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On Even Matroids

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This article is intended as a supplement to an earlier paper entitled "Lectures on matroids." The author takes this opportunity to correct some errors in "Lectures on Matroids." Theorems 4.31 and 4.372 are valid only for binary matroids, the plane of 4.281 must be connected, and the word "reductions" is used in 3.48 instead of "contractions."

Key Words: Binary, bridge-separable, even, graphic, matroid.

It is shown in [1][1] that every graphic matroid is regular ([1], 5.63) and even ([1], 9.23). Moreover a regular matroid can be characterized as a binary one which has no minor of either of the types called BI and BII. ([1], 7.51). In the present paper we establish a converse theorem: any even matroid which has no minor of Type BI must be graphic.

1. *Let Y be an atom of a binary matroid M, and suppose it to have the following properties.*

(i) *Y is bridge-separable*

(ii) *If B is any bridge of Y in M, then* $M \times (B \cup Y)$ *is graphic.*

Then M is graphic.

PROOF. If possible choose Y and M so that the theorem fails, and M has the least number of cells consistent with this condition.

Clearly there must be at least two bridges of Y in M. Since Y is bridge-separable we can arrange these bridges in two non-null disjoint classes P and Q so that no two members of the same class overlap. Let U_P be the union of the members of P, and let U_Q be defined analogously.

Now Y is an atom of $M \times (U_P \cup Y)$. Moreover the bridges of Y in $M \times (U_P \cup Y)$ are the members of P, and each determines the same partition of Y as in M. ([1], 8.53). Hence Y is totally bridge-separable in $M \times (U_P \cup Y)$. It follows, by the choice of Y and M that $M \times (U_P \cup Y)$ is graphic. Similarly Y is totally bridge-separable in $M \times (U_Q \cup Y)$, and this matroid is graphic.

We may now repeat the argument in the latter part of the proof of ([1], 9.41), with $U_P = S$ and $U_Q = T$. We thus find that M is graphic.

This contradiction establishes the theorem.

2. *Let Y be an atom of a binary matroid M on a set E. Let W be an atom of* $M \cdot (E-Y)$, *on a bridge B of Y in M, determining a partition* $\{S, T\}$ *of Y such that* $T \epsilon \pi(M, B, Y)$. *Let* Y_1 *denote the atom* $S \cup W$ *of* M, *and write* $M_1 = M \times (B \cup Y)$.

Then T is a bridge of Y_1 *in* M_1. *Moreover every other bridge of* Y_1 *in* M_1 *is also a bridge of* Y_1 *in M.*

PROOF. The set $Y \cup W = Y_1 \cup T$ is a line of M on the flat $B \cup Y$. Hence T is an atom of

$$(M \times (B \cup Y)) \cdot ((B \cup Y) - Y_1)$$

by ([1], 8.12). It is thus a subset of some bridge B_T of Y_1 in M_1.

Let C be any bridge of Y_1 in M_1. Suppose there is an atom Z of $M \cdot (E - Y_1)$ which meets $C - T$.

There is an atom X of M such that $X \cap (E - Y_1) = Z$.

There is an atom Z_1 of $M \cdot (E - Y)$ such that

$$Z_1 \subseteq X \cap (E - Y).$$

Since $C - T$ is a subset of $E - Y$ we may choose Z_1 to meet $C - T$, by ([1], 1.11). It then follows that Z_1 is on the bridge B of Y in M.

There is an atom X_1 of $M \times (B \cup Y)$ such that

$$X_1 \cap B = Z_1.$$

There is an atom Z_2 of $(M \times (B \cup Y)) \cdot ((B \cup Y) - Y_1)$ such that $Z_2 \subseteq X_1 \cap ((B \cup Y) - Y_1)$. By ([1], 1.11) we may choose Z_2 to meet $C - T$. We note that Z_2 is an atom of $(M \cdot (E - Y_1)) \times ((B \cup Y) - Y_1)$, by ([1], 3.334), and therefore of $M \cdot (E - Y_1)$. Further, Z_2 is on the bridge C of Y_1 in M_1. We thus have

$$Z_2 \subseteq X_1 \cap C = Z_1 \cap C$$

$$\subseteq X \cap (E - Y) \cap C \subseteq X \cap (C - T),$$

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[1] W. T. Tutte, Lectures on Matroids, J. Res. NBS **69B** (Math. and Math. Phys.), Nos. 1 & 2, 1-47 (1965).

since

$$T \cap (E-Y) = \phi,$$

$$\subseteq (Z \cup Y_1) \cap (C-T) = Z \cap (C-T).$$

Applying Axiom I to $M \cdot (E-Y_1)$ we deduce first that $Z_2 = Z$ and then that $Z \subseteq C-T$. Thus $C-T$ is a separator of $M \cdot (E-Y_1)$.

We deduce that $B_T = T$. Thus T is a bridge of Y_1 in M_1.

Suppose C is another bridge of Y_1 in M_1. The above result shows that $C (= C-T)$ is a separator of $M \cdot (E-Y_1)$.

To complete the proof we observe that

$$(M \cdot (E-Y_1)) \times C$$

is connected. For it is identical with

$$((M \cdot (E-Y_1)) \times ((B \cup Y) - Y_1)) \times C,$$

that is with $(M_1 \cdot ((B \cup Y) - Y_1)) \times C$ by ([1], 3.333), and C is a bridge of Y_1 in M_1. Hence C is an elementary separator of $M \cdot (E-Y_1)$, that is a bridge of Y_1 in M.

3. *Let* M *be any even matroid which has no minor of Type* BI. *Then* M *is graphic.*

PROOF. Assume that M is not graphic.

If Y is any atom of M there is a bridge B of Y in M such that $M \times (B \cup Y)$ is not graphic, by Theorem 1. Choose such a Y and B so that B has the least possible number of cells.

Choose $T \epsilon \pi(M, B, Y)$. Now B is nontrivial since $M \times (B \cup Y)$ is nongraphic. Hence there is an atom Z of $M \cdot (E-Y)$ on B determining the partition $\{T, Y-T\}$ of Y, by ([1], 8.62). By the definition of this partition the set $Y_1 = (Y-T) \cup Z$ is an atom of $M \times (B \cup Y)$.

One of the bridges of Y_1 in $M \times (B \cup Y)$ is T, by Theorem 2. Let the others, if any, be enumerated as $C_1, C_2, \ldots, C_k$. The matroid $M \times (T \cup Y_1)$ has rank 2, by ([1], 8.12). It is therefore graphic, by ([1], 9.41). Moreover any C_i is a bridge of Y_1 in M, by Theorem 2. Hence $M \times (C_i \cup Y_1)$ is graphic, by the choice of Y and B. But $M \times (B \cup Y)$ is nongraphic. It follows from Theorem 1 that Y_1 is not bridge-separable in $M \times (B \cup Y)$.

Now each C_i determines the same partition of Y_1 in M as in $M \times (B \cup Y)$, by ([1], 8.53). Hence if C_i and C_j overlap as bridges of Y_1 in $M \times (B \cup Y)$ they overlap also as bridges of Y_1 in M.

Since $Y \cup Z = Y_1 \cup T$ is a line of M the set T is an atom of $M \cdot (E-Y_1)$, by ([1], 8.12). Hence there is a bridge D of Y_1 in M such that $T \subseteq D$. This bridge is distinct from each of the bridges C_i. Each of the sets Z and $Y-T$ is either null or a union of members of $\pi(M, D, Y_1)$, by the definition of this partition. Hence if T and C_i overlap as bridges of Y_1 in $M \times (B \cup Y)$, then D and C_i overlap as bridges of Y_1 in M. For $\pi(M \times (B \cup Y), T, Y_1) = \{Z, Y-T\}$.

From the foregoing results we deduce that since Y_1 is not bridge-separable in $M \times (B \cup Y)$ it is also not bridge-separable in M. But this contradicts the definition of M as an even matroid. The theorem follows.

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