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Calibration Designs Based on Solutions to the Tournament Problem

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In high precision calibrations one measures differences between nominally equal objects or group of objects and establishes a value for the individuals with reference to one or more standards. The solutions to the classical tournament problem, which calls for arranging v individuals into teams of p players so that a player is teamed the same number of times with each of the other players and also that each player is pitted equally often against each of the other players, provide balanced designs for scheduling the measurements. These designs are useful in weighing and other measurements when the objects to be measured can be combined into groups without loss of precision or accuracy in the comparisons.

This paper presents solutions to the tournament problem for all $v \leq 13$ and for $p \leq \frac{v}{2}$. The statistical analysis, a worked example, and computational procedures are given.

Key Words: Calibration, calibration designs, combinatorial analysis, difference sets, incomplete block designs, statistical experiment designs, tournaments, weighing designs.

1. Introduction

In high precision calibration only differences between nominally equal objects (or groups of objects) can be measured, and the process of calibration consists of assigning the value for the "unknown" objects in terms of "known" or accepted standards. Where there are v objects and the intercomparisons can be made between groups of size p then one has a situation analogous to the classical tournament problem. Schedules for intercomparison which are balanced in the sense that each object (or player) is teamed up with each of the other objects (or players) an equal number of times and is in opposition to each of the other objects (or players) the same number of times are found in solutions to the tournament problem.

In a previous paper [6][1] solutions to the tournament problem for $p=2$ and $v \leq 50$ were given and this paper extends those results to include balanced weighing designs (BWD) for $v \leq 13$ and $p \leq v/2$. The statistical analyses appropriate when the designs are used in calibration, and an example from mass calibration are given.

The paper has two main parts; one related to the construction of the design, the other to their use and analysis. Those primarily interested in the use of the designs in measurement should begin with section 3.

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[1] Figures in brackets indicate the literature references at the end of this paper.

2. Construction of Balanced Weighing Designs

1. Let there be v players or objects. We have to arrange them in b blocks of size $2p$, each block consisting of two half-blocks of size p. Two objects appear in the same half-block λ_1 times, and in opposite half-blocks of the same block λ_2 times. Then

$$v,\ b,\ r,\ p,\ \lambda_1,\ \lambda_2$$

are said to be the parameters of the tournament design or balanced weight design (BWD). Here r is the number of blocks in which each object appears. It is readily shown [6] that

$$\lambda_1(v-1)=r(p-1), \quad \lambda_2(v-1)=rp.$$

Hence $r=\beta(v-1)$, where $\beta=\lambda_2-\lambda_1$. Then counting the number of objects in the b blocks in two different ways we have

$$2pb=vr=v(v-1)\beta \tag{2.1.1}$$

Hence

$$b=\beta v(v-1)/2p. \tag{2.1.2}$$

Let h be the highest common factor of $v(v-1)$ and $2p$, and let $2p=hn$, then $\beta=\lambda_2-\lambda_1$ must be divisible by n. Hence the least possible value of β is n, and in general $\beta=gn$ where g is a positive integer. If the design for $\beta=n$ exists, we shall call it minimal in the sense that no smaller number of blocks could possibly lead to a balanced design. The parameters of the design then are

$$v,\ b=v(v-1)/h,\ r=n(v-1),\ p,\ \lambda_1=n(p-1),\ \lambda_2=np$$

where h is the highest common factor of $v(v-1)$ and $2p=hn$.

It is known that a design with $\beta=n$ does not always exist. Such an example is given later in this paper. A BWD design will therefore be called minimal if g is the smallest positive integer such that a design with parameters

$$v,\ b=gv(v-1)/h,\ r=gn(v-1),$$
$$p,\ \lambda_1=gn(p-1),\ \lambda_2=gnp$$

exists. If the design with $g=1$ exists, then it is of course minimal.

Designs with $p=2$, $v\leq 50$ were studied in an earlier paper [6]. In this section we shall give some series of BWD designs for $p>2$, which include all minimal designs for $v\leq 13$, except the design

$$v=10,\ b=15,\ r=9,\ p=3,\ \lambda_1=2,\ \lambda_2=3.$$

It is not known whether this is combinatorially possible. However the corresponding design with $g=2$, i.e., the design

$$v=10,\ b=30,\ r=18,\ p=3,\ \lambda_1=4,\ \lambda_2=6$$

will be obtained.

Except for a few cases, the construction is based on the method of symmetrically repeated differences first used by Bose [3]. The theorems relevant to the construction of BWD designs have been given in Bose and Cameron [6], to which reference should be made. As in the earlier paper the notation

$$\{(a_1,\ a_2,\ \ldots,\ a_p),\ (b_1,\ b_2,\ \ldots,\ b_p)\}\oplus(c_1,\ c_2,\ \ldots,\ c_u)$$

will be used to denote the set of blocks

$$\{(a_1c_i,\ a_2c_i,\ \ldots,\ a_pc_i),\ (b_1c_i,\ b_2c_i,\ \ldots,\ b_pc_i)\},\quad i=1,2,\ldots,u;$$

where $a_1, a_2, \ldots, a_p, b_1, b_2, \ldots, b_p, c_1, c_2, \ldots, c_u$ are elements of a field or a commutative ring.

2. Let v be a prime power of the form $4t+3$. Let h be the H.C.F. of $2p$ and $(4t+2)(4t+3)$ and let $n=2p/h$. Then a design with parameters

$$v=4t+3,\quad b=(4t+2)(4t+3)/h,$$
$$r=n(4t+2),\quad p,\quad \lambda_1=n(p-1),\quad \lambda_2=np \tag{2.2.1}$$

is minimal if it exists.

(a) If p is relatively prime to $2t+1$ and $4t+3$, then $h=2$, $n=p$ and the design has parameters

$$v=4t+3,\quad b=(4t+3)(2t+1),\quad r=p(4t+2),\quad p,$$
$$\lambda_1=p(p-1),\quad \lambda_2=p^2. \tag{2.2.2}$$

A solution of this design is obtained by cyclically developing the initial blocks

$$[(a_1,\ a_2,\ \ldots,\ a_p),\ (b_1,\ b_2,\ \ldots,\ b_p)]\oplus(1,\ x,\ x^2,\ \ldots,\ x^{2t})$$

where $a_1, a_2, \ldots, a_p, b_1, b_2, \ldots, b_p$ are distinct elements of $GF(4t+3)$, and x is a primitive element. By cyclical development of, for example, $(a_1a_2)(b_1b_2)$, is meant the series of blocks $\{(a_1a_2)(b_1b_2)\}$, $\{(a_1+1, a_2+1), (b_1+1, b_2+1)\}\ \ldots\ \{(a_1+v-1, a_2+v-1)(b_1+v-1, b_2+v-1)\}$ reduced mod v where v is a prime. If v is a power of a prime, say p^n, then in place of $1, 2, \ldots v-1$ one adds $1, g_1, g_2, \ldots, g_{v-1}$ where the g_i are elements of the Galois field of order p^n. For example, for $v=9=3^2$ the elements of the field are $1, x, 2x+1, 2x+2, 2, 2x, x+2, x+1$ and the addition is carried on mod (x^2+x+2). A detailed discussion is given in reference 3.

The within half-block differences arising from the initial blocks are

$$\{\ldots,\ \pm(a_1-a_j),\ \ldots,\ \pm(b_1-b_j),\ \ldots\}\oplus(1,\ x,\ x^2,\ \ldots,\ x^{2t}).$$

Since x is a primitive element of GF$(4t+3)$, $x^{2t+1}=-1$. Hence the differences may be written as

$$\{\ldots,\ (a_1-a_j),\ \ldots\ (b_1-b_j),\ \ldots\}\oplus(1,\ x,\ x^2,\ \ldots,\ x^{4t+1}).$$

It is evident that each nonzero difference is repeated $\lambda_1=p(p-1)$ times.

Again the differences arising from the cross pairs, i.e., pairs belonging to opposite half-blocks within the same initial block are

$$\{\ldots,\ \pm(a_1-b_j),\ \ldots\}\oplus(1,\ x,\ x^2,\ \ldots,\ x^{2t})$$

and these may as before be written as

$$\{\ldots,\ (a_i-b_j),\ \ldots\}\oplus(1,\ x,\ x^2,\ \ldots,\ x^{4t+1})$$

so that each nonzero difference is repeated $\lambda_2=p^2$ times.

The proof follows as in [6].

Example (2.2.1). Let $t=2$, $p=3$. Let the objects be represented by elements of GF(11). Note that 2 is a primitive element of GF(11). A solution of the design

$$v=11,\quad b=55,\quad r=30,\quad p=3,\quad \lambda_1=6,\quad \lambda_2=9$$

is obtained by developing the initial blocks

$$\{(1,\ 2,\ 3),\quad (4,\ 5,\ 6)\}\oplus(1,\ 2,\ 4,\ 8,\ 5).$$

Example (2.2.2). Let $t=2$, $p=4$. As in the previous example let the objects be represented by elements of GF(11). A solution of the design

$$v=11,\quad b=55,\quad r=40,\quad p=4,\quad \lambda_1=12,\quad \lambda_2=16$$

is obtained by developing the initial blocks

$$\{(1,4,5,10),\quad (9,7,3,6)\}\oplus(1,2,4,8,5).$$

(b) Next suppose that $2t+1$ is a multiple of p say $2t+1=\beta p$ then $h=2p$ and $n=1$. Then from (2.2.1) the parameters of the design become

$$v=4t+3,\quad b=(4t+3)\beta,\quad r=4t+2,$$
$$p,\quad \lambda_1=p-1,\quad \lambda_2=p. \tag{2.2.3}$$

A solution of this design is obtained by developing the initial blocks

$$\{(1,x^{2\beta},x^{4\beta},\ldots,x^{2(p-1)\beta}),\quad (x^{\beta},x^{3\beta},\ldots,x^{(2p-1)\beta})\}$$
$$\oplus(1,x,x^2,\ldots,x^{\beta-1}).$$

The proof follows from the method of differences.

Example (2.2.3). Let $t=1$ and $p=3$. Then $\beta=1$. Let the objects be represented by elements of GF(7) and note that 3 is a primitive element. A solution of the design

$$v=7,\quad b=7,\quad r=6,\quad p=3,\quad \lambda_1=2,\quad \lambda_2=3$$

is obtained by developing the initial block

$$\{(1,2,4),\quad (3,6,5)\}.$$

Example (2.2.4). Let $t=2$ and $p=5$. Then $\beta=1$. Let the objects be represented by elements of GF(11). A solution of the design

$$v=11,\quad b=11,\quad r=10,\quad p=5,\quad \lambda_1=4,\quad \lambda_2=5$$

is obtained by developing the initial block

$$\{(1,4,5,9,3),\quad (2,8,10,7,6)\}$$

3. Let v be a prime power of the form $4t+1$. Let h be the H.C.F. of $2p$ and $4t(4t+1)$, and let $n=2p/h$. Then a design with parameters

$$v=4t+1,\quad b=4t(4t+1)/h,\quad r=4nt,\quad p$$
$$\lambda_1=n(p-1),\quad \lambda_2=np \tag{2.3.1}$$

is minimal if it exists.

(a) If p is relatively prime to $2t$ and $4t+1$, then $h=2$ and $n=p$. The parameters of the design become

$$v=4t+1,\quad b=2t(4t+1),\quad r=4tp,\quad p,$$
$$\lambda_1=p(p-1),\quad \lambda_2=p^2. \tag{2.3.2}$$

Let x be a primitive element GF($4t+1$). Then a solution of the design is obtained by developing the initial blocks

$$\{(a_1,a_2,\ldots,a_p),\quad (b_1,b_2,\ldots,b_p)\}$$
$$\oplus(1,x,x^2,\ldots,x^{2t-1})$$

where $a_1,a_2,\ldots,a_p,b_1,b_2,\ldots,b_p$ are distinct elements of GF($4t+1$).

The proof follows from the method of differences.

Example (2.3.1). Let $t=3$, $p=5$. Let the objects be represented by elements of GF(13). Note that 2 is a primitive element. A solution of the design

$$v=13,\quad b=78,\quad r=60,\quad p=5,\quad \lambda_1=20,\quad \lambda_2=25$$

is obtained by developing the initial blocks

$$\{(1,2,3,4,5),(6,7,8,9,10)\}\oplus(1,2,4,8,3,6).$$

(b) Next suppose that $2t$ is a multiple of p, say $2t=\beta p$, then $h=2p$ and $n=1$. Then from (2.3.1), the parameters of the design become

$$v=4t+1,\quad b=(4t+1)\beta,\quad r=4t,$$
$$p,\quad \lambda_1=p-1,\quad \lambda_2=p.$$

Let x be a primitive element of GF($4t+1$). Then a solution of the design is obtained by developing the initial blocks

$$\{(1,x^{2\beta},\ldots,x^{4t-2\beta}),\quad (x^{\beta},x^{3\beta},\ldots,x^{4t-\beta})\}$$
$$\oplus(1,x,\ldots,x^{\beta-1}).$$

The proof follows from the method of differences.

Example (2.3.2). Let $t=2$, $p=4$. Then $\beta=1$. Let the objects be represented by the elements of GF(3^2). A solution of the design

$$v=9,\quad b=9,\quad r=8,\quad p=4,\quad \lambda_1=3,\quad \lambda_2=4$$

is obtained by developing the initial block

$$\{(1,x^2,x^4,x^6),(x,x^3,x^5,x^7)\}$$

where x is a primitive element of GF(3^2).

Example (2.3.3). Let $t=3$, $p=3$. Then $\beta=2$. Let the objects be represented by the elements of GF(13). A solution of the design

$$v=13,\quad b=26,\quad r=12,\quad p=3,\quad \lambda_1=2,\quad \lambda_2=3$$

is obtained by developing the initial blocks

$$\{(1,3,9),(4,12,10)\}\oplus(1,2).$$

Example (2.3.4). Let $t=3$, $p=6$. Then $\beta=1$. Let the objects be represented by the elements of GF(13) as in the previous example. Then a solution of the design

$$v=13,\quad b=13,\quad r=12,\quad p=6,\quad \lambda_1=5,\quad \lambda_2=6$$

is obtained by developing the initial block

$$\{(1, 4, 3, 12, 9, 10), (2, 8, 6, 11, 5, 7)\}.$$

(c) If $p=4$ and t is odd, then the conditions assumed in neither (a) nor (b) are satisfied. In this case $h=4$, $n=2$. The parameters of the design (2.3.1) become

$$v=4t+1, \quad b=t(4t+1), \quad r=8t,$$
$$p=4, \quad \lambda_1=6, \quad \lambda_2=8.$$

As before let x be a primitive element of GF$(4t+3)$. A solution of the design is obtained by developing the initial blocks

$$\{(1, x^t, x^{2t}, x^{3t}), (x^2, x^{t+2}, x^{2t+2}, x^{3t+2})\}$$
$$\oplus(1, x, x^2, \ldots, x^{t-1}).$$

The proof follows from the method of symmetrically repeated differences.

Example (2.3.5). Let $t=3$, $p=4$. Let the objects be represented by elements of GF(13) as before. A solution of the design

$$v=13, \quad b=39, \quad r=24, \quad p=4, \quad \lambda_1=6, \quad \lambda_2=8$$

is obtained by developing the initial blocks

$$\{(1, 8, 12, 5), (4, 6, 9, 7)\}\oplus(1, 2, 4).$$

4. (a) If $6t+1$ is a prime power and $p=3$, there exists a minimal BWD with parameters

$$v=6t+1, \quad b=t(6t+1), \quad r=6t,$$
$$p=3, \quad \lambda_1=2, \quad \lambda_2=3 \qquad (2.4.1)$$

whose solution is obtained by developing the initial blocks

$$\{(1, x^{2t}, x^{4t}), \quad (x^t, x^{3t}, x^{5t})\}\oplus(1, x, x^2, \ldots, x^{t-1}). \qquad (2.4.2)$$

The proof follows at once by using the method of symmetrically repeated differences.

(b) We can modify the above solution to obtain a solution of the design

$$v=6t+2, \quad b=(3t+t)(6t+1), \quad r=3(6t+1),$$
$$p=3, \quad \lambda_1=6, \quad \lambda_2=9 \qquad (2.4.3)$$

when as in (a), $6t+1$ is a prime power. Let $6t+1$ objects be represented by elements of GF$(6t+1)$, and to these let us adjoin another object $\propto$. Let us take as initial blocks, the blocks

$$\{(1, x^{2t}, x^{4t}), \quad (x^t, x^{3t}, x^{5t})\}\oplus(x, x^2, \ldots, x^{t-1}), \qquad (2.4.4)$$

each repeated thrice, together with the four initial blocks

$$\{(1, x^{2t}, x^{4t}), \quad (x^t, x^{3t}, x^{5t})\} \qquad (2.4.5)$$
$$\{(\propto, 1, x^{2t}), \quad (x^t, x^{3t}, x^{5t})\} \qquad (2.4.6)$$
$$\{(\propto, 1, x^{4t}), \quad (x^t, x^{3t}, x^{5t})\} \qquad (2.4.7)$$
$$\{(\propto, x^{2t}, x^{4t}), \quad (x^t, x^{3t}, x^{5t})\}. \qquad (2.4.8)$$

Then by developing we shall obtain a solution of (2.4.3). Observe that when we develop (2.4.6), (2.4.7), (2.4.8), $\propto$ is replicated $3(6t+1)$ times, and occurs 6 times in the same half-block and 9 times in opposite half-blocks with each other object. Also any difference occurring in (2.4.4) occurs thrice in (2.4.5), (2.4.6), (2.4.7), (2.4.8).

Example (2.4.1). Let $t=1$. Let the objects be represented by $\propto$ and the elements of GF(7). Then the solution of the design

$$v=8, \quad b=28, \quad r=21, \quad p=3, \quad \lambda_1=6, \quad \lambda_2=9$$

is obtained by developing the initial blocks

$$\{(1, 2, 4), \quad (3, 5, 6)\}, \quad \{(\propto, 1, 2), \quad (3, 5, 6)\}$$
$$\{(\propto, 1, 4), \quad (3, 5, 6)\}, \quad \{(\propto, 2, 4), \quad (3, 5, 6)\}$$

Example (2.4.2). Let $t=2$. Let the objects be represented by $\propto$ and the elements of GF(13). Then a solution of design

$$v=14, \quad b=91, \quad r=39, \quad p=3, \quad \lambda_1=6, \quad \lambda_2=9$$

is obtained by developing the initial blocks

$$\{(1, 3, 9), \quad (4, 12, 10)\}$$
$$\oplus(1, 2, 2, 2), \quad \{(\propto, 1, 3), \quad (4, 12, 10)\}$$
$$\{(\propto, 1, 9), (4, 12, 10)\}, \quad \{(\propto, 3, 9), (4, 12, 10)\}.$$

5. A balanced incomplete block design (BIBD) is an arrangement of v^* objects in b^* blocks such that (i) each block contains exactly k^* different objects (ii) each object appears in exactly r^* blocks (iii) any pair of distinct objects appear together in exactly λ^* blocks. The BIB design is then said to have the parameters v^*, b^*, r^*, λ^*, k^*. Suppose the solution of a BIBD with parameters

$$v^*=4t+3, b^*=4t+3, r^*=2t+1, k^*=2t+1, \lambda^*=t. \qquad (2.5.1)$$

From this we can obtain a solution of a BWD with parameters

$$v=4t+4, b=4t+3, r=4t+3, p=2t+2, \lambda_1=2t+1,$$
$$\lambda_2=2t+2 \qquad (2.5.2)$$

in the following manner:

Let S denote the set of the $4t+3$ objects of the BIBD. Then for the objects of the BWD we take the set $\propto \cup S$, i.e., we adjoin new object $\propto$. If B_i is the set of objects in any block of a BIBD, then for the corresponding block of the BWD we take the set

$$\{(\propto \cup B_i), (S-B_i)\} \qquad i=1,2,\ldots,4t+3;$$

divided into two half-blocks as indicated. The $4t+3$ blocks $S-B_i$ form the design complementary to the given BIBD. It has parameters

$$v_\varphi^*=4t+3,\ b_\varphi^*=4t+3,\ r_\varphi^*=2t+2,\ k_\varphi^*=2t+2,$$
$$\lambda_1^*=t+1.$$

Then clearly $\propto$ occurs in $4t+3$ blocks and with each object of S, $r^*=2t+1=\lambda_1$ times in the same half-block, and with $r_1^*=2t+2=\lambda_2$ times in opposite half-blocks of the same block. Also any two elements of S occur in the same half-block $\lambda^*+\lambda_1^*=2t+1=\lambda_1$ times. Again since every block of the BWD contains all treatments exactly once, so every pair occurs once in each block and therefore $4t+3$ times in the whole design. Hence any two treatments occur in opposite half-blocks $4t+3-\lambda_1=2t+2$ times. This proves the required result.

If $4t+3$ is a prime power then the BIBD, with parameters given by (2.5.1) can be obtained [Bose, 3] by developing the initial block

$$(1, x^2, x^4, \ldots, x^{4t})$$

where x is a primitive element of GF$(4t+3)$. Hence a solution of the BWD with parameters (2.5.2) can be obtained by developing the initial block

$$\{(\propto, 1, x^2, \ldots, x^{4t}), (0, x, x^3, \ldots, x^{4t+1})\}.$$

Alternatively the BWD with parameters (2.5.2) can be obtained from a Hadamard matrix H of order $n=4t+4$ i.e., a matrix of order n each of whose elements is $+1$ or -1, and such that $HH^T=nI$. Hadamard matrices of order $n=2$ and $n=4t+4$ are known to exist for all values of $t \leqslant 200$ except for the unknown case, $n=188$ [1, 2, 9, 10]. Also the existence of a Hadamard matrix of order $n=4t+4$ is equivalent to the existence of a BIBD with parameters (2.5.1) [Bose and Shrikhande, 7]. Hence for any value of t for which a Hadamard matrix H of order $n=4t+4$ exists we can get a BWD with parameters given by (2.5.2). We can take $H=(h_{ij})$ in the normalized form in which the elements of the last row are all $+1$. Let the first $4t+3$ rows of H correspond to the blocks and let the columns of H correspond to the objects. Then the ith block of the BWD is obtained from the ith row of H by placing the object j in the first or the second half-block of the ith block, according as $h_{ij}=+1$ or -1.

Example (2.5.1). Let $t=1$. Let the objects be represented by the elements of GF(7) and $\propto$. Then a solution of

$$v=8,\ b=7,\ r=7,\ k=4,\ \lambda_1=3,\ \lambda_2=4$$

is obtained by developing the initial block

$$\{(\propto, 1, 2, 4), (0, 3, 6, 5)\}.$$

Alternatively the design can be obtained from a Hadamard matrix of order 8 by the method explained.

Example (2.5.2). Let $t=2$. Let the objects be represented by the elements of GF(11) and $\propto$. Then a solution of

$$v=12,\ b=11,\ r=11,\ p=6,\ \lambda_1=5,\ \lambda_2=6$$

is obtained by developing the initial block

$$\{(\propto, 1, 4, 5, 9, 3), (0, 2, 8, 10, 7, 6)\}.$$

Alternatively the design can be obtained from the Hadamard matrix of order 12.

6. Let $v=4t+2$, $p=2t+1$. Then $h=4t+2$, $n=1$. The minimal design if it existed would have the parameters

$$v=4t+2,\ b=4t+1,\ r=4t+1,\ p=2t+1,$$
$$\lambda_1=2t,\ \lambda_2=2t+1. \qquad (2.6.1)$$

We shall however show that a solution of (2.6.1) is impossible. Suppose, if possible, the design exists. Then the $8t+2$ half-blocks give a solution of the BIBD with parameters

$$v^*=4t+2,\ b^*=8t+2,\ r^*=4t+1,$$
$$k^*=2t+1,\ \lambda^*=2t. \qquad (2.6.2)$$

Since the two half-blocks of any block of (2.6.1) contain all the $4t+2$ objects, the BIBD (2.6.2) is resolvable in the sense of Bose [4]. Since $b^*=v^*+r^*-1$, the design is affine resolvable. Since $k^{*2}/v^*=(2t+1)/2$ must be integral, we have a contradiction.

We shall now give a construction for the BWD with parameters

$$v=4t+2,\ b=8t+2,\ r=8t+2,\ p=2t+1,$$
$$\lambda_1=4t,\ \lambda_2=4t+2 \qquad (2.6.3)$$

when $4t+1$ is a prime power. The design thus obtained is minimal according to our definition.

Let the objects be represented by the elements of GF$(4t+1)$ and $\propto$. Then a solution of the BWD with parameters (2.6.3) is obtained by developint the initial blocks

$$\{(\propto, 1, x^2, x^4, \ldots, x^{4t-2}), (0, x, x^3, \ldots, x^{4t-1})\}$$
$$\{(0, 1, x^2, x^4, \ldots, x^{4t-2}), (\propto, x, x^3, \ldots, x^{4t-1})\}.$$

The proof follows from the method of differences by noting [Bose, 5] that among the $2t(2t-1)$ mutual differences among

$$1, x^2, x^4, \ldots, x^{4t-2}$$

each nonzero square element (quadratic residue) of

GF$(4t+1)$ occurs $t-1$ times and each nonsquare element (nonquadratic residue) occurs t times.

Example (2.6.1). Let $t=1$. Let the objects be represented by the elements of GF(5) and $\propto$. Then a solution of

$$v=6,\ b=10,\ r=10,\ p=3,\ \lambda_1=4,\ \lambda_2=6$$

is obtained by developing the initial blocks

$$\{(\propto, 1, 4), (0, 2, 3)\},\ \{(0, 1, 4), (\propto, 2, 3)\}.$$

Example (2.6.2). Let $t=2$. Let the objects be represented by the elements of GF(3^2) and $\propto$. Then a solution of

$$v=10,\ b=18,\ r=18,\ p=5,\ \lambda_1=8,\ \lambda_2=10$$

is obtained by developing the initial blocks

$$\{(\propto, 1, x^2, x^4, x^6), (0, x, x^3, x^5, x^7)\}$$

$$\{(0, 1, x^2, x^4, x^6), (\propto, x, x^3, x^5, x^7)\}$$

where x is a primitive element of GF(3^2).

7. We shall next consider the BWD with parameters

$$v=9,\ b=12,\ r=8,\ p=3,\ \lambda_1=2,\ \lambda_2=3. \qquad (2.7.1)$$

Consider the BIBD with parameters

$$v^*=9,\ b^*=12,\ r^*=4,\ k^*=3,\ \lambda^*=1 \qquad (2.7.2)$$

This is a resolvable design (ismorphic with the finite affine plane EG(2, 3)). The blocks can be arranged in 4 sets, each set consisting of 3 blocks containing all the 9 treatments. Each set of blocks corresponds to a parallel pencil of EG(2, 3). The blocks can be written down by taking the rows, columns, and the diagonals of the scheme

1	2	3
4	5	6
7	8	9

Thus the blocks are

Set I	(1, 2, 3), (4, 5, 6), (7, 8, 9)
Set II	(1, 4, 7), (2, 5, 8), (3, 6, 9)
Set III	(1, 6, 8), (2, 4, 9), (3, 5, 7)
Set IV	(1, 5, 9), (2, 6, 7), (3, 4, 8)

Let us obtain the blocks of (2.7.1) by taking for half-blocks of the same block all possible pairs of blocks from the same set. Thus each set gives rise to 4 blocks. We thus get the design

$$\{(4,5,6),(7,8,9)\},\ \{(7,8,9),(1,2,3)\},\ \{(1,2,3),(4,5,6)\}$$

$$\{(2,5,8),(3,6,9)\},\ \{(3,6,9),(1,4,7)\},\ \{(1,4,7),(2,5,8)\}$$

$$\{(2,4,9),(3,5,7)\},\ \{(3,5,7),(1,6,8)\},\ \{(1,6,8),(2,4,9)\}$$

$$\{(2,6,7),(3,4,8)\},\ \{(3,4,8),(1,5,9)\},\ \{(1,5,9),(2,6,7)\}.$$

Since every block of (2.7.2), occurs as a half-block twice it is clear that we have $v=9$, $b=12$, $r=8$, $p=3$, $\lambda_1=2$. Also the design formed by the complete blocks is a BIBD complementary to (2.7.2), and therefore has parameters $v_1^*=9$, $b_1^*=12$, $r_1^*=8$, $k_1^*=6$, $\lambda_1^*=b^*-2r^*+\lambda^*=5$. Hence in the full blocks each pair occurs 5 times. Thus each pair occurs $\lambda_1^*-\lambda_1$ or 3 times in opposite half-blocks.

8. Let $v=10, p=3$. Then $h=(90, 6)=6$, $n=1$. Hence a BWD design with these values of v and p must have the parameters

$$v=10,\ b=15g,\ r=9g,\ p=3,\ \lambda_1=2g,\ \lambda_2=3g. \qquad (2.8.1)$$

If a combinatorial solution for $g=1$ is possible, then this would provide the minimal design. However no such solution is available and the question of its existence is open. We shall however give a solution of (2.8.1) with $g=2$. In this case the parameters are

$$v=10, b=30, r=18, p=3, \lambda_1=2, \lambda_2=3. \qquad (2.8.2)$$

Let the objects be represented by elements of GF(3^2) and $\propto$. We obtain 18 blocks by developing the initial blocks

$$\{(\propto, 1, x^3), (0, x, x^4)\},\ \{(\propto, x, x^4), (0, x^2, x^5)\}. \qquad (2.8.3)$$

Then clearly $\propto$ occurs in each of the 18 blocks, and occurs 4 times with every other treatment in the same half-block and 6 times with every other treatment in opposite half-blocks.

It is easily checked that every non-zero element of GF(3^2) occurs exactly twice among the differences obtained from all pairs formed from elements (other than $\propto$) occurring in the same half-blocks in (2.8.3) and exactly thrice among the differences obtained from all pairs formed from elements (other than $\propto$) occurring in the opposite half-blocks of (2.8.3).

Hence any pair of objects (other than $\propto$) occurs exactly twice in the same half-block, and exactly thrice in opposite half-blocks in the 18 blocks obtained by developing the two initial blocks (2.8.3). Also each object other than $\propto$ occurs exactly 10 times.

The required solution of (2.8.2) is now obtained by adding the 12 blocks of the design

$$v=9, b=12, r=8, p=3, \lambda_1=2, \lambda_2=3.$$

A solution of this has already been given in para. 7, but the objects there were called 1, 2, 3, . . ., 9. We can identify them with the elements of GF(3^2) by making the object i correspond to the element x^{i-1} of GF(3^2) for $i=1, 2, \ldots, 8$; and making the object 9 correspond to the element 0 of GF(3^2).

9. We give below the solutions for a number of minimal designs. In each case the proof depends on the method of differences.

(a) The solution of

$$v=12, b=22, r=11, p=3, \lambda_1=2, \lambda_2=3 \qquad (2.9.1)$$

is obtained by developing the initial blocks

$$(\infty, 1, 4), (5, 9, 3)\}, \{(0, 8, 1), (2, 7, 6)\} \qquad (2.9.2)$$

where the objects correspond to the elements of GF(11) and ∞.

Clearly ∞ occurs 11 times, and occurs twice with every other object in the same half-block, and thrice with every other object in opposite half-blocks.

Again every nonzero element of GF(11) occurs exactly twice among the differences obtained from pairs formed from elements (other than ∞) occurring in the same half-blocks in (2.9.2). This shows that $\lambda_1 = 2$. In the same way we show that $\lambda_2 = 3$.

(b) Consider the design

$$v = 10, b = 45, r = 36, p = 4, \lambda_1 = 12, \lambda_2 = 16.$$

Let the objects be represented by ∞ and the elements of GF(3^2). Then the solution is obtained by developing the initial blocks

$$\{(\infty, 0, 1, x^4), (x, x^3, x^5, x^7)\} \oplus (1, x, x^2, x^3)$$

$$\{(1, x^2, x^4, x^6), (x, x^3, x^5, x^7)\}.$$

(c) Consider the design

$$v = 12, b = 66, r = 55, p = 5, \lambda_1 = 20, \lambda_2 = 25.$$

The objects may be represented by the elements of GF(11) and ∞. A solution of the design is obtained by developing the initial blocks

$$\{(\infty, 2, 6, 7, 8), (1, 4, 5, 9, 3)\} \oplus (1, 2, 3, 4, 5)$$

$$\{(1, 4, 5, 9, 3), (2, 8, 10, 7, 6)\}.$$

(d) The solution of the design

$$v = 12, b = 33, r = 22, p = 4, \lambda_1 = 6, \lambda_2 = 8$$

is obtained by developing the initial blocks

$$\{(\infty, 5, 6, 8), (0, 1, 3, 7)\}$$

$$\{(\infty, 5, 6, 8), (2, 4, 9, 10)\}$$

$$\{(0, 1, 3, 7), (2, 4, 9, 10)\}$$

where as before the objects are represented by ∞ and the elements of GF(11).

3. The Use of Solutions to the Tournament Problem in Calibration

Calibration is the process of assigning to an object a value for its mass, length, angle, electrical resistance, capacitance or some other property by intercomparison with one or more accepted standards. For high precision calibration, these comparisons must be made between nominally equal objects (or groups of objects).

The balanced weighing designs of this paper give groupings into subsets of equal size so that the equality in nominal size is satisfied. The designs are especially appropriate in mass measurement but are equally applicable to other areas where the property being measured is additive without loss of precision of measurement.

The advantage of these designs can be illustrated by an example. If one had nine 1-gram weights, one could form $n(n-1)/2 = 36$ distinct pairings and could make the 36 measurements of the differences in value between elements of the pair. One can achieve the same precision in the estimate of the values (when the sum of all is known) with only 18 measurements by intercomparing subsets of size 2 as shown in design 10 of the appendix; with only 12 measurements using subsets of size 3 as shown in design 11; and with 9 measurements using subsets of size 4 as shown in design 12.

Statistical analysis. The v objects under study have unknown true values $\theta_1, \theta_2, \ldots \theta_v$. In a balanced weighing design one uses two distinct groups of p objects at a time, say $\theta_{i_1}, \theta_{i_2}, \ldots \theta_{i_p}$ and $\theta_{i_{p+1}}, \theta_{i_{p+2}}, \ldots \theta_{i_{2p}}$ and measures the difference between the values for the two groups so that the expected value for an observation is

$$E(y) = (\theta_{i_1} + \theta_{i_2} + \ldots \theta_{i_p}) - (\theta_{i_{p+1}} + \theta_{i_{p+2}} + \ldots \theta_{i_{2p}}).$$

In the complete design, b such observations will be made, each object being used r times.

For design 2 of the appendix, the 5 measurements of the quantities θ_1, θ_2, θ_3, θ_4 and θ_5 have expected values

$$\begin{aligned}
E(y_1) &= \theta_1 + \theta_4 - \theta_2 - \theta_3 \\
E(y_2) &= \theta_2 + \theta_5 - \theta_3 - \theta_4 \\
E(y_3) &= \theta_3 + \theta_1 - \theta_4 - \theta_5 \\
E(y_4) &= \theta_4 + \theta_2 - \theta_5 - \theta_1 \\
E(y_5) &= \theta_5 + \theta_3 - \theta_1 - \theta_2.
\end{aligned}$$

The normal equations will be

$$\begin{aligned}
4\theta_1 - \theta_2 - \theta_3 - \theta_4 - \theta_5 &= y_1 + y_3 - y_4 - y_5 \\
-\theta_1 + 4\theta_2 - \theta_3 - \theta_4 - \theta_5 &= y_2 + y_4 - y_5 - y_1 \\
-\theta_1 - \theta_2 + 4\theta_3 - \theta_4 - \theta_5 &= y_3 + y_5 - y_1 - y_2 \\
-\theta_1 - \theta_2 - \theta_3 + 4\theta_4 - \theta_5 &= y_4 + y_1 - y_2 - y_3 \\
-\theta_1 - \theta_2 - \theta_3 - \theta_4 + 4\theta_5 &= y_5 + y_2 - y_3 - y_4.
\end{aligned}$$

Because only differences are measured, the normal equations will be singular so that a restraint is needed for a unique solution. In calibration work this is provided by one or more standards or values derived from them. Let us denote this restraint by

$$k_1\theta_1 + k_2\theta_2 \ldots k_v\theta_v = m$$

or in matrix notation, by $K'\theta = m$. The normal equations then become [8]

$$\left[\begin{array}{c|c}(r+\beta)I-\beta J & K \\ \hline K' & 0\end{array}\right]\left[\begin{array}{c}\theta \\ \varphi\end{array}\right]=\left[\begin{array}{c}T \\ m\end{array}\right]$$

where $\beta=\lambda_2-\lambda_1$, I is the identity matrix, J is a matrix of all ones, φ is the Lagrangian multiplier entering in the minimization and T is the vector of sums of the observations for each object (the sign of the observation being changed if the object enters negatively in the equation for its expected value). It is worthwhile to discuss two cases in connection with the restraint; one in which the sum of all is given and the other in which the sum of the first t objects is known.

(a) Restraint that the sum of all is given

If the value, m, for the sum of all v objects is given, then the inverse of the matrix of normal equations is (letting $\mathbf{1}$ denote a vector of ones)

$$\left[\begin{array}{c|c}(r+\beta)I-\beta J & \mathbf{1} \\ \hline \mathbf{1}' & 0\end{array}\right]^{-1}=\frac{1}{v^2\beta}\left[\begin{array}{cc}vI-J & v\beta\mathbf{1} \\ v\beta\mathbf{1}' & 0\end{array}\right]$$

and the estimates for the unknowns are

$$\hat{\theta}_i=\frac{T_i}{v\beta}+\frac{m}{v}$$

and for the variance

$$s^2=\frac{1}{b-v+1}\sum \mathrm{dev}^2=\frac{1}{b-v+1}\left\{\sum y^2-\frac{\sum T^2}{v\beta}\right\}.$$

The variances of the estimates are

$$\mathrm{Var}\,(\hat{\theta})=\frac{(v-1)\sigma^2}{v^2\beta}$$

$$\mathrm{Var}\,(\widehat{\theta_i-\theta_j})=\frac{2\sigma^2}{v\beta}$$

(b) Restraint: sum of any number is known.

If the restraint is of the form

$$\sum_1^t \theta_i=m,$$

i.e., that the sum of the first t objects is known, then the inverse of the matrix of normal equations becomes (letting $\emptyset$ represent a vector of zeros)

$$\left[\begin{array}{ccc}(r+\beta)I-\beta J & -\beta J & \mathbf{1} \\ -\beta J & (r+\beta)I-\beta J & \emptyset \\ \mathbf{1} & \emptyset & 0\end{array}\right]^{-1}$$

$$=\frac{1}{tv\beta}\left[\begin{array}{ccc}tI-J & 0 & v\beta\mathbf{1} \\ 0 & tI+J & v\beta\mathbf{1} \\ v\beta\mathbf{1} & v\beta\mathbf{1} & 0\end{array}\right]$$

The estimates now become

$$\hat{\theta}_i=\frac{T_i}{v\beta}-\frac{S}{tv\beta}+\frac{m}{t}$$

where $S=\sum_1^t T_i$.

The variance estimate is the same as before but the variances of the $\hat{\theta}$ become

$$V(\hat{\theta}_i)=\frac{(t-1)\sigma^2}{tv\beta} \qquad i=1,2,\ldots t$$

$$V(\hat{\theta}_i)=\frac{(t+1)\sigma^2}{tv\beta} \qquad i=t+,t+2,\ldots v$$

$$V(\widehat{\theta_i-\theta_j})=\frac{2\sigma^2}{v\beta}.$$

4. Example and Computational Procedures

Weighing devices for large masses characteristically have groups of weights of the same nominal size (e.g., five 2000 Kg wt; ten 10,000 Kg, etc.). A typical configuration is that in use at the Instrument Development Branch, Test Laboratory, Marshall Space Flight Center at Huntsville, Ala., which has in its 25,000 Kg dead weight test machine a group of seven 1000 Kg weights, two of which were actually pairs of 500 Kg test weights which had been independently calibrated in terms of National Bureau of Standards weights. This assigned value for the sum of these weights is taken as the restraint in terms of which the other weights will be determined.

The measurements were made by using a load cell as a comparator so that the "observations" are the values for the differences between two nominally equal masses. For the group of seven 1000 Kg weights the design involving comparisons between pairs of weights was used and the results shown in table 1 were obtained following the order given in Design Number 5 of the appendix.

TABLE 1. *Values of mass difference in calibration of seven 1000 Kg weights.*

Restraint: sum of first two weights $m=-0.0014$ Kg.

Design: Weight No. 1	2	3	4	5	6	7	Observations	Deviations (observed-calculated)
							Kg	*Kg*
+	+	−	−				0.1846	−0.032493
	+	+	−	−			−.0018	.038115
		+	+	−	−		−.0286	−.060285
			+	+	−	−	.1500	−.071522
−				+	+	−	−.0400	.059799
−	−				+	+	−.3451	−.029307
+	−	−				+	−.0016	.013193
	−	+		−	+		−.4471	−.086864
		−	+		−	+	.1700	−.040314
+			−	+		−	.0730	−.006322
−	+			−	+		−.1079	−.023050
	−	+			−	+	−.0612	.055151
+		−	+			−	.2062	.005942
−	+		−	+			−.0038	−.075343
−	+		+		−		.3031	.077993
	−	+		+		−	−.1704	−.032629
−		−	+		+		−.0388	.006135
	−		−	+		+	−.1182	.029836
+		−		−	+		−.0228	−.001536
	+		−		−	+	.1355	−.035842
+		+		−		−	−.0070	.037443
Sum of squares							0.62545461	0.0451268243

Computational procedure. The following steps refer to table 2 and indicate the order of the calculations.

1. Form sums corresponding to each weight, i.e., add or subtract those observations involving the weight depending on the sign given in the design.

$$T_1=\{(.1846)-(-.0400)-(-.3451)+(-.0016)$$
$$+(.0730)-(-.1079)+(.2062)-(-.0038)$$
$$-(-.3031)-(-.0388)+(-.0228)+(-.0070)\}$$
$$=.6649$$

$$T_2=\{(.1846)+(-.0018)-(-.3451)\ \ldots\ \text{etc.}\}=1.6533.$$

These sums are shown in column 2 of table 2 and have a check sum of zero, i.e., $\Sigma T_i=0$.

2. Form the sum, $S=\sum_1^t T_i$, of the t totals involved in the restraint.

In this example $t=2$ and

$$S=T_1+T_2=0.6649+1.6533=2.3182$$

3. Form the differences tT_i-S which will have as a check sum $-vS$ which in this example is $-7(2.3182)=-16.2274$.

4. Divide tT_i-S by $tv\beta$ (in this example $tv\beta=28$).

5. Add the restraint value $\frac{m}{t}$ (in this example $-\frac{0.0014}{2}=-0.007$).

6. The calculated value for each observation is computed by substituting the estimates in the design as illustrated below for the first two observations.

$$(\text{Calculated value})_1=(-0.036)+(0.0346)$$
$$-(-0.170186)-(-0.048307)=0.217093$$

$$(\text{Calculated value})_2=(0.0346)+(0.170186)$$
$$-(-0.048307)-(-0.047364)=-0.039915$$

The deviations are computed from

$$(\text{Deviation})_1=(\text{Observed})_1-(\text{Calculated})_1$$
$$=(0.1846)-(0.21709)=-0.03249$$

$$(\text{Deviation})_2=(\text{Observed})_2-(\text{Calculated})_2$$
$$=(-0.0018)-(-0.03991)=0.03811$$

etc. and are entered in table 1.

7. The standard deviation, s, may be calculated as

$$s=\sqrt{\frac{\Sigma(\text{deviations})^2}{b-v+1}}$$

where b is the number of observations and $b-v+1$ is the number of degrees of freedom, or from

$$s=\sqrt{\frac{1}{b-v+1}\left\{\sum(\text{observations})^2-\sum T_i^2/v\beta\right\}}$$

the former being preferred for machine computation.

8. The standard deviation of the estimates are

$$\text{s.d. (weight in the restraint)}=\sqrt{\frac{t-1}{tv\beta}}\,\sigma=\frac{\sigma}{\sqrt{28}}$$

$$\text{s.d. (other weights)}=\sqrt{\frac{t+1}{tv\beta}}\,\sigma=\frac{\sigma}{\sqrt{28/3}}.$$

The standard deviation for the difference between the two weights in the restraint is $\sqrt{\frac{2}{v\beta}}\,\sigma$ or $\frac{\sigma}{\sqrt{7}}$ for the example.

TABLE 2. *Computational form for analysis of data from balanced weighing design: Design 5 for 7 weights in 21 measurements*

$v=7$ $\quad b=21$ $\quad \beta=2$ $\quad$ d.f. $=b-v+1=15.$

Restraint: Sum of first $t=2$ weights is -0.0014, i.e., $\theta_1+\theta_2=m=-0.0014$

Weight Number i	1 Sums T_i	3 tT_i-S $2T_i-S$	4 $B=\frac{tT_i-S}{tv\beta}$ $B=(2T_i-S)/28$	5 $\hat{\theta}=B+\frac{m}{t}=B-0.0007$
1	0.6649	−0.9884	−0.0353	−0.036
2	1.6533	.9884	.0353	.0346
3	−1.2137	−4.7456	−.169486	−.170186
4	.4926	−1.3330	−.047607	−.048307
5	.5058	−1.3066	−.046664	−.047364
6	−1.6705	−5.6592	−.202114	−.202814
7	−.4324	−3.1830	−.113679	−.114379
Sum	0	−16.2274	−.57955	−.58445
Check	0	$-vS$ $=-7(2.3182)$ $=-16.2274$	$-S/t\beta$ $=-(2.3182)/4$ $=-.57955$	$-S/t\beta+vm/t$ $=-(2.3182)/4+7(-.0014)/2$ $=-.58445$

② $S=T_1+T_2\ .\ .\ .\ T_t=T_1+T_2=0.6949+1.6533=2.3182$

⑥ Deviations are shown in table 1.

Standard deviation:

⑦ $$s^2=\frac{1}{b-v+1}\left\{\sum(\text{obs})^2-\sum T_i^2/v\beta\right\} \qquad \text{d.f.}=b-v+1=15$$

$$s^2=\frac{1}{15}\left\{0.62545461-(8.124589)/14\right\}=0.0451268243/15$$

$$s=0.05485$$

Alternatively

$$s=\sqrt{\frac{1}{15}\sum(\text{deviations})^2}=\sqrt{\frac{0.0451268243}{15}}=0.05485.$$

⑧ Standard deviation of estimates

Weights inside the restraint $\sigma\sqrt{\frac{t-1}{tv\beta}}=\frac{\sigma}{\sqrt{28}}=0.18890\sigma.$

Weights outside the restraint $\sigma\sqrt{\frac{t+1}{tv\beta}}=\frac{\sigma}{\sqrt{28/3}}=0.32733\sigma.$

Standard deviation of difference of two weights

$$\sigma\sqrt{\frac{2}{v\beta}}=\frac{\sigma}{\sqrt{7}}=0.37796\sigma.$$

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APPENDIX: *Balanced Weighing Designs for* $v \leq 13$, $p \leq v/2$

Design Number	v	b	r	p	λ_1	λ_2	$\beta = \lambda_2 - \lambda_1$	d.f.
1	4	3	3	2	1	2	1	0
2	5	5	4	2	1	2	1	1
3	6	15	10	2	2	4	2	10
4	6	10	10	3	4	6	2	5
5	7	21	12	2	2	4	2	15
6	7	7	6	3	2	3	1	1
7	8	14	7	2	1	2	1	7
8	8	28	21	3	6	9	3	21
9	8	7	7	4	3	4	1	0
10	9	18	8	2	1	2	1	10
11	9	12	8	3	2	3	1	4
12	9	9	8	4	3	4	1	1
13	10	45	18	2	2	4	2	36
14	10	30	18	3	2	3	1	21
15	10	45	36	4	12	16	4	36
16	10	18	18	5	8	10	2	9
17	11	55	20	2	2	4	2	45
18	11	55	30	3	6	9	3	45
19	11	55	40	4	12	16	4	45
20	11	11	10	5	4	5	1	1
21	12	33	11	2	1	2	1	22
22	12	22	11	3	2	3	1	11
23	12	33	22	4	6	8	2	22
24	12	66	55	5	20	25	5	55
25	12	11	11	6	5	6	1	0
26	13	39	12	2	1	2	1	27
27	13	26	6	3	2	3	1	14
28	13	39	24	4	6	8	2	27
29	13	78	60	5	20	25	5	66
30	13	13	12	6	5	6	1	1

1. $v=4$ $p=2$ $\lambda_1=1$ d.f.$=0$
$r=3$ $b=3$ $\lambda_2=2$

1 2	3 4
1 3	4 2
1 4	2 3

2. $v=5$ $p=2$ $\lambda_1=1$ d.f.$=1$
$r=4$ $b=5$ $\lambda_2=2$

1 4	2 3
2 5	3 4
3 1	4 5
4 2	5 1
5 3	1 2

3. $v=6$ $p=2$ $\lambda_1=2$ d.f.$=10$
$r=10$ $b=15$ $\lambda_2=4$

1 4 6 2	6 1 2 3	2 3 1 4
2 5 6 3	6 2 3 4	3 4 2 5
3 1 6 4	6 3 4 5	4 5 3 1
4 2 6 5	6 4 5 1	5 1 4 2
5 3 6 1	6 5 1 2	1 2 5 3

4. $v=6$ $p=3$ $\lambda_1=4$ d.f.$=5$
$r=10$ $b=10$ $\lambda_2=6$

6 1 4	5 2 3	5 1 4	6 2 3
6 2 5	1 3 4	1 2 5	6 3 4
6 3 1	2 4 5	2 3 1	6 4 5
6 4 2	3 5 1	3 4 2	6 5 1
6 5 3	4 1 2	4 5 3	6 1 2

5. $v=7$ $p=2$ $\lambda_1=2$ d.f.$=15$
$r=12$ $b=21$ $\lambda_2=4$

1 2 3 4	3 6 2 5	2 4 6 1
2 3 4 5	4 7 3 6	3 5 7 2
3 4 5 6	5 1 4 7	4 6 1 3
4 5 6 7	6 2 5 1	5 7 2 4
5 6 7 1	7 3 6 2	6 1 3 5
6 7 1 2	1 4 7 3	7 2 4 6
7 1 2 3	2 5 1 4	1 3 5 7

APPENDIX: *Balanced Weighing Designs for* $v \leq 13$, $p \leq v/2$—Continued

6. $v=7$ $p=3$ $\lambda_1=2$ d.f.$=1$
$r=6$ $b=7$ $\lambda_2=3$

1 2 4	3 6 5
2 3 5	4 7 6
3 4 6	5 1 7
4 5 7	6 2 1
5 6 1	7 3 2
6 7 2	1 4 3
7 1 3	2 5 4

7. $v=8$ $p=2$ $\lambda_1=1$ d.f.$=7$
$r=7$ $b=14$ $\lambda_2=2$

7 1 4 2	1 4 2 8
1 2 5 3	2 5 3 8
2 3 6 4	3 6 4 8
3 4 7 5	4 7 5 8
4 5 1 6	5 1 6 8
5 6 2 7	6 2 7 8
6 7 3 1	7 3 1 8

8. $v=8$ $p=3$ $\lambda_1=6$ d.f.$=21$
$r=21$ $b=28$ $\lambda_2=9$

8 1 2 3 6 5	8 3 5 1 2 4	1 2 4 8 3 6	1 2 4 3 6 5
8 2 3 4 7 6	8 4 6 2 3 5	2 3 5 8 4 7	2 3 5 4 7 6
8 3 4 5 1 7	8 5 7 3 4 6	3 4 6 8 5 1	3 4 6 5 1 7
8 4 5 6 2 1	8 6 1 4 5 7	4 5 7 8 6 2	4 5 7 6 2 1
8 5 6 7 3 2	8 7 2 5 6 1	5 6 1 8 7 3	5 6 1 7 3 2
8 6 7 1 4 3	8 1 3 6 7 2	6 7 2 8 1 4	6 7 2 1 4 3
8 7 1 2 5 4	8 2 4 7 1 3	7 1 3 8 2 5	7 1 3 2 5 4

9. $v=8$ $p=4$ $\lambda_1=3$ d.f.$=0$
$r=7$ $b=7$ $\lambda_2=4$

1 2 3 4	5 6 7 8
1 2 5 6	3 4 7 8
1 3 5 7	2 4 6 8
1 2 7 8	3 4 5 6
1 3 6 8	2 4 5 7
1 4 5 8	2 3 6 7
1 4 6 7	2 3 5 8

10. $v=9$ $p=2$ $\lambda_1=1$ d.f.$=10$
$r=8$ $b=18$ $\lambda_2=2$

1 5 3 7	2 6 4 8
5 9 4 2	8 3 6 7
8 7 1 4	6 9 5 3
4 6 7 9	1 8 2 5
6 3 2 1	5 7 8 9
9 1 6 8	7 4 3 2
3 4 8 5	9 2 7 1
2 8 9 3	4 5 1 6
7 2 5 6	3 1 9 4

11. $v=9$ $p=3$ $\lambda_1=2$ d.f.$=4$
$r=8$ $b=12$ $\lambda_2=3$

1 2 3	4 5 6
1 2 3	7 8 9
4 5 6	7 8 9
1 4 7	2 5 8
1 4 7	3 6 9
2 5 8	3 6 9
1 5 9	2 6 7
1 5 9	3 4 8
2 6 7	3 4 8
1 6 8	2 4 9
1 6 8	3 5 7
2 4 9	3 5 7

12. $v=9$ $p=4$ $\lambda_1=3$ d.f.$=1$
$r=8$ $b=9$ $\lambda_2=4$

1 3 5 7	2 4 6 8
5 4 9 2	8 6 3 7
8 1 7 4	6 5 9 3
4 7 6 9	1 2 8 5
6 2 3 1	5 8 7 9
9 6 1 8	7 3 4 2
3 8 5 4	9 7 2 1
2 9 8 3	4 1 5 6
7 5 2 6	3 9 1 4

13. $v=10$ $p=2$ $\lambda_1=2$ d.f.$=36$
$r=18$ $b=45$ $\lambda_2=4$

1 5 10 2	2 6 10 1	1 5 2 6
5 9 10 8	8 3 10 5	5 9 8 3
8 7 10 6	6 9 10 8	8 7 6 9
4 6 10 1	1 8 10 4	4 6 1 8
6 3 10 5	5 7 10 6	6 3 5 7
9 1 10 7	7 4 10 9	9 1 7 4
3 4 10 9	9 2 10 3	3 4 9 2
2 8 10 4	4 5 10 2	2 8 4 5
7 2 10 3	3 1 10 7	7 2 3 1

3 7	4 8	3 7	4 8
4 2	6 7	4 2	6 7
1 4	5 3	1 4	5 3
7 9	2 5	7 9	2 5
2 1	8 9	2 1	8 9
6 8	3 2	6 8	3 2
8 5	7 1	8 5	7 1
9 3	1 6	9 3	1 6
5 6	9 4	5 6	9 4

14. $v=10$ $p=3$ $\lambda_1=2$ d.f. $=21$
$r=18$ $b=30$ $\lambda_2=3$

10 1 4	9 2 5	10 2 5	9 3 6	1 4 7	2 5 8
10 5 6	1 8 9	10 8 9	1 4 3	2 5 8	3 6 9
10 8 5	2 6 7	10 6 7	2 1 9	3 6 9	1 4 7
10 4 2	3 1 6	10 1 6	3 7 8	9 1 5	7 2 6
10 6 8	4 5 3	10 5 3	4 2 7	7 2 6	8 3 4
10 9 3	5 7 1	10 7 1	5 6 4	8 3 4	9 1 5
10 3 7	6 9 4	10 9 4	6 8 2	6 8 1	4 9 2
10 2 1	7 4 8	10 4 8	7 9 5	4 9 2	5 7 3
10 7 9	8 3 2	10 3 2	8 5 1	5 7 3	6 8 1
				1 2 3	4 5 6
				4 5 6	7 8 9
				7 8 9	1 2 3

15. $v=10$ $p=4$ $\lambda_1=12$ d.f. $=36$
$r=36$ $b=45$ $\lambda_2=16$

10 9 1 5	2 4 6 8	10 9 2 6	3 5 7 1
10 1 5 9	8 6 3 7	10 1 8 3	4 9 2 5
10 2 8 7	6 5 9 3	10 2 6 9	1 7 4 8
10 3 4 6	1 2 8 5	10 3 1 8	7 6 9 4
10 4 6 3	5 8 7 9	10 4 5 7	2 3 1 6
10 5 9 1	7 3 4 2	10 5 7 4	6 1 8 9
10 6 3 4	9 7 2 1	10 6 9 2	8 4 5 3
10 7 2 8	4 1 5 6	10 7 4 5	9 8 3 2
10 8 7 2	3 9 1 4	10 8 3 1	5 2 6 7

10 9 3 7	4 6 8 2	10 9 4 8	5 7 1 3
10 1 4 2	6 3 7 8	10 1 6 7	9 2 5 4
10 2 1 4	5 9 3 6	10 2 5 3	7 4 8 1
10 3 7 9	2 8 5 1	10 3 2 5	6 9 4 7
10 4 2 1	8 7 9 5	10 4 8 9	3 1 6 2
10 5 6 8	3 4 2 7	10 5 3 2	1 8 9 6
10 6 8 5	7 2 1 9	10 6 7 1	4 5 3 8
10 7 9 3	1 5 6 4	10 7 1 6	8 3 2 9
10 8 5 6	9 1 4 3	10 8 9 4	2 6 7 5

1 3 5 7	2 4 6 8
5 4 9 2	8 6 3 7
8 1 7 4	6 5 9 3
4 7 6 9	1 2 8 5
6 2 3 1	5 8 7 9
9 6 1 8	7 3 4 2
3 8 4 5	9 7 2 1
2 9 8 3	4 1 5 6
7 5 2 6	3 9 1 4

16. $v=10$ $p=5$ $\lambda_1=8$ d.f. $=9$
$r=18$ $b=18$ $\lambda_2=10$

10 1 3 5 7	9 2 4 6 8	9 1 3 5 7	10 2 4 6 8
10 5 4 9 2	1 8 6 3 7	1 5 4 9 2	10 8 6 3 7
10 8 1 7 4	2 6 5 9 3	2 8 1 7 4	10 6 5 9 3
10 4 7 6 9	3 1 2 8 5	3 4 7 6 9	10 1 2 8 5
10 6 2 3 1	4 5 8 7 9	4 6 2 3 1	10 5 8 7 9
10 9 6 1 8	5 7 3 4 2	5 9 6 1 8	10 7 3 4 2
10 3 8 4 5	6 9 7 2 1	6 3 8 4 5	10 9 7 2 1
10 2 9 8 3	7 4 1 5 6	7 2 9 8 3	10 4 1 5 6
10 7 5 2 6	8 3 9 1 4	8 7 5 2 6	10 3 9 1 4

17. $v=11$ $p=2$ $\lambda_1=2$ d.f. $=45$
$r=20$ $b=55$ $\lambda_2=4$

1 2	3 4
2 3	4 5
3 4	5 6
4 5	6 7
5 6	7 8
6 7	8 9
7 8	9 10
8 9	10 11
9 10	11 1
10 11	1 2
11 1	2 3

44 other blocks developed cyclically from the initial blocks

{(2 4)(6 8)}
{(4 8)(1 5)}
{(8 5)(2 10)}
{(5 10)(4 9)}

18. $v=11$ $p=3$ $\lambda_1=6$ d.f. $=45$
$r=30$ $b=55$ $\lambda_2=9$

1 2 3	4 5 6
2 3 4	5 6 7
3 4 5	6 7 8
4 5 6	7 8 9
5 6 7	8 9 10
6 7 8	9 10 11
7 8 9	10 11 1
9 10 11	11 1 2
10 11 1	1 2 3
11 1 2	2 3 4

44 other blocks developed cyclically from the initial blocks

{(2 4 6)(8 10 1)}
{(4 8 1)(5 9 2)}
{(8 5 2)(10 7 4)}
{(5 10 4)(9 3 8)}

APPENDIX: *Balanced Weighing Designs for* $v \leq 13$, $p \leq v/2$ – Continued

19. $v=11$ $p=4$ $\lambda_1=12$ d.f. $=45$
$r=40$ $b=55$ $\lambda_2=16$

1 4 5 10	9 7 3 6
2 5 6 11	10 8 4 7
3 6 7 1	11 9 5 8
4 7 8 2	1 10 6 9
5 8 9 3	2 11 7 10
6 9 10 4	3 1 8 11
7 10 11 5	4 2 9 1
8 11 1 6	5 3 10 2
9 1 2 7	6 4 11 3
10 2 3 8	7 5 1 4
11 3 4 9	8 6 2 5

44 other blocks developed cyclically from the initial blocks
{(2 8 10 9)(7 3 6 1)}
{(4 5 9 7)(3 6 1 2)}
{(8 10 7 3)(6 1 2 4)}
{(5 9 3 6)(1 2 4 8)}

20. $v=11$ $p=5$ $\lambda_1=4$ d.f. $=1$
$r=10$ $b=11$ $\lambda_2=5$

1 4 5 9 3	2 8 10 7 6
2 5 6 10 4	3 9 11 8 7
3 6 7 11 5	4 10 1 9 8
4 7 8 1 6	5 11 2 10 9
5 8 9 2 7	6 1 3 11 10
6 9 10 3 8	7 2 4 1 11
7 10 11 4 9	8 3 5 2 1
8 11 1 5 10	9 4 6 3 2
9 1 2 6 11	10 5 7 4 3
10 2 3 7 1	11 6 8 5 4
11 3 4 8 2	1 7 9 6 5

21. $v=12$ $p=2$ $\lambda_1=1$ d.f. $=22$
$r=11$ $b=33$ $\lambda_2=2$

11 1	3 7	4 9	2 10	12 5	6 8
1 2	4 8	5 10	3 11	12 6	7 9
2 3	5 9	6 11	4 1	12 7	8 10
3 4	6 10	7 1	5 2	12 8	9 11
4 5	7 11	8 2	6 3	12 9	10 1
5 6	8 1	9 3	7 4	12 10	11 2
6 7	9 2	10 4	8 5	12 11	1 3
7 8	10 3	11 5	9 6	12 1	2 4
8 9	11 4	1 6	10 7	12 2	3 5
9 10	1 5	2 7	11 8	12 3	4 6
10 11	2 6	3 8	1 9	12 4	5 7

22. $v=12$ $p=3$ $\lambda_1=2$ d.f. $=11$
$r=11$ $b=22$ $\lambda_2=3$

12 2 5	6 10 4	1 9 11	3 8 7
12 3 6	7 11 5	2 10 1	4 9 8
12 4 7	8 1 6	3 11 2	5 10 9
12 5 8	9 2 7	4 1 3	6 11 10
12 6 9	10 3 8	5 2 4	7 1 11
12 7 10	11 4 9	6 3 5	8 2 1
12 8 11	1 5 10	7 4 6	9 3 2
12 9 1	2 6 11	8 5 7	10 4 3
12 10 2	3 7 1	9 6 8	11 5 4
12 11 3	4 8 2	10 7 9	1 6 5
12 1 4	5 9 3	11 8 10	2 7 6

23. $v=12$ $p=4$ $\lambda_1=6$ d.f. $=22$
$r=22$ $b=33$ $\lambda_2=8$

12 5 6 8	11 1 3 7	12 5 6 8	2 4 9 10	11 1 3 7	2 4 9 10
12 6 7 9	1 2 4 8	12 6 7 9	3 5 10 11	1 2 4 8	3 5 10 11
12 7 8 10	2 3 5 9	12 7 8 10	4 6 11 1	2 3 5 9	4 6 11 1
12 8 9 11	3 4 6 10	12 8 9 11	5 7 1 2	3 4 6 10	5 7 1 2
12 9 10 1	4 5 7 11	12 9 10 1	6 8 2 3	4 5 7 11	6 8 2 3
12 10 11 2	5 6 8 1	12 10 11 2	7 9 3 4	5 6 8 1	7 9 3 4
12 11 1 3	6 7 9 2	12 11 1 3	8 10 4 5	6 7 9 2	8 10 4 5
12 1 2 4	7 8 10 3	12 1 2 4	9 11 5 6	7 8 10 3	9 11 5 6
12 2 3 5	8 9 11 4	12 2 3 5	10 1 6 7	8 9 11 4	10 1 6 7
12 3 4 6	9 10 1 5	12 3 4 6	11 2 7 8	9 10 1 5	11 2 7 8
12 4 5 7	10 11 2 6	12 4 5 7	1 3 8 9	10 11 2 6	1 3 8 9

24. $v=12$ $p=5$ $\lambda_1=20$ d.f. $=55$
$r=55$ $b=66$ $\lambda_2=25$

Develop the following initial blocks (mod 11)

[(∝ 2 6 7 8), (1 4 5 9 3)]
[(∝ 4 1 3 5), (2 8 10 7 6)]
[(∝ 6 7 10 2), (3 1 4 5 9)]
[(∝ 8 2 6 10), (4 5 8 3 1)]
[(∝ 10 8 2 7), (5 9 3 1 3)]
[(1 4 5 9 3), (2 8 10 7 6)]

and replace the symbol ∝ by 12.

25. $v=12$ $p=6$ $\lambda_1=5$ d.f. $=0$
$r=11$ $b=11$ $\lambda_2=6$

1	3	4	5	9	12	2	6	7	8	10	11
2	4	5	6	10	12	3	7	8	9	11	1
3	5	6	7	11	12	4	8	9	10	1	2
4	6	7	8	1	12	5	9	10	11	2	3
5	7	8	9	2	12	6	10	11	1	3	4
6	8	9	10	3	12	7	11	1	2	4	5
7	9	10	11	4	12	8	1	2	3	5	6
8	10	11	1	5	12	9	2	3	4	6	7
9	11	1	2	6	12	10	3	4	5	7	8
10	1	2	3	7	12	11	4	5	6	8	9
11	2	3	4	8	12	1	5	6	7	9	10

26. $v=13$ $p=2$ $\lambda_1=1$ d.f. $=27$
$r=12$ $b=39$ $\lambda_2=2$

1	12	8	5	2	11	3	10	4	9	6	7
2	13	9	6	3	12	4	11	5	10	7	8
3	1	10	7	4	13	5	12	6	11	8	9
4	2	11	8	5	1	6	13	7	12	9	10
5	3	12	9	6	2	7	1	8	13	10	11
6	4	13	10	7	3	8	2	9	1	11	12
7	5	1	11	8	4	9	3	10	2	12	13
8	6	2	12	9	5	10	4	11	3	13	1
9	7	3	13	10	6	11	5	12	4	1	2
10	8	4	1	11	7	12	6	13	5	2	3
11	9	5	2	12	8	13	7	1	6	3	4
12	10	6	3	13	9	1	8	2	7	4	5
13	11	7	4	1	10	2	9	3	8	5	6

27. $v=13$ $p=3$ $\lambda_1=2$ d.f. $=14$
$r=6$ $b=26$ $\lambda_2=3$

1	3	9	4	12	10	2	6	5	8	11	7
2	4	10	5	13	11	3	7	6	9	12	8
3	5	11	6	1	12	4	8	7	10	13	9
4	6	12	7	2	13	5	9	8	11	1	10
5	7	13	8	3	1	6	10	9	12	2	11
6	8	1	9	4	2	7	11	10	13	3	12
7	9	2	10	5	3	8	12	11	1	4	13
8	10	3	11	6	4	9	13	12	2	5	1
9	11	4	12	7	5	10	1	13	3	6	2
10	12	5	13	8	6	11	2	1	4	7	3
11	13	6	1	9	7	12	3	2	5	8	4
12	1	7	2	10	8	13	4	3	6	9	5
13	2	8	3	11	9	1	5	4	7	10	6

APPENDIX: *Balanced Weighing Designs for* $v \leq 13$, $p \leq v/2$—Continued

28. $v=13$ $p=4$ $\lambda_1=6$ d.f. $=27$
$r=24$ $b=39$ $\lambda_2=8$

1	8	12	5	4	6	9	7	2	3	11	10	8	12	5	1	4	6	9	7	3	11	10	2
2	9	13	6	5	7	10	8	3	4	12	11	9	13	6	2	5	7	10	8	4	12	11	3
3	10	1	7	6	8	11	9	4	5	13	12	10	1	7	3	6	8	11	9	5	13	12	4
4	11	2	8	7	9	12	10	5	6	1	13	11	2	8	4	7	9	12	10	6	1	13	5
5	12	3	9	8	10	13	11	6	7	2	1	12	3	9	5	8	10	13	11	7	2	1	6
6	13	4	10	9	11	1	12	7	8	3	2	13	4	10	6	9	11	1	12	8	3	2	7
7	1	5	11	10	12	2	13	8	9	4	3	1	5	11	7	10	12	2	13	9	4	3	8
8	2	6	12	11	13	3	1	9	10	5	4	2	6	12	8	11	13	3	1	10	5	4	9
9	3	7	13	12	1	4	2	10	11	6	5	3	7	13	9	12	1	4	2	11	6	5	10
10	4	8	1	13	2	5	3	11	12	7	6	4	8	1	10	13	2	5	3	12	7	6	11
11	5	9	2	1	3	6	4	12	13	8	7	5	9	2	11	1	3	6	4	13	8	7	12
12	6	10	3	2	4	7	5	13	1	9	8	6	10	3	12	2	4	7	5	1	9	8	13
13	7	11	4	3	5	8	6	1	2	10	9	7	11	4	13	3	5	8	6	2	10	9	1

29. $v=13$ $p=5$ $\lambda_1=20$ d.f. $=66$
$r=60$ $b=78$ $\lambda_2=25$

The initial blocks to be developed cyclically are

{(1 2 3 4 5)(6 7 8 9 10)}
{(2 4 6 8 10)(12 1 3 5 7)}
{(4 8 12 3 7)(11 2 6 10 1)}
{(8 3 11 6 1)(9 4 12 7 2)}
{(3 6 9 12 2)(5 8 11 1 4)}
{(6 12 5 11 4)(10 3 9 2 8)}

30. $v=13$ $p=6$ $\lambda_1=5$ d.f. $=1$
$r=12$ $b=13$ $\lambda_2=6$

1	4	3	12	9	10	2	8	6	11	5	7
2	5	4	13	10	11	3	9	7	12	6	8
3	6	5	1	11	12	4	10	8	13	7	9
4	7	6	2	12	13	5	11	9	1	8	10
5	8	7	3	13	1	6	12	10	2	9	11
6	9	8	4	1	2	7	13	11	3	10	12
7	10	9	5	2	3	8	1	12	4	11	13
8	11	10	6	3	4	9	2	13	5	12	1
9	12	11	7	4	5	10	3	1	6	13	2
10	13	12	8	5	6	11	4	2	7	1	3
11	1	13	9	6	7	12	5	3	8	2	4
12	2	1	10	7	8	13	6	4	9	3	5
13	3	2	11	8	9	1	7	5	10	4	6

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