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E–Transforms (II) *

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The following class of integral transform pairs is established

$$g(x)=\int_0^\infty E\left(\begin{matrix}\nu-ix, \nu+ix, \alpha_1, \ldots, \alpha_p : \frac{1}{y}\\ \beta_1, \ldots, \beta_q\end{matrix}\right) f(y)dy, \tag{1}$$

$$f(x)=\frac{x^{\nu-1}}{i\pi^2}\int_0^\infty yg(y)\left[\frac{1}{i}\sum_{i,-i} ix^{iy}\sin(iy+\nu)\pi E\left(\begin{matrix}1-\nu-iy, \beta_1-\nu-iy, \ldots, \beta_q-\nu-iy : x\\ 1-2iy, \alpha_1-\nu-iy, \ldots, \alpha_p-\nu-iy\end{matrix}\right)\right]dy. \tag{2}$$

The kernel in the transform (1) is MacRobert's E-function and integration is performed with respect to the argument of this function. In the inversion formula (2), the kernel is likewise an E-function, but the integration is performed with respect to its parameters.

Known special cases of this general transform pair is the Kantorovich-Lebedev transforms pair:

$$g(x)=\frac{2}{\pi^2}x\sinh(\pi x)\int_0^\infty y^{-1}K_{ix}(y)f(y)dy,$$

$$f(x)=\int_0^\infty K_{iy}(x)g(y)dy,$$

and the generalized Mehler transform pair

$$g(x)=\frac{x}{\pi}\sinh(\pi x)\Gamma\left(\frac{1}{2}-k+ix\right)\Gamma\left(\frac{1}{2}-k-ix\right)\int_0^\infty P^k_{ix-1/2}(y)f(y)dy,$$

$$f(x)=\int_0^\infty P^k_{iy-1/2}(x)g(y)dy.$$

Key Words: E-functions, integral transforms, inversion formulas, kernels.

1. Introduction

In this paper we establish the following class of integral transforms:

$$g(x)=\int_0^\infty E\left(\begin{matrix}\nu-ix, \nu+ix, \alpha_1, \ldots, \alpha_p : \frac{1}{y}\\ \beta_1, \ldots, \beta_q\end{matrix}\right) f(y)dy \tag{1}$$

$$f(x)=\frac{x^{\nu-1}}{i\pi^2}\int_0^\infty yg(y)\left[\frac{1}{i}\sum_{i,-i} ix^{iy}\sin(iy+\nu)\pi E\left(\begin{matrix}1-\nu-iy, \beta_1-\nu-iy, \ldots, \beta_q-\nu-iy : x\\ 1-2iy, \alpha_1-\nu-iy, \ldots, \alpha_p-\nu-iy\end{matrix}\right)\right]dy \tag{2}$$

where the integrals are convergent and the symbol $\sum_{i,-i}$ means that in the expression following it i is to be replaced by $-i$ and the two expressions are to be added.

The kernel in the transform (1) is MacRobert's E-function whose definitions and properties

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are given in [1] [1] pp. 348–358, and which will be discussed further in section 2. The integration in this transform is performed with respect to the argument of the E-function. In the inversion formula (2), the kernel is likewise an E-function, but the integration is performed with respect to its parameters. Known special cases of our general transform pair are the Kantorovich-Lebedev transform pair (see [2], pp. 175–177: [3], pp. 229–241 and [4], pp. 33–40)

$$g(x)=\frac{2}{\pi^2}x\sinh\pi x\int_0^\infty y^{-1}K_{ix}(y)f(y)\,dy, \tag{3}$$

$$f(x)=\int_0^\infty K_{iy}(x)g(y)\,dy; \tag{4}$$

and the generalized Mehler transform pair (see [5] pp. 57–59 and [6]).

$$g(x)=\frac{x}{\pi}\sinh\pi x\Gamma\left(\frac{1}{2}-k+ix\right)\Gamma\left(\frac{1}{2}-k-ix\right)\int_1^\infty P^k_{ix-\frac{1}{2}}(y)f(y)\,dy \tag{5}$$

$$f(x)=\int_0^\infty P^k_{iy-\frac{1}{2}}(x)g(y)\,dy. \tag{6}$$

Section 2 contains a treatment of the E-function and our main transform pair is derived in section 3. Section 4 contains the derivation of the Kantorovich-Lebedev and Mehler transforms and other new integral transforms.

The Mellin transform ([7], p. 7)

$$g(s)=\int_0^\infty x^{s-1}f(x)\,dy \tag{7}$$

and its inversion formula

$$f(x)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+\infty} x^{-s}g(s)\,ds \tag{8}$$

will be utilized in the proofs.

Also the following formulas are required in the proofs: ([1], p. 374):

$$E(p\colon \alpha_r : q\colon \rho_t : z)=\frac{1}{2\pi i}\int\frac{\Gamma(\zeta)\prod_{r=1}^{p}\Gamma(\alpha_r-\zeta)}{\prod_{t=1}^{q}\Gamma(\rho_t-\zeta)}z^{\zeta}\,d\zeta, \tag{9}$$

where $|\mathrm{amp}\ z|<\frac{1}{2}(p-q+1)\pi$ and the contour of integration is of Barnes's type with loops, if necessary, to separate the pole at the origin from the poles at $\alpha_1, \alpha_2, \ldots, \alpha_p$:
([1], p. 257):

$$K_n(z)=\frac{\pi}{2\sin n\pi}\{I_{-n}(z)-I_n(z)\}; \tag{10}$$

[1], p. 347:

$$I_n(z)=\frac{1}{\Gamma(n+1)}\left(\frac{1}{2}z\right)^n e^{-z}F\left(\begin{matrix}n+\frac{1}{2};\ 2z\\ 2n+1\end{matrix}\right); \tag{11}$$

[1] Figures in brackets indicate the literature references at the end of this paper.

and [1], p. 262:

$$F\begin{pmatrix}-n, n+1: -z\\ 1-m\end{pmatrix}=\Gamma(1-m)\left(\frac{z}{1+z}\right)^{\frac{1}{2}m}P_n^m(2z+1). \tag{12}$$

2. Properties of the E-Function

If $p \leqslant q$ then the E-function is defined as

$$E(p;\ \alpha_r : q;\ \rho_t : z)=\frac{\Gamma(\alpha_1)\ .\ .\ .\ \Gamma(\alpha_p)}{\Gamma(\rho_1)\ .\ .\ .\ \Gamma(\rho_q)}\,F\left(p;\ \alpha_r : q;\ \rho_t : -\frac{1}{z}\right). \tag{13}$$

When $p \geqslant q+1$, $|\arg z| < \pi$, then the E-function (see [1], p. 353) can be shown to be

$$E(p;\ \alpha_r : q;\ \rho_t : z)=\sum_{r=1}^{p}\prod_{r=1}^{p\ \prime}\Gamma(\alpha_s-\alpha_r)\left\{\prod_{t=1}^{q}\Gamma(\rho_t-\alpha_r)\right\}^{-1}\Gamma(\alpha_r)Z^{\alpha_r}$$

$$\times\, {}_{q+1}F_{p-1}\begin{pmatrix}\alpha_r,\ \alpha_r-\rho_1+1,\ .\ .\ .,\ \alpha_r-p_q+1:\ (-1)^{p-q}z\\ \alpha_r-\alpha_1+1,\ .\ .\ .\ *\ .\ .\ .,\ \alpha_r-\alpha_p+1\end{pmatrix}, \tag{14}$$

where the asterisk means that the factor $\alpha_r-\alpha_r+1$ is omitted. To familarize ourselves with the E-function, the following relations may be worth noting: From the definition (13) it is clear that the E-function is immediately related to the generalized hypergeometric function

$${}_pF_q\begin{pmatrix}\alpha_r : z\\ \rho_t\end{pmatrix}=\sum_{n=0}^{\infty}\frac{(\alpha_1;\ n)\ .\ .\ .\ (\alpha_p;\ n)}{(1;\ n)(\rho_1;\ n)\ .\ .\ .\ (\rho_q;\ n)}\,z^n,$$

and reduces to single expressions in the ordinary or Gauss hypergeometric function when $p=2$, $q=1$. For $p=1$, $q=1$ it is evident that the E-function reduces to the confluent hypergeometric function or Kummer's function. The case $p=1$, $q=0$ yields the relation

$$E(\alpha :: z)=\Gamma(\alpha)(1+1/z)^{-\alpha}. \tag{15}$$

The case $p=0$, $q=1$ gives the relation

$$E(:\ \nu+1 :: z)=z^{1/\frac{\nu}{2}}I_\nu(2z^{-1/2}). \tag{16}$$

The case $p=2$, $q=0$ yields the relations (see [1], p. 351)

$$\cos n\pi E\left(\frac{1}{2}+n,\ \frac{1}{2}-n :: 2z\right)=(2\pi z)^{\frac{1}{2}z}e^zK_n(z), \tag{17}$$

$$E\left(\frac{1}{2}-k+m,\ \frac{1}{2}-k-m :: z\right)=\Gamma\left(\frac{1}{2}-k-m\right)\Gamma\left(\frac{1}{2}-k+m\right)z^{-k}e^{\frac{1}{2}z}W_{k,m}(z), \tag{18}$$

where $K_n(z)$ and $W_{k,m}(z)$ are the modified Bessel function and Whittaker function respectively. Also it is evident from the definitions of the E-function that for $p=q=0$

$$E(:: z)=\exp\ (-1/z). \tag{19}$$

More complicated parameters in the E-function lead to the equivalence of the E-function with products of Whittaker functions, Hankel functions, Lommel functions and other special functions. Some examples are

$$W_{k,m}(2iz)W_{k,m}(-2iz)=\pi^{-1/2}\left(\frac{z}{2}\right)^{2k}\left\{\Gamma\left(\frac{1}{2}-k+m\right)\Gamma\left(\frac{1}{2}-k-m\right)\right\}^{-1}$$

$$\times E\left(\frac{1}{2}-k+m,\ \frac{1}{2}-k-m,\ \frac{1}{2}-k,\ 1-k : 1-2k : \frac{z^2}{4}\right), \tag{20}$$

$$e^{-\frac{1}{2}z}W_{k,m}(z)=\frac{1}{2\pi}\sum_{i,-i}\frac{1}{i}E\left(\frac{1}{2}+m,\frac{1}{2}-m,1:1-k:e^{i\pi}z\right), \tag{21}$$

$$H_\nu^{(1)}(z)H_\nu^{(2)}(z)=2\pi^{-5/2}\cos(\nu\pi)z^{-1}E\left(\frac{1}{2}+\nu,\frac{1}{2}-\nu,\frac{1}{2}::z^2\right), \tag{22}$$

$$J_\nu(z)J_{-\nu}(z)=\{\Gamma(1-\nu)\Gamma(1+\nu)\}^{-1}{}_1F_2\left(\frac{1}{2};1-\nu,1+\nu;-z^2\right), \tag{23}$$

$$J_\nu^2(z)=\pi^{-1/2}z^{2\nu}E\left(\frac{1}{2}+\nu:1+\nu,1+2\nu:\frac{1}{z^2}\right), \tag{24}$$

$$S_{\mu,\nu}(z)=2^{\mu-1}\left\{\Gamma\left(\frac{1}{2}-\frac{1}{2}\mu-\frac{1}{2}\nu\right)\Gamma\left(\frac{1}{2}-\frac{1}{2}\mu+\frac{1}{2}\nu\right)\right\}^{-1}\left(\frac{z}{2}\right)^{\mu-1}$$

$$\times E\left(1,\frac{1}{2}-\frac{1}{2}\mu+\frac{1}{2}\nu,\frac{1}{2}-\frac{1}{2}\mu-\frac{1}{2}\nu::\frac{1}{4}z^2\right), \tag{25}$$

$$M_{k,m}(iz)M_{k,m}(-iz)=z^{2m+1}{}_2F_3\begin{pmatrix}\frac{1}{2}+m+k,\frac{1}{2}\quad,m,k;-\frac{z^2}{4}\\ \frac{1}{2}+m,1+m,1+2m\end{pmatrix}, \tag{26}$$

$$z^{2\mu}K_{2\nu}(ze^{\frac{\pi i}{4}})K_{2\nu}(ze^{-\pi i/4})=2^{3\mu-4}\pi^{-3/2}$$

$$\times\sum_{i,-i}\frac{1}{i}E\left(\frac{1}{2}\mu+\nu,\frac{1}{2}\mu-\nu,\frac{1}{2}\mu,\frac{1}{2}\mu+\frac{1}{2},1::e^{i\pi}\frac{z^4}{64}\right), \tag{27}$$

$$z^\mu K_\nu^2(z)=2^{-2}\pi^{1/2}\sum_{i,-i}\frac{1}{i}E\left(\nu+\frac{1}{2}\mu,-\nu+\frac{1}{2}\mu,\frac{1}{2}\mu,1:\frac{1}{2}\mu+\frac{1}{2}:e^{i\pi}z^2\right), \tag{28}$$

$$z^\mu K_\nu(z)=2^{\mu-2}\pi^{-1}\sum_{i,-i}\frac{1}{i}E\left(\frac{1}{2}\mu+\frac{1}{2}\nu,\frac{1}{2}\mu-\frac{1}{2}\nu,1:e^{i\pi}\frac{z^2}{4}\right), \tag{29}$$

$$[e^{-\frac{\pi i}{4}}H_{a-b}^{(1)}(z^{1/2})H_{a+b}^{(2)}(z^{1/2})+e^{\frac{\pi i}{4}}H_{a+b}^{(1)}(z^{1/2})H_{a-b}^{(2)}(z^{1/2})]$$

$$=4\pi^{-5/2}\cos(a\pi)\cos(b\pi)z^{-1/2}E(a+\tfrac{1}{2},b+\tfrac{1}{2},-a+\tfrac{1}{2},-b+\tfrac{1}{2}:\tfrac{1}{2}:z), \tag{30}$$

$$M_{k,m}(iz)M_{k,m}(-iz)=z^{2m+1}{}_2F_3\begin{pmatrix}\frac{1}{2}+m+k,\frac{1}{2}+m-k;-\frac{z^2}{4}\\ \frac{1}{2}+m,1+m,1+2m\end{pmatrix}, \tag{31}$$

$$e^{-2m\pi i}\{\Gamma(1+m+n)\Gamma(m-n)\}^{-1}Q_n^m[(1+z^2)^{1/2}]Q_{-n-1}^m[(1+z^2)^{1/2}]$$

$$=\frac{\pi}{2z}\left\{\Gamma\left(\frac{1}{2}-n\right)\Gamma\left(\frac{3}{2}+n\right)\right\}^{-1}{}_3F_2\begin{pmatrix}\frac{1}{2}-m,\frac{1}{2}+m,\frac{1}{2};-\frac{1}{z^2}\\ \frac{1}{2}-n,\frac{3}{2}+n\end{pmatrix}, \tag{32}$$

$$H_\nu(z)-Y_\nu(z)=\pi^{-\nu-1}\cos\nu\pi\, z^{\nu-1}E\left(\frac{1}{2},1,\frac{1}{2}-\nu::\frac{z^2}{4}\right), \tag{33}$$

$$H_{b-a}^{(1)}(z^{1/2})H_{b-a}^{(2)}(z^{1/2})=2\pi^{-5/2}\cos(b-a)\pi z^{-1/2}E(\tfrac{1}{2},\tfrac{1}{2}+b-a,\tfrac{1}{2}+a-b::z), \tag{34}$$

$$K_{2(b-a)}(2^{3/2}z^{1/4}e^{\pi i/4})K_{2(b-a)}(2^{3/2}z^{1/4}e^{-\pi i/4})=2^{-4}\pi^{-\frac{3}{2}}z^{-q}$$

$$\times\sum_{i,\,-i}\frac{1}{i}E\left(1,\,a,\,b,\,a+\frac{1}{2},\,2a-b::ze^{i\pi}\right). \tag{35}$$

3. Proof of the Main Theorem

From (7), (8), and (9), we arrive at the formula

$$\int_0^\infty z^{s-1}E(p\colon\alpha_r:q\colon\rho_t\colon z)dz=\frac{\Gamma(-s)\prod_{r=1}^{p}\Gamma(\alpha_r+s)}{\prod_{t=1}^{q}\Gamma(\rho_t+s)}, \tag{36}$$

where $R(\alpha_r+s)>0(r=1,2,\ldots,p)$ and $R(s)<0$.

We wish to solve the integral equation (1)

$$g(x)=\int_0^\infty E\begin{pmatrix}\nu-ix,\,\nu+ix,\,\alpha_1,\,\alpha_2,\,\ldots,\,\alpha_p:\dfrac{1}{y}\\ \beta_1,\,\beta_2,\,\ldots,\,\beta_q\end{pmatrix}f(y)d(y) \tag{37}$$

where the conditions on the parameters and the function $f(y)$ are such that the integral is convergent.

Replace x by $-i\xi$. Multiply both sides of (37) by $x^{-\xi}d\xi$ and integrate from $c-i\infty$ to $c+i\infty$, so getting

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}x^{-\xi}g(-i\xi)d\xi=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}x^{-\xi}\int_0^\infty E\begin{pmatrix}\nu-\xi,\,\nu+\xi,\,\alpha_1,\,\ldots,\,\alpha_p:\dfrac{1}{y}\\ \beta_1,\,\ldots,\,\beta_q\end{pmatrix}f(y)dyd\xi$$

$$=\frac{-1}{4\pi^2}\int_0^\infty f(y)dy\int_{c-i\infty}^{c+i\infty}x^{-\xi}d\xi\int_{\gamma-i\infty}^{\gamma+i\infty}\Gamma(\zeta)\Gamma(\nu+\xi-\zeta)\Gamma(\nu-\xi-\zeta)\frac{\prod_{r=1}^{p}\Gamma(\alpha_r-\zeta)}{\prod_{t=1}^{q}\Gamma(\beta_t-\zeta)}y^{-\zeta}d\zeta,$$

by (9).

Now change the order of integration, so that the integral with respect to ξ becomes the last and the last expression becomes

$$-\frac{1}{4\pi^2}\int_0^\infty f(y)dy\int_{\gamma-i\infty}^{\gamma+i\infty}\Gamma(\zeta)\frac{\prod_{r=1}^{p}\Gamma(\alpha_r-\zeta)}{\prod_{t=1}^{q}\Gamma(\beta_t-\zeta)}y^{-\zeta}d\zeta\int_{c-i\infty}^{c+i\infty}\Gamma(\nu-\zeta-\xi)\Gamma(\nu-\zeta+\xi)x^{-\xi}d\xi$$

$$=-\frac{1}{4\pi^2}\int_0^\infty f(y)dy\int_{\gamma-i\infty}^{\gamma+i\infty}\Gamma(\zeta)\frac{\prod_{r=1}^{p}\Gamma(\alpha_r-\zeta)}{\prod_{t=1}^{q}\Gamma(\beta_t-\zeta)}y^{-\zeta}x^{\nu-\zeta}d\zeta\times\int_{c-i\infty}^{c+i\infty}\Gamma(\xi)\Gamma(2\nu-2\zeta-\xi)x^{-\xi}d\xi$$

$$=-\frac{1}{4\pi^2}\int_0^\infty f(y)dy\int_{\gamma-i\infty}^{\gamma+i\infty}\Gamma(\zeta)\frac{\prod_{r=1}^{p}\Gamma(\alpha_r-\zeta)}{\prod_{t=1}^{q}\Gamma(\beta_t-\zeta)}x^{\nu}(xy)^{-\zeta}d\zeta E\left(2\nu-2\zeta:\frac{1}{x}\right),$$

by (9) again.

Now apply (15) and the last expression becomes

$$\frac{1}{2\pi i}\int_0^\infty f(y)dy\int_{\gamma-i\infty}^{\gamma+i\infty}\Gamma(\zeta)\Gamma(\nu-\zeta)\Gamma\left(\nu+\frac{1}{2}-\zeta\right)2^{2\nu-2\zeta-1}\frac{\prod_{r=1}^{p}\Gamma(\alpha_r-\zeta)}{\pi^{1/2}\prod_{t=1}^{q}\Gamma(\beta_t-\zeta)}x^\nu(1+x)^{-2\nu+2\zeta}(xy)^{-\zeta}d\zeta$$

$$=\frac{1}{2(\pi)^{1/2}}\left(\frac{4x}{(1+x)^2}\right)^\nu\int_0^\infty E\begin{pmatrix}\nu,\ \nu+\frac{1}{2},\ \alpha_1,\ .\ .\ .,\ \alpha_p:\dfrac{(1+x)^2}{4xy}\\ \beta_1,\ .\ .\ .,\ \beta_q\end{pmatrix}f(y)dy,$$

by (9).

Now let $\bar{x}=\dfrac{4x}{(1+x)^2}$ and

$$\bar{g}(x)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}x^{+\xi}g(-i\xi)d\xi, \tag{38}$$

and the last expression becomes

$$2(\pi)^{1/2}\bar{x}^{-\nu}\bar{g}(x)=\int_0^\infty E\begin{pmatrix}\nu,\ \nu+\frac{1}{2},\ \alpha_1,\ .\ .\ .,\ \alpha_p:\dfrac{1}{xy}\\ \beta_1,\ .\ .\ .,\ \beta_q\end{pmatrix}f(y)dy \tag{39}$$

Using the notation of ([7], p. 315), we have

$$\mathscr{G}(s)=2(\pi)^{1/2}\int_0^\infty \bar{x}^{s-\nu-1}\bar{g}(x)d\bar{x}=2(\pi)^{1/2}4^{s-\nu}\int_0^{-1}(1-x)x^{s-\nu-1}(1+x)^{2\nu-2s-1}\bar{g}(x)dx. \tag{40}$$

Here write $1-s$ for s, apply (38) and get

$$\mathscr{G}(1-s)=\frac{4^{1-s-\nu}}{i(\pi)^{1/2}}\int_{c-i\infty}^{c+i\infty}g(-i\xi)d\xi\int_0^{-1}(1-x)x^{-s-\nu-\xi}(1+x)^{2s+2\nu-3}dx$$

$$=\frac{e^{i\pi(-s-\nu)}}{i(\pi)^{1/2}}\frac{\Gamma(2s+2\nu-2)}{4^{s+\nu-1}}\int_{c-i\infty}^{c+i\infty}-g(-i\xi)\left[\frac{\Gamma(1-s-\nu-\xi)}{\Gamma(s+\nu-\xi-1)}+\frac{\Gamma(2-s-\nu-\xi)}{\Gamma(s+\nu-\xi)}\right]e^{i\pi\xi}d\xi$$

$$=\frac{e^{-i\pi(s+\nu)}}{i\pi}\Gamma\left(s+\nu-\frac{1}{2}\right)\Gamma(s+\nu-1)\int_{c-i\infty}^{c+i\infty}e^{-\pi i\xi}g(-i\xi)\frac{\Gamma(1-s-\nu-\xi)}{\Gamma(s+\nu-\xi)}d\xi.$$

Thus

$$\mathscr{G}(1-s)=\frac{e^{-i\pi(s+\nu)}}{i\pi}\Gamma\left(s+\nu-\frac{1}{2}\right)\Gamma(s+\nu-1)\int_{c-i\infty}^{c+i\infty}e^{-\pi i\xi}g(-i\xi)\frac{\Gamma(1-s-\nu-\xi)}{\Gamma(s+\nu-\xi)}d\xi. \tag{A}$$

Also from (36) and (39)

$$R(1-s)=\Gamma(1-s)\Gamma(\nu-1+s)\Gamma\left(\nu-\frac{1}{2}+s\right)\frac{\prod_{r=1}^{p}\Gamma(\alpha_r-1+s)}{\prod_{t=1}^{q}\Gamma(\beta_t-1+s)}, \tag{41}$$

and so by [7], p. 316,

$$f(x)=\frac{1}{2\pi i}\int_{c'-i\infty}^{c'+i\infty}\frac{\mathscr{G}(1-s)}{\mathscr{R}(1-s)}x^{-s}ds=\frac{1}{2(\pi i)^2}\int_{c'-i\infty}^{c'+i\infty}x^{-s}ds\int_{c-i\infty}^{c+i\infty}e^{-\pi i(s+\nu+\xi)}g(-i\xi)$$

$$\times\frac{\Gamma(1-s-\nu-\xi)\prod_{t=1}^{q}\Gamma(\beta_t-1+s)}{\Gamma(s+\nu-\xi)\Gamma(1-s)\prod_{r=1}^{p}\beta(\alpha_r-1+s)}\,d\xi=\frac{1}{\pi i}\int_{c-i\infty}^{c+i\infty}e^{-\pi i(\nu+\xi)}\xi g(-i\xi)\,d\xi$$

$$\times\frac{1}{2\pi i}\int_{c'-i\infty}^{c'+i\infty}\frac{\Gamma(1-\nu-\xi+s)\prod_{t=1}^{q}\Gamma(\beta_t-1-s)}{\Gamma(\nu-\xi-s)\Gamma(1+s)\prod_{r=1}^{p}\Gamma(\alpha_r-1-s)}(e^{i\pi}x)^s ds, \tag{A}$$

using the definition of the generalized E-function (see [1], p. 419) namely

$$E\begin{pmatrix}p;\ \alpha_r\Big|m;\ \rho_{q+s}:x\\ q;\ \rho_s\Big|l+1,\ \alpha_{p+r},\ 1\end{pmatrix}=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\prod_{r=1}^{p}\Gamma(\alpha_r-\zeta)\prod_{s=1}^{m}\Gamma(\zeta-\rho_{q+s}+1)}{\prod_{s=1}^{q}\Gamma(\rho_s-\zeta)\prod_{r=1}^{l}\Gamma(\zeta-\alpha_{p+r}+1)}x^{\zeta}d\zeta$$

$$=\pi^{m-l-1}\sum_{s=1}^{m}\frac{\prod_{r=1}^{l}\sin(\rho_{q+s}-\alpha_{p+r})\pi}{\prod_{s=1}^{m}{}'\sin(\rho_{q+s}-\rho_{q+t})\pi}x^{\rho_{q+s}-1}\times E\begin{pmatrix}p+l;\ \alpha_r-\rho_{q+s}+1:\omega x\\ \rho_1-\rho_{q+s}+1,\ .\ .\ .\ *\ .\ .\ .,\ \rho_{q+m}-\rho_{q+s}+1\end{pmatrix}, \tag{42}$$

where $\omega=e^{\pm i\pi}$ or 1 according as $l+m$ is even or odd; the expression (A) becomes

$$\frac{-x^{\nu-1}}{i(\pi)^2}\int_{c-i\infty}^{c+i\infty}\xi g(-i\xi)x^{\xi}\sin(\nu+\xi)\pi E\begin{pmatrix}1-\nu-\xi,\ \beta_1-\nu-\xi,\ .\ .\ .,\ \beta_t-\nu-\xi:x\\ 1-2\xi,\ \alpha_1-\nu-\xi,\ .\ .\ .,\ \alpha_p-\nu-\xi\end{pmatrix}d\xi.$$

Here put $c=0$, note that $g(x)=g(-x)$, so getting

$$f(x)=\frac{x^{\nu-1}}{i(\pi)^2}\int_0^{\infty}yg(y)\left[\frac{1}{i}\sum_{i,-i}ix^{iy}\sin(i\pi y+\nu\pi)E\begin{pmatrix}1-\nu-iy,\ \beta_1-\nu-iy,\ .\ .\ .,\ \beta_q-\nu-iy:x\\ 1-2iy,\ \alpha_1-\nu-iy,\ .\ .\ .,\ \alpha_p-\nu-iy\end{pmatrix}\right]dy$$

With (43) established, we have the transform pair (1) and (2). (43)

4. Derivation of the Kantorovich-Lebedev and Generalized Mehler Transforms

In the transform pair (1) and (2) take $p=q=0$, $\nu=\frac{1}{2}-k$, apply (18) and (13), so getting the transform pair

$$g(x)=\Gamma\left(\frac{1}{2}-k-ix\right)\Gamma\left(\frac{1}{2}-k+ix\right)\int_0^{\infty}e^{1/2y}y^kW_{k,\,ix}\left(\frac{1}{y}\right)f(y)\,dy, \tag{44}$$

$$f(x)=\frac{x^{-k-1/2}}{i(\pi)^2}e^{-1/x}\int_0^{\infty}yg(y)\left[\frac{1}{i}\sum_{i,-i}ix^{iy}\sin\left(\frac{1}{2}-k+iy\right)\pi\frac{\Gamma(\frac{1}{2}+k-iy)}{\Gamma(1-2iy)}{}_1F_1\begin{pmatrix}\frac{1}{2}-k-iy;\ \frac{1}{x}\\ 1-2iy\end{pmatrix}\right]dy, \tag{45}$$

where we have used the Kummer transformation

$${}_1F_1\begin{pmatrix}\alpha;\ x\\ \rho\end{pmatrix}=e^x{}_1F_1\begin{pmatrix}\rho-\alpha;\ -x\\ \rho\end{pmatrix} \tag{46}$$

Now apply the relations ([1], p. 351 and p. 352)

$$ {}_1F_1(\tfrac{1}{2}-k-iy;\ 1-2iy;\ x)=x^{-1/2+iy}e^{\frac{1}{2}x}\ M_{k,\,-iy}(x), \tag{47}$$

$$ W_{k,\,iy}(x)=\sum_{y,\,-y}\frac{\Gamma(-2iy)}{\Gamma(\frac{1}{2}-k-iy)}M_{k,\,iy}(x); \tag{48}$$

and so get the transform pair

$$ g(x)=\Gamma\left(\frac{1}{2}-k-ix\right)\Gamma\left(\frac{1}{2}-k+ix\right)\int_0^\infty W_{k,\,ix}\left(\frac{1}{y}\right)e^{1/2y}y^k f(y)\,dy \tag{49}$$

$$ f(x)=x^{-k}e^{-1/2x}\frac{1}{i(\pi)^2}\int_0^\infty yg(y)W_{k,\,iy}\left(\frac{1}{x}\right)\sin\,(2iy)\pi dy \tag{50}$$

In (49) replace y by $\dfrac{1}{y}$, in (50), x by $\dfrac{1}{x}$ and then replace $e^{y/2}y^{-k}f\left(\dfrac{1}{y}\right)$ by $f(y)$. The transform pair is

$$ g(x)=\Gamma\left(\frac{1}{2}-k-ix\right)\Gamma\left(\frac{1}{2}-k+ix\right)\int_0^\infty W_{k,\,ix}(y)f(y)\,dy, \tag{51}$$

$$ f(x)=\frac{1}{(x\pi)^2}\int_0^\infty y\sinh\,(2y\pi)W_{k,\,iy}(y)g(y)\,dy. \tag{52}$$

In (51), (52) put $k=0$, apply the relation

$$ W_{0,\,m}(x)=\left(\frac{x}{\pi}\right)^{1/2}K_m\left(\frac{x}{2}\right), \tag{53}$$

and so obtain the Kantorovich-Lebedev transform (3) and (4). For a study how such transforms arise from second order differential equations, see [8] and [9].

In the following table, we give a short list of integrals corresponding to formula (4).

TABLE 1. *Kantorovich-Lebedev transforms*

$g(x)$	$f(x)=\int_0^\infty K_{iy}(x)g(y)\,dy$
$\cosh\,(\alpha x)\cos\,(zx)$ $\lvert\mathrm{Im}\ \alpha\rvert+\lvert\mathrm{Im}\ z\rvert<\frac{\pi}{2}$	$\frac{\pi}{2}e^{-x\cosh z\cos\alpha}\cos\,(x\sinh z\sin\alpha)$
$\sinh\,(\alpha x)\sin\,(zx)$ $\lvert\mathrm{Im}\ \alpha\rvert+\lvert\mathrm{Im}\ z\rvert<\frac{\pi}{2}$	$\frac{\pi}{2}e^{-x\cosh z\cos\alpha}\sin\,(x\sinh z\sin\alpha)$
$\cosh\left(\frac{\pi}{2}x\right)\cos\,(zx)$ $\lvert\mathrm{Im}\ z\rvert<\frac{\pi}{2}$	$\frac{\pi}{2}\cos\,(x\sinh z)$

$g(x)$	$f(x)=\int_0^\infty K_{iy}(x)g(y)dy$
$\sinh\left(\frac{\pi}{2}x\right)\sin(zx)$ $\lvert\mathrm{Im}\, z\rvert<\frac{\pi}{2}$	$\frac{\pi}{2}\sin(x\sinh z)$
$\cosh\left(\frac{\pi}{4}x\right)\cos(zx)$ $\lvert\mathrm{Im}\, z\rvert<\frac{\pi}{2}$	$\frac{\pi}{2}e^{-\frac{x}{\sqrt{2}}\cosh z}\cos\left(\frac{x}{\sqrt{2}}\sinh z\right)$
$x\sinh(\pi x)E\left(\frac{k+ix}{2},\frac{k-ix}{2},\alpha_1,\ldots,\alpha_p:\frac{z}{4}\right.$ $\left.\rho_1,\ldots,\rho_q\right)$ $\lvert\arg z\rvert<\pi$	$\frac{\pi^2}{2^{k-1}}x^kE\left(p;\alpha_r:q;\rho_s:\frac{z}{x^2}\right)$
$x\sinh\left(\frac{1}{2}\pi x\right)K_{ix/2}\left(\frac{z}{8}\right)$ $\lvert\arg z\rvert<\frac{\pi}{2}$	$\pi^{3/2}z^{-1/2}\exp\left(-\frac{z}{8}-\frac{x^2}{z}\right)$
$x\sinh(\pi x)W_{k,ix/2}\left(\frac{z}{4}\right)\times\Gamma\left(\frac{1}{2}-k-\frac{ix}{2}\right)\Gamma\left(\frac{1}{2}-k-\frac{ix}{2}\right)$ $R\left(\frac{1}{2}-k\right)>0,\ \lvert\arg z\rvert<\frac{\pi}{2}$	$2^{2k}\pi^2\left(\frac{xz}{4}\right)^{1-2k}\exp\left(-\frac{z}{8}-\frac{x^2}{z}\right)$
$x\sinh(\pi x)\Gamma\left(\frac{k+ix}{2}\right)\Gamma\left(\frac{k-ix}{2}\right)$ $\times{}_2F_1\left(\frac{k+ix}{2},\frac{k-ix}{2};1+\nu;-\frac{4}{z}\right)$ $R(k)>0$ z is real and positive	$\frac{\pi^2}{2^{k-1}}\Gamma(1+\nu)x^k\left(\frac{z}{x^2}\right)^{\frac{\nu}{2}}J_\nu\left(\frac{2x}{\sqrt{z}}\right)$
$x\sinh(\pi x)E\left(\frac{k+ix}{2},\frac{k-ix}{2},\alpha:\frac{z}{4}\right)$ z is real and positive	$x^k\Gamma(\alpha)\left(1+\frac{x^2}{z}\right)^{-\alpha}$
$x\sinh(\pi x)\Gamma(m+ix)\Gamma(m-ix)\times\begin{cases}P^{1/2-m}_{ix-1/2}(z) & \lvert z\rvert>1\\ T^{1/2-m}_{ix-1/2}(z), & \lvert z\rvert<1\end{cases}$ $R(m)>0,\ R(z)>-1.$	$\frac{\pi^{3/2}}{2^{1/2}}(z^2-1)^{1/\frac{m}{2}-\frac{1}{4}}x^me^{-xz}$
$x\tanh(\pi x)P_{ix-1/2}(z)$	$\left(\frac{1}{2}\pi x\right)^{1/2}e^{-xz}$
$x\tanh(\pi x)P_{ix-1/2}(z)K_{ix}(\alpha)$ $\lvert\arg\alpha\rvert<\frac{\pi}{2},\ \lvert\arg(z-1)\rvert<\pi$	$\frac{\pi}{2}(\alpha x)^{1/2}(x^2+\alpha^2+2\alpha zx)^{-1/2}\exp\left[-(x^2+\alpha^2+2\alpha zx)^{1/2}\right]$
$x\sinh(\pi x)E(\frac{1}{2}+m+ix,\ \frac{1}{2}+m-ix,\ l:m+1:2z)$ $\lvert\arg z\rvert<\frac{\pi}{2}$	$\pi^{3/2}2^{m-1/2}\Gamma(l)z^le^{-x}x^{m+1/2}(z+x)^{-l}$

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$x \tanh(\pi x) K_{ix}(z)$	$\frac{1}{2}\pi (zx)^{1/2}(z+x)^{-1} \exp(-z-x)$
$x \sinh(\pi x) E\left(\begin{matrix}\lambda+ix,\ \lambda-ix,\ \alpha_1,\ \ldots,\ \alpha_p : z\\ \lambda+\frac{1}{2},\ \rho_1,\ \ldots,\ \rho_q\end{matrix}\right)$ $R(\lambda)>0,\ \lvert\arg z\rvert<\frac{\pi}{2}$	$2^{\lambda-1}\pi^{3/2}x^{\lambda}e^{-x}E\left(p;\ \alpha_r : q;\ \rho_s : \frac{z}{2x}\right)$
$x \sinh(\pi x) K_{2ix}(\alpha)$ $\lvert\arg \alpha\rvert<\frac{\pi}{4}$	$\frac{\pi^{3/2}\alpha}{2^{7/2}x^{1/2}} \exp\left(-x-\frac{\alpha^2}{8x}\right)$
$x \sinh(\pi x)\Gamma(\frac{1}{2}-k+ix)\Gamma(\frac{1}{2}-k-ix)\times W_{k,\,ix}(z)$ $R(\frac{1}{2}-k)>0,\ \lvert\arg z\rvert<\frac{\pi}{2}$	$2^{-k-1/2}\pi^{3/2}\Gamma(1-k)x^{1/2-k}z^{k}\,[\exp(-x-\frac{1}{2}z)]\left(1+\frac{2x}{z}\right)^{-1+k}$
$x \sinh(\pi x) E\left(\begin{matrix}\lambda+ix,\ \lambda-ix,\ \frac{1}{2}+n,\ \frac{1}{2}-n : 4z\\ \lambda+\frac{1}{2}\end{matrix}\right)$ $R(\lambda > 0,\ \lvert\arg z\rvert<\frac{\pi}{2}$	$2^{\lambda-1/2}\pi^{2}\sec(n\pi)x^{\lambda-1/2}z^{1/2}\exp\left(-x+\frac{z}{x}\right)K_n\left(\frac{z}{x}\right)$
$x \sinh(\pi x) E\left(\begin{matrix}\lambda+ix,\ \lambda-ix,\ \alpha : z\\ \lambda+\frac{1}{2}\end{matrix}\right)$ $R(\lambda)>0,\ \lvert\arg z\rvert<\frac{\pi}{2}$	$2^{\lambda-1}\pi^{3/2}x^{\lambda}e^{-x}\Gamma(\alpha)\left(1+\frac{2x}{z}\right)^{-\alpha}$
$x \sinh(\pi x) E\left(\begin{matrix}\lambda+ix,\ \lambda-ix,\ \frac{1}{2}-k+m,\ \frac{1}{2}-k-m : z\\ \lambda+\frac{1}{2}\end{matrix}\right)$ $R(\lambda)>0,\ \lvert\arg z\rvert<\frac{\pi}{2}$	$2^{\lambda+k-1}\pi^{3/2}\Gamma(\frac{1}{2}-k+m)\Gamma(\frac{1}{2}-k-m)\Gamma(\frac{1}{2}-m)$ $\times z^{-k}x^{\lambda+k}\left[\exp\left(-x+\frac{z}{4x}\right)\right]W_{k,\,m}\left(\frac{z}{2x}\right)$
$x \sinh(\pi x)\Gamma(\lambda+ix)\Gamma(\lambda-ix)\,{}_2F_2\left(\begin{matrix}\lambda+ix,\ \lambda-ix;\ -\frac{1}{z}\\ \lambda+\frac{1}{2},\ \nu+1\end{matrix}\right)$ $R(\lambda)>0$, z is real and positive	$2^{\lambda-\frac{\nu}{2}-1}\pi^{3/2}x^{\lambda-1/\frac{\nu}{2}}z^{1/\frac{\nu}{2}}\Gamma\left(\lambda+\frac{1}{2}\right)\Gamma(1+\nu)J_\nu\left[\left(\frac{8x}{z}\right)^{1/2}\right]$
$x \sinh(\pi x)\Gamma(\frac{1}{2}+m+ix)\times\Gamma(\frac{1}{2}+m-ix)M_{ix,\,m}(iz)M_{ix,\,m}(-iz)$ $R(m)>-\frac{1}{2}$ and z is real and positive	$2^{m-1/2}\pi^{3/2}x^{1/2+m}z^{-1-2m}\,{}_0F_2\left(:\frac{1}{2}+m,\ 1+m;\ -\frac{xz^2}{2}\right)$
$x \sinh(\pi x) K_{2ix}(\alpha)K_{ix}(\beta)$ $2\lvert\arg \alpha\rvert+\lvert\arg \beta\rvert<\pi$	$\frac{\pi^{3/2}\alpha}{16(\beta x)^{1/2}}\left(\frac{4\beta x}{4\beta x+\alpha^2}\right)^{1/2}\exp\left[-(\beta+x)\left(\frac{4\beta x+\alpha^2}{4\beta x}\right)^{1/2}\right]$

To derive the generalized Mehler transform pair (5) and (6) take in (1), (2), $p=0$, $q=1$, $\nu=\frac{1}{2}$ with $\beta_1=1-k$, so getting

$$g(x)=\frac{\pi}{\Gamma(1-k)\cosh(\pi x)}\int_0^\infty {}_2F_1\left(\begin{matrix}\frac{1}{2}-ix,\ \frac{1}{2}+ix;-y\\ 1-k\end{matrix}\right)f(y)dy, \tag{54}$$

$$f(x)=\frac{x^{-1/2}}{i\pi^2}\int_0^\infty yg(y)\left[\frac{1}{i}\sum_{i,-i} ix^{iy}\sin\left(\frac{\pi}{2}+i\pi y\right)\frac{\Gamma(\frac{1}{2}-iy)\Gamma(\frac{1}{2}-k-iy)}{\Gamma(1-2iy)}\right.$$

$$\left.\times {}_2F_1\left(\begin{matrix}\frac{1}{2}-iy,\ \frac{1}{2}-k-iy;-\frac{1}{x}\\ 1-2iy\end{matrix}\right)\right]dy. \tag{55}$$

Now write $\frac{1}{2}(y-1)$ for y in (54) and $\frac{1}{2}(x-1)$ for x in (55) and get

$$g(x)=\frac{\pi}{2\Gamma(1-k)\cosh\pi x}\int_1^\infty {}_2F_1\left(\begin{matrix}\frac{1}{2}-ix,\ \frac{1}{2}+ix;\ \frac{1-y}{2}\\ 1-k\end{matrix}\right)f\left(\frac{y-1}{2}\right)dy, \tag{56}$$

$$f\left(\frac{x-1}{2}\right)=\frac{1}{i\pi^2}\left(\frac{x-1}{2}\right)^{-1/2}\int_0^\infty yg(y)\left[\frac{1}{i}\sum_{i,-i}\left(\frac{x-1}{2}\right)^{iy}\sin\left(\frac{\pi}{2}+i\pi y\right)\right.$$

$$\left.\times\frac{\Gamma(\frac{1}{2}-iy)\Gamma(\frac{1}{2}-k-iy)}{\Gamma(1-2iy)}\,{}_2F_1\left(\begin{matrix}\frac{1}{2}-iy,\ \frac{1}{2}-k-iy;\ \frac{2}{1-x}\\ 1-2iy\end{matrix}\right)\right]dy. \tag{57}$$

If we use the relationships ([1], pp. 303 and 391)

$$P_n^{-m}(z)=\frac{1}{\Gamma(m+1)}\left(\frac{z-1}{z+1}\right)^{\frac{1}{2}m}{}_2F_1\left(\begin{matrix}-n,\ n+1;\ \frac{1-z}{2}\\ m+1\end{matrix}\right), \tag{58}$$

$$\Gamma(\tfrac{1}{2})(z^2-1)^{-\frac{1}{2}m}P_n^{-m}(z)=\frac{2^n\Gamma(n+\frac{1}{2})}{\Gamma(n+m+1)}(z-1)^{n-m}{}_2F_1\left(\begin{matrix}-n,\ m-n;\ \frac{2}{z-1}\\ -2n\end{matrix}\right)$$

$$+\frac{2^{-n-1}\Gamma(-n-\frac{1}{2})}{\Gamma(m-n)}(z-1)^{-n-m-1}{}_2F_1\left(\begin{matrix}n+1,\ n+m+1;\ \frac{2}{z-1}\\ 2n+2\end{matrix}\right), \tag{59}$$

in (56) and (57) respectively we obtain the generalized Mehler transform pair (5) and (6). A short list of integrals corresponding to formula (6) is given in the following table:

TABLE 2. *Generalized Mehler transforms*

$g(x)$	$f(x)=\int_0^\infty P^k_{iy-1/2}(x)g(y)dy$
$x\sinh(\pi x)\Gamma(\frac{1}{2}-k+ix)\Gamma(\frac{1}{2}-k-ix)K_{ix}(z)$ $R(\frac{1}{2}-k)>0$, $\lvert\arg z\rvert<\frac{1}{2}\pi$	$2^{-1/2}\pi^{3/2}z^{1/2-k}e^{-zx}(x^2-1)^{-\frac{1}{2}k}$* *compare the formula p. 15
$x\sinh(2\pi x)\Gamma(\frac{1}{2}-k+ix)\Gamma(\frac{1}{2}-k-ix)$ $E(\frac{1}{2}m+\frac{1}{2}n,\ \frac{1}{2}m-\frac{1}{2}n,\ \frac{1}{2}+ix,\ \frac{1}{2}-ix:1-k:z)$ $R\left(\frac{1}{2}-k\right)>0$, $\lvert\arg z\rvert<\frac{\pi}{2}\cdot R(m)>0$	$\pi^2 2^{3-m/2}z^{m/2}(x-1)^{k/2+m/2-1}\quad(x+1)^{-k/2}K_n[\{2z(x-1)\}^{1/2}]$.

TABLE 2. *Generalized Mehler transforms*—Continued

$g(x)$	$f(x)=\int_0^\infty P^k_{iy-1/2}\,(x)g(y)dy$
$x\sinh(2\pi x)\Gamma(\tfrac12-k+ix)\Gamma(\tfrac12-k-ix)$ $\times\left[\Gamma(\tfrac12+m+ix)\Gamma(\tfrac12+m-ix)\,{}_2F_1\left(\begin{matrix}\frac12+m+ix,\ \frac12+m-ix;\ \frac1z\\ 1-m\end{matrix}\right) - z^m\frac{\pi}{\cosh(\pi x)\Gamma(1+m)}\,{}_2F_1\left(\begin{matrix}\frac12+ix,\ \frac12-ix;\ \frac1z\\ 1+m\end{matrix}\right)\right]$ $R(\tfrac12-k)>0$, $R(1\pm m)>0$ and z is real and positive	$2^{1-m}\sin(m\pi)(x+1)^{-\frac12 k}$ $\times(x-1)^{\frac12 k+m-1}\left[1+\frac{2}{z(x-1)}\right]^{-1+k+m}$
$x\sinh(2\pi x)\Gamma(\tfrac12-k+ix)\Gamma(\tfrac12-k-ix)$ $\times\left[i^{m-n}E(\tfrac12 m+\tfrac12 n,\ \tfrac12 m-\tfrac12 n,\ \tfrac12+ix,\ \tfrac12-ix:1-k:e^{-i\pi}z) - i^{n-m}E(\tfrac12 m+\tfrac12 n,\ \tfrac12 m-\tfrac12 n,\ \tfrac12+ix,\ \tfrac12-ix,\ 1-k:e^{-i\pi}z)\right]$ $R\left(\frac12-k\right)>0,\ \lvert\arg z\rvert<\frac{\pi}{2}$	$i\pi^3 z^{2-\frac m2}z^{\frac m2}$ $\times(x-1)^{\frac m2+\frac k2-1}(x+1)^{-\frac12 k}$ $\times J_n[\{2z(x-1)\}^{\frac12}]$
$x\sinh(2\pi x)\Gamma(\tfrac12-k+ix)\Gamma(\tfrac12-k-ix)$ $\times\left[\frac{\pi z^m}{\cosh(\pi x)\Gamma(1-k)\Gamma(1-m)}\,{}_2F_2\left(\begin{matrix}\frac12+ix,\ \frac12-ix:\ \frac1z\\ 1-k,\ 1-m\end{matrix}\right) - \frac{\Gamma(\frac12-k+ix)\Gamma(\frac12-k-ix)}{\Gamma(1-k+m)\Gamma(1+m)}\,{}_2F_2\left(\begin{matrix}\frac12-k+ix,\ \frac12-k-ix;\ \frac1z\\ 1-k-m,\ 1+m\end{matrix}\right)\right]$ $R(\tfrac12-k)>0$, z is real and positive	$\pi 2^{1-m}\sin(m\pi)$ $\times(x-1)^{\frac12 k-m-1}(x+1)^{-\frac12 k}$ $\exp\left[-\frac{2}{z(x-1)}\right]$
$x\sinh(2\pi x)\Gamma(\tfrac12-k+ix)\Gamma(\tfrac12-k-ix)$ $\times E(\tfrac12+ix,\ \tfrac12-ix,\ l-k:1-k:2z)$ $R\left(\frac12-k\right)>0,\ \lvert\arg z\rvert<\frac{\pi}{2},\ R(l+m)>0$	$2\pi^2 z^{k-l}(x-1)^{l-1}(x^2-1)^{-\frac k2}\exp[-z(x-1)]$
$x\sinh(\pi x)E\left(\begin{matrix}\frac12-k+ix,\ \frac12-k-ix,\ l:2z\\ 1-k\end{matrix}\right)$ $R\left(\frac12-k\right)>0,\ R(l)>0,\ \lvert\arg z\rvert<\frac{\pi}{2}$	$2^{-k}z^l(x-1)^{l-1}(x^2-1)^{\frac k2}\exp[-z(x-1)]$
$x\sinh(\pi x)\Gamma(\tfrac12-k+ix)\Gamma(\tfrac12-k-ix)$ $\times\Gamma(\tfrac12+m+ix)\Gamma(\tfrac12+m-ix)P^{-m}_{ix-1/2}(z)$ $R(\tfrac12-k)>0,\ R(\tfrac12+m)>0,\ R(z)>1$	$\pi\Gamma(m-k+1)\,\frac{(x^2-1)^{-\frac k2}(z^2-1)^{m/2}}{(x+z)^{m-k+1}}$
$x\sinh(2\pi x)\Gamma(\tfrac12-k+ix)\Gamma(\tfrac12-k-ix)$ $\times E\left(\begin{matrix}\gamma+m,\ \delta+m,\ \frac12+ix,\ \frac12-ix:z\\ \gamma+\delta+m,\ 1-k\end{matrix}\right)$ $R\left(\frac12-k\right)>0,\ \lvert\arg z\rvert<\frac{\pi}{2},\ R(\gamma+m)>0,\ R(\delta+m)>0$	$\pi^2 2^{1-m}z^m\{\Gamma(\gamma)\Gamma(\delta)\}^{-1}$ $\times(x-1)^{\frac k2+m-1}(x+1)^{-\frac k2}$ $\times E\left[\gamma,\delta::\frac z2(x-1)\right]$
$x\sinh(2\pi x)\Gamma(\tfrac12-k+ix)\Gamma(\tfrac12-k-ix)$ $\times E\left(\begin{matrix}\frac12+m+n,\ \frac12+m-n,\ \frac12+ix,\ \frac12-ix:z\\ 1+m,\ 1-k\end{matrix}\right)$	$2^{1/2-m}\pi^{3/2}z^{1/2+m}(x-1)^{\frac k2+m-1/2}\exp\left[\frac z4(x-1)\right]$ $\times K_n\left[\frac z4(x-1)\right].$

$R\left(\frac{1}{2}-k\right)>0,\ \lvert\arg z\rvert<\frac{\pi}{2}$	
$x\sinh(2\pi x)\Gamma(\frac{1}{2}-k+ix)\Gamma(\frac{1}{2}-k-ix) \times E\left(\begin{matrix}\frac{1}{2}-k'+m'+m,\ \frac{1}{2}-k'-m'+m,\ \frac{1}{2}+ix,\ \frac{1}{2}-ix:z\\ 1-2k'+m,\ 1-k\end{matrix}\right)$ $R\left(\frac{1}{2}-k\right)>0,\ \lvert\arg z\rvert<\frac{\pi}{2}$	$\pi^2 2^{1+k'-m}z^{m-k'}(x-1)^{-\frac{1}{2}k+m-k'-1}(x+1)^{-\frac{k}{2}} \times\exp\left[\frac{z}{4}(x-1)\right]W_{k',m'}\left[\frac{z}{2}(x-1)\right]$

Other applications can be obtained from (1) and (2) by special choices of the parameters. Thus (1) and (2) in combination with (25) yield the transform pair

$$g(x)=\int_0^\infty S_{2\mu,\,2ix}(y)f(y)dy, \tag{60}$$

$$f(x)=-\frac{2^{2\mu-2}}{i\pi x}\int_0^\infty yg(y)\left[\frac{\Gamma(\frac{1}{2}-\mu-iy)}{\Gamma(\frac{1}{2}+\mu-iy)}J_{-2iy}(x)-\frac{\Gamma(\frac{1}{2}-\mu+iy)}{\Gamma(\frac{1}{2}+\mu+iy)}J_{2iy}(x)\right]dy, \tag{61}$$

which may be called S-transforms.

When $p=0,\ q=1$ then the E-functions in (1) and (2) reduce to the ordinary hypergeometric functions of Gauss and the following transform pair is obtained:

$$g(x)=\int_0^\infty {}_2F_1\left(\begin{matrix}\nu-ix,\ \nu+ix;-y\\ \beta\end{matrix}\right)f(y)dy, \tag{62}$$

$$f(x)=\frac{\Gamma(\beta)x^{\nu-1}}{i\pi^2}\int_0^\infty \Gamma(\nu-iy)\Gamma(\nu+iy)yg(y)$$

$$\times\left[\frac{1}{i}\sum_{i,-i} ix^{iy}\frac{\sin(i\pi y+\nu\pi)\Gamma(1-\nu-iy)\Gamma(\beta-\nu-iy)}{\Gamma(1-2iy)}\,{}_2F_1\left(\begin{matrix}1-\nu-iy,\ \beta-\nu-iy;-\frac{1}{x}\\ 1-2iy\end{matrix}\right)\right]dy. \tag{63}$$

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