JOURNAL OF RESEARCH of the National Bureau of Standards—B. Mathematics and Mathematical Physics
Vol. 71B, Nos. 2 and 3, April–September 1967

Construction of *EPr* Generalized Inverses by Inversion of Nonsingular Matrices*

John Z. Hearon**

(April 20, 1967)

Any matrix B such that $ABA=A$ is called a C_1-inverse of A and a C_1-inverse of A such that $BAB=B$ is called a C_2-inverse of A. Some properties of such inverses are established. It is shown that if A is p-square of rank $q<p$ and P is any positive semidefinite matrix, whose rank is the nullity of A, such that $U=A+P$ is nonsingular, then $B=U^{-1}AU^{-1}$ is a C_2-inverse of A with the property that null space B = null space B^*. That such a P exists for arbitrary square A is shown. The relation between this result and the work of Goldman and Zelen is discussed.

Key Words: *EPr* matrices, generalized inverse, matrix.

1. Introduction

Goldman and Zelen [1] [1] have shown how to construct a generalized inverse (of a kind made precise in what follows) of a real symmetric matrix A by inversion of a nonsingular matrix formed from A. It is inherent in the assumption that A is symmetric that the resulting generalized inverse is also symmetric.

We show that if a complex matrix A and its conjugate transpose have the same null space (i.e., A is an *EPr* matrix [3]) then there always exists a generalized inverse of the kind discussed which is also an *EPr* matrix. It is then shown that the construction given by Goldman and Zelen [1] goes through, essentially step for step, when the condition that A be real symmetric is replaced by the condition that A be an *EPr* matrix, and that the resulting generalized inverse is an *EPr* matrix.

It is further shown that with no restrictions on A the Goldman-Zelen procedure produces a generalized inverse which is *EPr*, although in this case the details of the construction are somewhat different. The (rather surprising) implication is that an arbitrary square complex matrix always possesses a generalized inverse, of the type discussed, which is an *EPr* matrix. In any given case a generalized inverse of this character can be obtained in principle by the Zelen-Goldman procedure, i.e., by inverting a certain nonsingular matrix and selecting from it a specified submatrix.

2. Some Properties of Generalized Inverses

All matrices considered have complex entries. We use the symbols $\rho(V)$, $N(V)$, $R(V)$, and V^* to denote the rank, null space, range and conjugate transpose of the matrix V. When V is square, $|V|$ denotes the determinant of V. For two subspaces S_1 and S_2, $S_1\cdot S_2$ is the intersection of S_1 and S_2; $S_1\leqslant S_2$ denotes that S_1 is a subspace of S_2; the dimension of S_1 is written dim S_1. The symbol I denotes an identity matrix of whatever order is appropriate in the context.

For a given arbitrary matrix A we define by $C_1(A)$ the set of all matrices B such that $ABA=A$. We call any matrix in $C_1(A)$ a C_1-inverse of A. We define by $C_2(A)$ the set of all matrices B such that $B\epsilon C_1(A)$ and $BAB=B$. Thus $B\epsilon C_2(A)$ if and only if $B\epsilon C_1(A)$ and $A\epsilon C_1(B)$; and $B\epsilon C_2(A)$ if and only if $A\epsilon C_2(B)$. We call any matrix in $C_2(A)$ a C_2-inverse of A. C_1-inverses and C_2-inverses have been termed by Rohde [4] generalized inverses and reflexive generalized inverses respectively. We begin with several lemmas regarding these kinds of generalized inverses.

The first lemma, which we state for ready reference, is due to Rohde [4].

LEMMA 1. *If* A *is any matrix and* B *is a* C_1*-inverse of* A *then* $\rho(B)\geqslant\rho(A)=\rho(AB)=\rho(BA)$.

The next lemma was first proved by Rohde [4]. We here give a shortened proof of a quite different character.

LEMMA 2. *Let* A *be any matrix and* B *any* C_1*-inverse of* A. *Then* B *is a* C_2*-inverse of* A *if and only if* $\rho(B)=\rho(A)$.

PROOF. Assume $B\epsilon C_2(A)$ then since $B\epsilon C_1(A)$ we have, by Lemma 1, $\rho(B)\geqslant\rho(A)$; and since also $A\epsilon C_1(B)$ we have, by Lemma 1, $\rho(B)\leqslant\rho(A)$. Thus $\rho(A)=\rho(B)$. Conversely, assume $B\epsilon C_1(A)$ and $\rho(B)=\rho(A)=r$. From $ABA=A$ the matrix AB is idempotent and by Lemma 1, $\rho(AB)=r$. There are then linearly independent vectors x_i, $1\leqslant i\leqslant r$, such that $ABx_i=x_i$. If there are n columns in B and y_i, $1\leqslant i\leqslant n-r$, are any basis of $N(B)=N(AB)$ then $ABy_i=0$. We then have $BABx_i=Bx_i$ and $BABy_i$

*An invited paper.

**Present address: Mathematical Research Branch, NIAMD, National Institutes of Health, Bethesda, Md. 20014.

[1] Figures in brackets indicate the literature references at the end of this paper.

$=By_i$, from which it follows [2] that $BAB=B$ and that $B\epsilon C_2(A)$.

LEMMA 3. *Let* P *be an* n × m *matrix,* Q *and* R *be* m × n *matrices. If* PQ *is idempotent,* $\rho(PQ)=\rho(Q)$ *and* $N(R)=N(Q)$, *then* $RPQ=R$.

PROOF. If PQ is idempotent and $\rho(PQ)=\rho(Q)=r$, then as in the proof of Lemma 1, we have x_i, $1\leqslant i\leqslant r$, and y_i, $1\leqslant i\leqslant n-r$, linearly independent and such that $PQx_i=x_i$, $PQy_i=0$ and $y_i\epsilon N(Q)$. If $N(R)=N(Q)$ then $RPQx_i=Rx_i$ and $RPQy_i=Ry_i$ from which the conclusion follows.

It has been seen that if $B\epsilon C_1(A)$ then BA is idempotent and has the rank of A. The following corollary of Lemma 3 shows the converse of this to be true.

COROLLARY 1. *The matrix* B *is a* C_1*-inverse of* A *if and only if* BA *is idempotent and* $\rho(BA)=\rho(A)$; *and if and only if* AB *is idempotent and* $\rho(AB)=\rho(A)$.

PROOF. If $B\epsilon C_1(A)$ then from $ABA=A$, AB, and BA are idempotent and that they have the rank of A is given by Lemma 1. Conversely, assume BA idempotent, $\rho(BA)=\rho(A)$, and in Lemma 3 take $P=B$, $R=Q=A$. Then $ABA=A$ and $B\epsilon C_1(A)$. If AB is idempotent and $\rho(AB)=\rho(A)$ then $\rho(B^*A^*)=\rho(A^*)$ and B^*A^* is idempotent. By Lemma 3 (with $P=B^*$, $R=Q=A^*$) we have $A^*B^*A^*=A^*\Rightarrow ABA=A\Rightarrow B\epsilon C_1(A)$.

The following corollary of Lemma 3 gives a relation between an EPr matrix and a C_1-inverse of that matrix.

COROLLARY 2. $A^*=A^*BA$ *if and only if* B *is a* C_1*-inverse of* A *and* $N(A)=N(A^*)$. *Further,* $A^*=ABA^*$ *if and only if* $A^*=A^*BA$.

PROOF. If $B\epsilon C_1(A)$ then BA is idempotent and has the rank of A. If further $N(A)=N(A^*)$, then Lemma 3 (with $P=B$, $Q=A$, $R=A^*$) gives $A^*BA=A^*$. Conversely, if $A^*BA=A^*$ then $N(A)\leqslant N(A^*)$, and hence $N(A)=N(A^*)$, and this being so $A^*BA=A^*$ $\Rightarrow A^*(I-BA)=0\Rightarrow A(I-BA)=0\Rightarrow B\epsilon C_1(A)$. That $B\epsilon C_1(A)$ and $N(A)=N(A^*)$ are necessary and sufficient for $A^*=ABA^*$ is proved in the same way.

REMARK: Corollary 2 can in fact be proved without recourse to Lemma 3. The "if part" follows at once from the fact [3] that $N(A^*)=N(A^*)$ if and only if $R(A)=R(A^*)$.

The next lemma shows that C_2-inverses can be constructed from C_1-inverses.

LEMMA 4. *Let* B_1 *and* B_2 *be any two (not necessarily distinct)* C_1*-inverses of* A. *Then* $B=B_1AB_2$ *is a* C_2*-inverse of* A.

PROOF. Given that B_1 and B_2 are in $C_1(A)$ we have $ABA=(AB_1A)B_2A=AB_2A=A$, or that $B\epsilon C_1(A)$. By Lemma 1, $\rho(B)\geqslant\rho(A)$, but $\rho(B)\leqslant\rho(B_1A)=\rho(A)$ also follows from Lemma 1 and $B_1\epsilon C_1(A)$. Thus we have $B\epsilon C_1(A)$ and $\rho(B)=\rho(A)$, and Lemma 2 gives the conclusion $B\epsilon C_2(A)$.

A C_i-inverse, $i=1, 2$, of a hermitian matrix is not necessarily hermitian but that a hermitian matrix always possesses at least one hermitian C_i-inverse is known [3] [4]. We observe that the existence of a hermitian C_1-inverse of a hermitian matrix insures, by Lemma 4, the existence of hermitian C_2-inverse. For if $B_1=B_1^*\epsilon C_1(A)$ then $B=B_1AB_1\epsilon C_2(A)$ and is hermitian whenever A is hermitian. There is in fact a considerable list of properties such that by using Lemma 4 we can assert: If there exists a C_1-inverse with one of the properties then there exists a C_2-inverse with that property.

The next lemma shows that every EPr matrix possesses a C_2-inverse which is an EPr matrix.

LEMMA 5. *Let* A *be a matrix with the property* $N(A)=N(A^*)$. *Then there exist matrices* B *such that* $B\epsilon C_2(A)$ *and* $N(B)=N(B^*)$. *In fact* $B=B_1A^*B_1^*$, *where* B_1 *is any* C_2*-inverse of* A, *is such a matrix.*

PROOF. If $N(A)=N(A^*)$ and $B_1\epsilon C_2(A)$ then by Corollary 2, we have $A^*B_1A=A^*$ and $A^*B_1^*A=A$. Let $B=B_1A^*B_1^*$, then $ABA=AB_1(A^*B_1^*A)=AB_1A=A$. Thus $B\epsilon C_1(A)$. By Lemma 1, $\rho(B)\geqslant\rho(A)$. On the other hand $\rho(B)\leqslant\rho(A^*B_1^*)=\rho(A)$ follows from Lemma 1 and the construction of B. Hence by Lemma 2, $B\epsilon C_2(A)$. Clearly $N(B_1^*)\leqslant N(B)$ and $N(B_1^*)\leqslant N(B^*)$. But $\rho(B_1)=\rho(B)$, for we have just proved $\rho(A)=\rho(B)$ and $\rho(A)=\rho(B_1)$ follows from Lemma 2. Hence $N(B)=N(B^*)$.

3. C_2-inverses by Inversion of a Nonsingular Matrix

Let A be a $p\times p$ matrix, $\rho(A)=q$, K be a $p\times r$ matrix and define the matrices M and U as follows:

$$M=\begin{bmatrix} A & K \\ K^* & 0 \end{bmatrix} \tag{1}$$

$$U=A+KK^*. \tag{2}$$

We further denote by S and S^* the subspaces $S=N(A)\cdot N(K^*)$ and $S^*=N(A^*)\cdot N(K^*)$. We then prove the following theorem.

THEOREM 1. *Let* M *and* U *be as in (1) and (2). If* $r=p-q$ *then any one of the following statements implies the other two:* (i) $S=0$ *and* $S^*=0$, (ii) M^{-1} *exists,* (iii) U^{-1} *exists.*

PROOF. (i)$\Leftrightarrow$(ii). It has been shown [2] that $S^*=0$ is equivalent to the existence of a matrix H such that $H^*A=0$ and $|H^*K|\neq 0$. Assume (i) and let $z^T=(x^T, y^T)$ be a suitably partitioned vector such that $z\epsilon N(M)$. Then $Ax+Ky=0$ and $K^*x=0$. The first of these two equalities shows $H^*Ky=0\Rightarrow y=0$. Given $y=0$ we are left with $Ax=0$ and $K^*x=0$ so that $x\epsilon S$ and hence $x=0$. Thus (i)$\Rightarrow$(ii). It has been shown [2] that $S\neq 0$ $\Rightarrow|M|=0$ and this same argument shows that $S^*\neq 0$ $\Rightarrow|M^*|=0$. We then have $S=0\Leftarrow|M|\neq 0\Leftrightarrow|M^*|$ $\neq 0\Rightarrow S^*=0$. Hence (ii)$\Rightarrow$(i).

(i)$\Leftrightarrow$(iii). Assume (i). Let $x\epsilon N(U)$, then $Ax+KK^*x$ $=0$. There is an H such that $H^*Ax+H^*KK^*x=H^*KK^*x$ $=0\Rightarrow K^*x=0$. But then $x\epsilon S$ and $x=0$. Thus (i)$\Rightarrow$(iii).

[2] We observe that the x_i, $1\leqslant i\leqslant r$, and y_i, $1\leqslant i\leqslant n-r$ are a complete set of engenvectors of the projection $E=AB$ and are thus linearly independent. For proof, let $z=\Sigma\alpha_ix_i+\Sigma\beta_jy_j$. Then $Ez=\Sigma\alpha_ix_i$. If $z=0$, then $\alpha_i=0$, and given this, $z=\Sigma\beta_jy_j=0$, and $\beta_j=0$.

[3] The argument in [4], and in Lemma 5 to follow, assumes the existence of a C_1-inverse. The existence of a C_1-inverse for an arbitrary matrix A was given constructive proof by Bose in 1959 and is given in [5] in detail.

Now assume (i) false. If (i) is false due to $S \neq 0$ then $|U| = 0$, since any $x \epsilon S$ is in $N(U)$; if (i) is false due to $S^* \neq 0$ then $|U^*| = 0$, since any $x \epsilon S^*$ is in $N(U^*)$. Thus (iii) $\Rightarrow$(i).

Whenever M^{-1} exists[4] we partition this matrix in the same manner as M and write

$$M^{-1} = \begin{bmatrix} B & B_{12} \\ B_{21} & B_2 \end{bmatrix}. \tag{3}$$

Regarding the relation of the blocks in M to those in M^{-1} we have the following theorem.

THEOREM 2. *Let* M *be as in (1). Assume* M^{-1} *to exist and be as in (3). Then,* (i) $A \epsilon C_1(B)$, (ii) $N(B) = N(B^*)$, (iii) B *is a* C_2*-inverse of* A *if and only if* $r = p - q$, (iv) $B_{21} \epsilon C_2(K)$, (v) $B_{12} \epsilon C_2(K^*)$, (vi) *if* $r = p - q$, $B_2 = 0$, (vii) *if* $r = p - q$ *and* $N(A) = N(A^*)$ *then* $B_{12}^* = B_{21}$.

PROOF. It is known [2] that if M^{-1} exists then $\rho(K) = r$ and $r \geqslant p - q$. Assuming M^{-1} exists we obtain from $MM^{-1} = I$ and $M^{-1}M = I$

$$AB + KB_{21} = I \tag{4}$$

$$K^*B = 0 \tag{5}$$

$$BA + B_{12}K^* = I \tag{6}$$

$$AB_{12} + KB_2 = 0. \tag{7}$$

From (4) and (5) we have at once $B^*AB = B^*$ and (i) and (ii) follow from Corollary 2. Given (ii), (5) implies $BK = 0$ and this with (4) gives $KB_{21}K = K$. Thus $B_{21} \epsilon C_1(K)$ and, by Lemma 1, $\rho(KB_{21}) = \rho(K) = r$. Since KB_{21} is idempotent of rank r we now have from (4) that $AB = I - KB_{21}$ is idempotent of rank $p - r$. But $\rho(AB) = \rho(B) = p - r$ follows from (i) and Lemma 1. Thus by Lemma 2, $A \epsilon C_2(B)$ and hence $B \epsilon C_2(A)$ if and only if $\rho(A) = q = \rho(B) = p - r$, and (iii) is proved. We have just seen that $B_{21} \epsilon C_1(K)$. Hence, by Lemma 1, $\rho(B_{21}) \geqslant \rho(K) = r$, but B_{21} has r rows and thus $\rho(B_{21}) = r$ and (iv) follows from Lemma 2. From (5) and (6), $K^*B_{12}K^* = K^*$ and $B_{12} \epsilon C_1(K^*)$ implying, by Lemma 1, $\rho(B_{12}) \geqslant \rho(K) = r$. But B_{12} has r columns, thus $\rho(B_{12}) = r$ and (v) follows from Lemma 2. If $r = p - q$, then by Theorem 1, $S^* = 0$ and [2] there exists an H such that $H^*A = 0$, $|H^*K| \neq 0$. This being so, (7) gives $H^*KB_2 = 0$ and (vi) is proved. If $r = p - q$ then from (4), $H^*KB_{21} = H^*$ gives $B_{21} = (H^*K)^{-1}H^*$. If further, $N(A) = N(A^*)$ then from (6), $B_{12}K^*H = H$ gives $B_{12} = H(K^*H)^{-1}$ and (vii) is evident.

The next theorem gives an explicit formula for the matrix B of Theorem 2 in terms of the matrix U in (2).

THEOREM 3. *Let* U *be as in (2) and* $r = p - q$. *If* U^{-1} *exists then* $B = U^{-1}AU^{-1}$ *is a* C_2*-inverse of* A *with the property* $N(B) = N(B^*)$. *If further* $N(A) = N(A^*)$ *then* $UBU^* = U^*BU = A^*$.

[4] Although not needed in Theorem 1, it is a fact that the existence of M^{-1} implies both $\rho(K) = r$ and $r \geqslant p - q$ [2], cf. alternative proof of Theorem 3.

PROOF. If U^{-1} exists and $r = p - q$ then, by Theorem 1, M^{-1} exists. This being the case, according to Theorem 2, the block B in (3) is a C_2-inverse of A with the property $N(B) = N(B^*)$. Further B obeys $K^*B = 0$ and $BK = 0$. From these last two equalities and the definition (3) of U we have $BU = BA$ and $UB = AB$ which imply $UBU = ABA = A$ and hence $B = U^{-1}AU^{-1}$. We also have from $UB = AB$ and $BU^* = BA^*$ that $UBU^* = ABA^*$; and if $N(A) = N(A^*)$ then $UBU^* = A^*$ follows from Corollary 2. Similarly from $BU = BA$ and $U^*B = A^*B$ we have $U^*BU = A^*BA$ and if $N(A) = N(A^*)$ then $U^*BU = A^*$.

It is of some interest to prove Theorem 3 without recourse to the existence of M^{-1}.

ALTERNATE PROOF. If U^{-1} exists and $r = p - q$ then $\rho(K) = r$. For if $\rho(K) < r$, then $\dim N(K^*) = p - \rho(K) > p - r$ and $\dim N(A) + \dim N(K^*) > p - q + p - r = p \Rightarrow S \neq 0$ and hence, by Theorem 1, $|U| = 0$. Now $U^{-1}A = I - U^{-1}KK^*$ shows that if $x \epsilon N(A)$ then $U^{-1}KK^*x = x$, and clearly $y \epsilon N(K^*)$ implies $U^{-1}KK^*y = 0$. Thus given $\dim N(A) = p - q = r$ and $\dim N(K^*) = p - r = q$, we have that $U^{-1}KK^*$ is idempotent of rank r and so $U^{-1}A$ is idempotent of rank q. By Corollary 1, $U^{-1} \epsilon C_1(A)$ and $U^{-1} \epsilon C_1(KK^*)$. By Lemma 4, $B = U^{-1}AU^{-1}$ is a C_2-inverse of A. From $B = (U^{-1}A)U^{-1} = (I - U^{-1}KK^*)U^{-1}$ and $U^{-1} \epsilon C_1(KK^*)$ it follows that $KK^*B = BKK^* = 0$ and hence (since $\rho(K) = \rho(K^*K) = r$) that $B^*K = BK = 0$. But $\dim N(B) = p - q = r = \rho(K)$ and hence $N(B) = N(B^*)$. The remainder of the proof is as given above.

We observe from Theorem 2 that when $p - q = r$, every block in M^{-1} is a C_2-inverse of an appropriate block in M (we agree that trivially a zero square matrix is its own C_2-inverse), and that if additionally $N(A) = N(A^*)$ then M and M^{-1} are of the same form in that $B_{21} = B_{12}^*$. Furthermore we have from Theorem 3 that $U^{-1}AU^{-1} = B \epsilon C_2(A)$, $U^{-1} \epsilon C_1(A)$, as noted in the alternative proof, and $U \epsilon C_1(B)$, which follows from $BU = BA$. This last set of relations among generalized inverses is a special instance of the following lemma.

LEMMA 6. *Let* B_1 *be any* C_1*-inverse of* A, *then* $B = B_1AB_1$ *is a* C_2*-inverse of* A *and any* C_1*-inverse of* B_1 *is a* C_1*-inverse of* B.

PROOF. If $B_1 \epsilon C_1(A)$ then $B_1AB_1 = B \epsilon C_2(A)$ is given by Lemma 4. Let $Q \epsilon C_1(B_1)$ then $BQB = B_1AB_1QB_1AB_1 = B_1AB_1AB_1 = B_1AB_1 = B$ and hence $Q \epsilon C_1(B)$.

In Theorems 2 and 3 it has been shown for $r = p - q$ that the existence of a K such that M^{-1} and U^{-1} exist implies the existence of a B such that B is an EPq matrix and $B \epsilon C_2(A)$. If it is shown that such a K exists when A is an arbitrary p-square matrix of rank q, then we will have the conclusion that every square matrix possesses a C_2-inverse which is an EP matrix. The next theorem shows this to be the case.

Let X be the first block row of M, $X = (A, K)$ and Y the first block column of M, $Y^* = (A^*, K)$. It is clear that $\rho(X) = p$ if and only if $S^* = 0$, for $N(X^*) = S^*$. Similarly $\rho(Y) = p$ if and only if $S = 0$, since $N(Y) = S$. By Theorem 1, $\rho(X) = \rho(Y) = p$ is necessary and sufficient for M^{-1} to exist and necessary and sufficient for U^{-1} to exist. This established, we need only to have the following theorem, called to the author's attention by John W. Evans (Mathematical Research Branch, NIAMD, NIH) who kindly furnished the proof which follows:

THEOREM 4. *Let* A *be a* p-*square matrix*, $\rho(A) = q < p$, $p - q = r$. *Then there exists a* $p \times r$ *matrix* K *such that* $X = (A, K)$ *and* $Y = (A^*, K)$ *have rank* p.

PROOF: We first show that given any two *proper* subspaces T and L, it is possible to select a vector x such that $x \notin T$ and $x \notin L$. In the group theoretic context this result is well known, viz, the union of two proper subgroups of a given group cannot be that group. Let $\mathscr{E}$ be the set of all vectors x such that either $x \in T$ or $x \in L$. There are two cases to be considered. First, either $T < L$ or $L < T$ and then there are certainly vectors not in $\mathscr{E}$. Second $T \cdot L < T$ and $T \cdot L < L$. In this case let $\mathscr{L}_T$ be the set of all vectors x such that $x \in T$ and $x \notin T \cdot L$ and let $\mathscr{L}_L$ be the set of all vectors x such that $x \in L$ and $x \notin T \cdot L$. Now let $x \in \mathscr{L}_T$ and $y \in \mathscr{L}_L$. Then $z = x + y$ is not in $\mathscr{E}$, since for example if $z \in T$ then $z - x = y \in T$ which contradicts $y \in \mathscr{L}_L$.

Let $a_1, a_2, \ldots, a_p$ be the columns of A and $\alpha_1, \alpha_2, \ldots, \alpha_p$ the columns of A^*. Define $T_0 = \{a_1, a_2, \ldots, a_p\}$ to be the subspace spanned by the a_i, $1 \leq i \leq p$, and when appropriate vectors $k_1, k_2, \ldots, k_j$ have been selected define $T_j = \{a_1, a_2, \ldots, a_p, k_1, k_2, \ldots, k_j\}$, $1 \leq j \leq r$, to be subspaces spanned by the vectors $a_1, \ldots, a_p, k_1, k_2, \ldots, k_j$. Similarly define $L_0 = \{\alpha_1, \alpha_2, \ldots, \alpha_p\}$ and $L_j = \{\alpha_1, \alpha_2, \ldots, \alpha_p, k_1, k_2, \ldots, k_j\}$, $1 \leq j \leq r$. Select a vector k_1 such that $k_1 \notin T_0$ and $k_1 \notin L_0$; select k_2 such that $k_2 \notin T_1$ and $k_2 \notin L_1$; continue this process up to the selection of k_r such that $k_r \notin T_{r-1}$ and $k_r \notin L_{r-1}$. Now this selection process is always possible, for clearly $\dim T_0 = \dim L_0 = q$ and at each stage of the process $\dim T_j = \dim L_j = q + j < p$ for $0 \leq j \leq r - 1$. Assuming the above selection process completed, then $\dim T_r = \dim L_r = p$ and $\rho(X) = \rho(Y) = p$ as asserted.

The following observation is due to A. J. Goldman (National Bureau of Standards): Since the proof of Theorem 4 nowhere makes use of the assumption that the α_i are the columns of A^*, we have in fact proved that if A and C are p-square matrices of rank $q < p$ and $r = p - q$, then there exists a $p \times r$ matrix K such that $[A, K]$ and $[C, k]$ have rank p.

4. References

[1] A. J. Goldman and M. Zelen, Weak generalized inverses and minimum variance linear unbiased estimation, J. Res. NBS **68B** (Math. and Math. Phys.) No. 4, 151–172 (1964).

[2] J. Z. Hearon, On the singularity of a certain bordered matrix. Accepted for publication in J. Appl. Math. SIAM.

[3] I. J. Katz and M. H. Pearl, On *EPr* and normal *EPr* matrices, J. Res. NBS **70B** (Math. and Math. Phys.) No. 2, 47–77 (1966).

[4] C. A. Rohde, Some results on generalized inverses, SIAM Rev. **8**, 201–205 (1966).

[5] C. A. Rohde, Contributions to the theory, computation and applications of generalized inverses, Ph.D. Dissertation, North Carolina State University, Raleigh, 1964.

(Paper 71B2&3–198)