} d(A_{jj}(\lambda)). \tag{4.9}$$

In (4.9) $\overline{A}_{ii}$ is the matrix obtained by deleting the last row and column of A_{ii} if $i=1$; otherwise it is the matrix obtained by deleting the first row and column of A_{ii}.

Since $\alpha > 3$, it follows immediately from (4.9) that $A \epsilon Q - Q^{n-1}$ has the multiple eigenvalue λ_0 if λ_0 is an eigenvalue of at least three of the submatrices A_{ii}. Suppose, on the other hand, that λ_0 is an eigenvalue of A of multiplicity at least two. Then λ_0 must be an eigenvalue of the principal submatrix of A, $A(p)$, i.e., λ_0 must be an eigenvalue of A_{ii} for some value of i, say for $i=k$. Thus

$$0 = d(P\ A\ P' - \lambda_0 I) = -\hat{a}(p, q_k)\, d(\overline{A}_{kk}(\lambda_0)) \prod_{\substack{j=1 \\ j \neq k}}^{\alpha} d(A_{jj}(\lambda_0)).$$

Hence, either $A_{jj}(\lambda_0) = 0$ for some value of $j \neq k$, or $(\overline{A}_{kk}(\lambda_0)) = 0$. In the latter case the block A_{kk} is not a Jacobi matrix and must itself have λ_0 as a multiple eigenvalue. Clearly the argument up to this point can be repeated for the block A_{kk}, so we may confine ourselves to the case where $A_{jj}(\lambda_0) = 0$ for some value of $j \neq k$. Thus after a finite number of steps we must arrive at a matrix having the form (4.8) such that each diagonal block is a Jacobi matrix. But it is easy to see that such a matrix can have a multiple eigenvalue λ_0 only if at least three of the blocks have λ_0 as an eigenvalue.

Now, if a diagonal block A_{ii} in the representation (4.8) is not a Jacobi matrix, there is some row of A, say the qth row, having at least three nonzero off-diagonal elements. This row could have been used in the beginning to set up the form (4.8). Moreover, if this is done, we see that λ_0 a multiple eigenvalue of A implies λ, is an eigenvalue of at least three of the diagonal blocks.

This sketch establishes the theorem of Parter which we here formulate quite differently than his original graph-theoretic formulation.

THEOREM 4.7 (*Parter*): *Let* $A \epsilon Q - Q^{n-1}$ *be of semipositive type. Then* A *has the multiple eigenvalue* λ_0 *if and only if there exists a permutation matrix* P *such that* P A P' *has the form* (4.8) *with* $\alpha \geqslant 3$ *and* λ_0 *an eigenvalue of at least three of the diagonal blocks* A_{ii}.

On the basis of this theorem it is obvious that the assertions made regarding the matrices A_1 and A_2 in subsection 4.2 are valid.

The theorem on interlacing proved above, theorem 4.4, for the elements of Q^{n-1} is of considerable interest in the applications to mechanics. That it does not extend in precisely this form to the elements of $Q-Q^{n-1}$ is quite clear. Nevertheless one may ask when λ_0 can be an eigenvalue of $A \epsilon Q-Q^{n-1}$ and also an eigenvalue of a certain type of principal submatrix. We shall not pursue this question in the present paper. It will be taken up elsewhere when we study the application of matrices in $\mathscr{J}_2$ of semi-positive type to problems in mechanical vibrations. Similarly the results on eigenvectors are more strongly motivated in a mechanical setting. On the other hand, some of the results of section 3 have special application to the semipositive elements of $\mathscr{J}_2$ and we take up this subject next.

We assume for the remainder of this section that A is positive semidefinite. This will be the case, for example, if A is diagonally dominant. Lemmas 3.4 and 3.5 now imply

THEOREM 4.8: *Let* $A \epsilon Q$ *be of semipositive type and positive semidefinite. Then* cof A *has no zero elements and is a matrix of positive cyclic type. In particular, if* A *is positive definite, then* A^{-1} *has no zero elements and is of positive cyclic type.*

It should be mentioned at this point that some of the theory of matrices of positive cyclic type is worked out in [2]. In the literature of mathematical economics these matrices are usually called Morishima matrices in honor of Professor Michio Morishima who first characterized them. The analogue of the Perron-Frobenius theorem is valid for such matrices. Since the largest eigenvalue of A^{-1} is related to the smallest eigenvalue of A we have

COROLLARY 4.9: *Let* $A \epsilon Q$ *be of semipositive type and suppose* A *is positive definite. Then if*

$$\lambda_1 \leqslant \lambda_2 \leqslant . . . \leqslant \lambda_n$$

are the eigenvalues of A *in increasing order, we have*

$$0 < \lambda_1 < \lambda_i < \lambda_n, \; i=2, . . ., n-1$$

i.e., λ_1 and λ_n are simple eigenvalues of A. Moreover if any element of A^{-1} is increased, then λ_1 is strictly decreased.

We feel the relationship between theorem 4.7 and corollary 4.9 should be stressed. If $A \epsilon Q-Q^{n-1}$, is semipositive and positive definite, and if A has a multiple eigenvalue λ^*, then λ^* cannot be equal to λ_i or to λ_n. Thus for the matrix A_2 of subsection 4.2 the double eigenvalue $\lambda=2$ is neither the smallest nor the largest eigenvalue.

The fact that A^{-1} is of positive cyclic type enables us to say even more about A itself. Every matrix of positive cyclic type is similar to a positive matrix via a diagonal matrix D. Thus, if the conditions of corollary 4.9 are satisfied there exists a diagonal matrix D such that the matrix $B^{-1}=D^{-1}A^{-1}D$ satisfies

$$B^{-1} > 0. \tag{4.10}$$

Now (4.10) implies that B itself is monotone, i.e., if $Bx > 0$ then $x > 0$. Consequently, we deduce that A is similar by the diagonal matrix D to a monotone matrix.

In the special case where $A_{ij} \leqslant 0$, $i \neq j$, and A is positive cyclic type, then A is both monotone and of positive cyclic type. In this case the inverse of A is positive, $A^{-1} > 0$. That $A^{-1} \geqslant 0$ follows from the well-known connection between monotone matrices and nonnegative matrices. In order to conclude that $A^{-1} > 0$ we must appeal to the cofactor formula (3.3), and lemma 3.4. This result is a generalization of a theorem of Gautmacher and Krein.

In conclusion a few remarks on total positivity are in order. We cannot use lemma 3.7 directly in the same way as lemma 3.5, to conclude that $\Lambda^p A$ is of positive cyclic type, since lemma 3.7 is weaker than 3.5. In fact, we have seen above that there exist elements $A \epsilon Q-Q^{n-1}$ having multiple eigenvalues. Since totally positive matrices enjoy the interlacing property characterized in theorem 4.4., such elements of $Q-Q^{n-1}$ cannot be totally positive. This question therefore remains open. It may be formulated as follows: Which elements $A \epsilon Q$ of semipositive type have the property that $\Lambda^p A$ is of nonnegative cyclic type for $p=1, . . ., n$?(Of course, we know that every such A must be positive semidefinite.)

5. Seminegative Elements of $\mathscr{J}_2$

The seminegative elements of $\mathscr{J}_2$ first became of interest through the work of Quirk and Ruppert [10] in qualitative economics. We restrict ourselves here to real matrices in $\mathscr{J}_2$. As usual we also confine ourselves to the consideration of elements of Q.

Definition 5.1: $A \epsilon Q$ is of seminegative type if every nonzero 2-cycle of A is negative. A is principally negative if A is of seminegative type and

$$a_{jj} \leqslant o, \; j=1. . . ., n.$$

The following result is established by an argument entirely similar to that used to prove a portion of theorem 2 in [1]. We omit the proof.

LEMMA 5.2: *Let* $A \epsilon Q$ *be of seminegative type and set* $A=E+A^*$ *where* $E=diag \; \{a_{11}, . . ., a_{nn}\}$ *and* $A^*=\{a^*_{jk}\}$ *with* $a^*_{jk}=a_{jk}$ *if* $j \neq k$ *and* $a^*_{jj}=0$. A^* *is similar to a skew-symmetric matrix by a diagonal matrix* D, *i.e., there exists a diagonal matrix* D *such that*

$$D^{-1}AD=E+B,$$

where B *is skew symmetric.*

Lemma 5.2 leads immediately to the following result:

LEMMA 5.3: *Let* $A \epsilon Q$ *be of seminegative type and have a zero principal diagonal. Then all of the eigenvalues of* A *lie on the imaginary axis in the complex plane.*

We shall conclude the paper by formulating the Quirk-Ruppert theorem somewhat more elegantly than it appears in [10]; (see, however, [9] where the result is also better formulated).

Definition 5.2: If $A=(a_{jk})_1^n$ is a real matrix we associate with it the matrix sgn $A=(a_{jk}^*)_1^n$ where $a_{jk}^*=$ sgn a_{jk}, j, $k=1, \ldots n$, and the equivalence class Q_A defined by

$$Q_A=\{B|\operatorname{sgn} B=\operatorname{sgn} A\}.$$

The equivalence class Q_A may be called a qualitative matrix.

Definition 5.3: A is a sign stable matrix if $B \epsilon Q_A$ implies B is a stable matrix, i.e., all the eigenvalues of B lie in the open left half of the complex plane.

THEOREM 5.3: (*Quirk-Ruppert*) *Let* A *be an irreducible matrix. Then* A *is sign stable if and only if*

(i) AϵQ *and* A *is principally negative,*
(ii) $a_{jj}<0$ *for some value of* j, *and*
(iii) d(A) $\neq$ 0.

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