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Poincaré's Conjecture Is Implied by a Conjecture on Free Groups*

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Poincaré's conjecture is implied by a single group-theoretic conjecture. The converse is also valid modulo a hypothesis on the uniqueness of decomposition of the 3-sphere as the union of two handlebodies intersecting in a torus.

Key Words: Free group, handlebody, homeomorphism, 3-manifold, simply connected, 3-sphere, topology.

1. Introduction

We are concerned here with Poincaré's conjecture for dimension three: every simply connected closed [1] 3-manifold is a 3-sphere; i.e., a homotopy 3-sphere is a 3-sphere (Papakyriakopoulos [9],[2] p. 250). The main result of this paper is that a purely algebraic conjecture implies the Poincaré conjecture. The converse situation is also considered: Poincaré's conjecture implies our algebraic conjecture modulo another topological hypothesis.

This work was inspired both by the papers of Dr. Papakyriakopoulos and by a remark he made to the author, that more must be learned about free groups in order to understand the Poincaré conjecture.

The notation is as follows: $(w_1, \ldots, w_n)$ denotes the group freely generated by $w_1, \ldots, w_n$; $\langle w_1, \ldots, w_n\rangle$ denotes the normal subgroup generated by $w_1, \ldots, w_n$ (i.e., the smallest normal subgroup containing the w_i); in cases where confusion can arise, we shall write $\langle w_1, \ldots, w_n\rangle\, G$, meaning the smallest normal subgroup of the group G containing the w_i; $[x_i, y_i]$ means $x_i y_i x_i^{-1} y_i^{-1}$.

We shall prove, then, that Poincaré's conjecture is implied by:

CONJECTURE 1: *Suppose* n *is any positive integer. Let* μ *be an isomorphism,* μ: $(x_1, \ldots, x_n, y_1, \ldots, y_n) \to (\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n)$. *Let* ν_1 *and* ν_2 *be homomorphisms,* ν_1: $(x_1, \ldots, x_n, y_1, \ldots, y_n) \to (X_1, \ldots, X_n)$, ν_2: $(\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n) \to (A_1, \ldots, A_n)$. *Suppose that:*

(1) μ *induces an isomorphism* ν: $\left\langle \prod_{i=1}^{n} [x_i, y_i] \right\rangle \to \left\langle \prod_{i=1}^{n} [\alpha_i, \beta_i] \right\rangle$.

(2) $\nu_1(x_r) = X_r$, $\nu_1(y_r) = 1$, $\nu_2(\alpha_r) = A_r$, $\nu_2(\beta_r) = 1$; $(r = 1, \ldots, n)$.

(3) $(A_1, \ldots, A_n) = \langle \nu_2\mu(y_1), \ldots \nu_2\mu(y_n)\rangle$;
$(X_1, \ldots, X_n) = \langle \nu_1\mu^{-1}(\beta_1), \ldots, \nu_1\mu^{-1}(\beta_n)\rangle$.

Then μ *differs by an inner automorphism of* $(\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n)$ *from an isomorphism* ϕ *for which it is possible to choose* $\tilde{\alpha}_1, \ldots, \tilde{\alpha}_n \in (\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n)$ *and* $\tilde{x}_1, \ldots, \tilde{x}_n \in (x_1, \ldots, x_n, y_1, \ldots, y_n)$ *so that*

(4) $(\nu_1(\tilde{x}_1), \ldots, \nu_1(x_n)) = (X_1, \ldots, X_n)$, $(\nu_2(\tilde{\alpha}_1), \ldots, \nu_2(\tilde{\alpha}_n)) = (A_1, \ldots, A_n)$, $\nu_1\phi^{-1}(\tilde{\alpha}_r) = 1$, $\nu_2\phi(\tilde{x}_r) = 1$; $(r = 1, \ldots, n)$.

(5) $(\tilde{\alpha}_1, \ldots, \tilde{\alpha}_n, \phi(\tilde{x}_1), \ldots, \phi(\tilde{x}_n)) = (\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n)$.

(6) $\left\langle \prod_{i=1}^{n} [\tilde{\alpha}_i, \phi(\tilde{x}_i)] \right\rangle = \left\langle \prod_{i=1}^{n} [\alpha_i, \beta_i] \right\rangle$.

The proof depends on the following theorem:

THEOREM 1: (Handlebody theorem) *Consider* T *and* T′, *spheres with* n *handles bounding handlebodies* Q *and* Q′ *respectively. Let* h *be a homeomorphism,* h : T → T′. *A necessary and sufficient condition for* h *to have an extension to a homeomorphism* $\bar{h}$: Q → Q′, *is that* $h_{\#}{}^3$ *maps kernel* $\nu_{\#}$: $\pi_1(T) \to \pi_1(Q)$ *onto kernel* $\eta_{\#}$: $\pi_1(T') \to \pi_1(Q')$; *where* $\nu_{\#}$ *and* $\eta_{\#}$ *are induced by inclusions.*

This theorem was observed by Smale (Neuwirth [4]). A proof is contained in McMillan's demonstration of the following theorem (McMillan [3]).

THEOREM 1′: *Let* H *be a solid torus (handlebody). Then any two sets of generators for* π_1(H) *are equivalent. (A set of generators for* π_1(H) *is a set of simple closed disjoint curves in the boundary of* H *which generate* π_1(H). *Two sets are equivalent if there is an autohomeomorphism of* H *carrying one set onto the other.)*

A proof of Theorem 1 is included in the appendix.

Everything in this paper is taken from the semilinear point of view.

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[1] Closed means compact without boundary.
[2] Figures in brackets indicate the literature references at the end of this paper.

2. Algebraic Preliminaries

A. By a well-known result (Seifert and Threlfall [10], p. 219) every closed orientable 3-manifold M^3 has a Heegard splitting: that is, M^3 is the union of two handlebodies, H_1 and H_2 each of genus n, with their boundaries, T_1 and T_2, identified by a homeomorphism $h : T_1 \to T_2$. Thus, Poincaré's conjecture is true if and only if, for each positive integer n, given two handlebodies of genus n and a homeomorphism between their boundaries, then the resulting identification space M^3 is simply connected implies M^3 is a 3-sphere. That this is the case for $n=1$ is a well-known fact (Papakyriakopoulos [9], pp. 251–252).

For the rest of this paper, n is to be a fixed positive integer.

Let S be a 3-sphere, $S=Q_1 \cup Q_2$, $Q_1 \cap Q_2 = T$; where Q_1 and Q_2 are handlebodies of genus n and T is a torus of genus n.

We choose sets of simple circuits $\{x_1, \ldots, x_n, y_1, \ldots, y_n\}$, $\{\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n\}$, and $\{\gamma_1, \ldots, \gamma_n, \delta_1, \ldots, \delta_n\}$ having the following properties:

(1) $x_i, y_i \subset T_1$; $\alpha_i, \beta_i \subset T_2$; $\gamma_i, \delta_i \subset T$; $(i=1, \ldots, n)$.

(2) y_i is contractible in H_1; β_i is contractible in H_2; γ_i is contractible in Q_2; δ_i is contractible in Q_1; $(i=1, \ldots, n)$.

(3) $x_1 \cup \ldots \cup x_n$ is a deformation retract of H_1; $\alpha_1 \cup \ldots \cup \alpha_n$ is a deformation retract of H_2; $\gamma_1 \cup \ldots \gamma_n$ is a deformation retract of Q_1; $\delta_1 \cup \ldots \cup \delta_n$ is a deformation retract of Q_2.

(4) $\{x_1, \ldots, x_n, y_1, \ldots, y_n\}$ is a set of generators for $\pi_1(T_1) = (x_1, \ldots, x_n, \quad y_1, \ldots, y_n\colon \prod_{i=1}^{n} [x_i, y_i])$. Similarly for T_2 and T. (We regard, say, x_i both as a circuit and as an element of $\pi_1(T_1)$. We also regard, say, x_i as a member either of the fundamental group of the torus or as a member of the "overlying" free group, i.e., the group freely generated by the standard generators of the fundamental group.) $x_1 \cap \ldots \cap x_n \cap y_1 \cap \ldots \cap y_n = \{p\}$, a single point. Similarly for T_2 and T.

See Papakyriakopoulos ([7, 9], p. 260).

Let $i_p : T_p \to H_p$ and $j_q : T \to Q_q$ denote inclusions $(p, q=1, 2)$.

Let $i_1(x_r)=X_r$, $i_2(\alpha_r)=A_r$; $(r=1, \ldots, n)$. Then $\{X_1, \ldots, X_n\}$ and $\{A_1, \ldots, A_n\}$ freely generate $\pi_1(H_1)$ and $\pi_1(H_2)$ respectively. (Again X_r and A_r denote the circuit and the element of the fundamental group.)

F_1, F_2, G_1, and G_2 shall denote the free groups $(X_1, \ldots, X_n)$, $(A_1, \ldots, A_n)$, $(x_1, \ldots, x_n, y_1, \ldots, y_n)$, and $(\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n)$ respectively.

B. Let $h_\#$ be the isomorphism from $\pi_1(T_1)$ to $\pi_1(T_2)$ induced by h. Then $h_\#$ is induced by an isomorphism $\mu\colon G_1 \to G_2$; (Nielsen [5], Zieschang [13]). Clearly μ establishes an isomorphism ν: $\left\langle \prod_{i=1}^{n} [x_i, y_i] \right\rangle \to \left\langle \prod_{i=1}^{n} [\alpha_i, \beta_i] \right\rangle$.

[3] $h_\#$ is the induced homomorphism of fundamental groups.

We have the following diagram:

$$\begin{array}{ccccc} & G_1 & \to & G_2 & \\ \nu_1 & \downarrow & & \downarrow & \nu_2 \\ & F_1 & & F_2 & \end{array}$$

where ν_1 and ν_2 are induced by $i_{1\#}$ and $i_{2\#}$; that is, $\nu_1(x_r)=X_r$, $\nu_1(y_r)=1$, $\nu_2(\alpha_r)=A_r$, $\nu_2(\beta_r)=1$; $(r=1, \ldots, n)$.

C. By van Kampen's theorem (van Kampen [12], Hilton and Wylie [1], p. 243) a presentation of $\pi_1(M^3)$ is $(X_1, \ldots, X_n, \quad A_1, \ldots, A_n : \quad i_{1\#}(x_1) = i_2\, h_\#(x_1), \ldots, i_{1\#}(x_n) = i_{2\#}h_\#(x_n), \quad i_{1\#}(y_1) = i_{2\#}h_\#(y_1), \ldots, i_{1\#}(y_n) = i_{2\#}h_\#(y_n))$ or $(X_1, \ldots, A_n : X_1 = \nu_2\mu(x_1), \ldots, X_n = \nu_2\mu(x_n), 1 = \nu_2\mu(y_1), \ldots, 1 = \nu_2\mu(y_n))$.

If this group is trivial, i.e., *if M^3 is simply connected*, X_r and A_r must vanish as a consequence of the $2n$ relations $(r=1, \ldots, n)$. Since $\nu_2\mu(x_j)$ and $\nu_2\mu(y_j)$ are words in the A_r, it follows that $F_2 = \langle \nu_2\mu(y_1), \ldots, \nu_2\mu(y_n)\rangle$. Thus ν_2 carries $\langle \mu(y_1), \ldots, \mu(y_n)\rangle G_2$ onto F_2.

Similarly, $F_1 = \langle \nu_1\mu^{-1}(\beta_1), \ldots, \nu_1\mu^{-1}(\beta_n)\rangle$; and so ν_1 carries $\langle \mu^{-1}(\beta_1), \ldots, \mu^{-1}(\beta_n)\rangle G_1$ onto F_1.

D. Let $\tilde{x}_j$ be such that $\nu_1(\tilde{x}_j) = X_j$ $(j=1, \ldots, n)$; $\tilde{x}_j = \zeta_j x_j$ where $\zeta_j \in$ kernel $\nu_1 = \langle y_1, \ldots, y_n \rangle G_1$. Thus $\mu(\tilde{x}_j) = \mu(\zeta_j) \cdot \mu(x_j) = \bar{\zeta}_j \cdot \mu(x_j)$, where $\bar{\zeta}_j \in \langle \mu(y_1), \ldots, \mu(y_n)\rangle G_2$.

By C and noting that (i) ζ_j is an arbitrary element of $\langle y_1, \ldots, y_n \rangle G_1$ and (ii) μ is an isomorphism, we can choose ζ_j so that $\nu_2\mu(\tilde{x}_j) = 1$.

Similarly, we can find $\theta_j \in \langle \beta_1, \ldots, \beta_n \rangle G_2$; $\tilde{\alpha}_j = \theta_j \alpha_j$; $\nu_1\mu^{-1}(\tilde{\alpha}_j) = 1$; $\nu_2(\tilde{\alpha}_j) = A_j$.

E. The argument of D is unchanged if we replace μ by ϕ, an isomorphism from G_1 to G_2 differing from μ by an inner automorphism of G_2. Hence, we can find for any such ϕ, elements $\tilde{\alpha}_j$ and $\tilde{x}_j$ satisfying (4) of Conjecture 1.

3. The Main Result

THEOREM 2: Poincaré's conjecture is true if Conjecture 1 is.

PROOF: In the light of section 2, it is sufficient to show that M^3 considered above is a 3-sphere if it is possible, for some ϕ differing from μ by an inner automorphism, to choose $\tilde{x}_j$ and $\tilde{\alpha}_j$ $(j=1, \ldots, n)$ so that they satisfy not only (4) of Conjecture 1, but in addition (5) $(\tilde{\alpha}_1, \ldots, \tilde{\alpha}_n, \quad \phi(\tilde{x}_1), \ldots, \phi(\tilde{x}_n)) = (\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n)$; and (6) $\left\langle \prod_{i=1}^{n} [\tilde{\alpha}_i, \phi(\tilde{x}_i)] \right\rangle = \left\langle \prod_{i=1}^{n} [\alpha_i, \beta_i] \right\rangle$.

Suppose then that this is the case. The correspondences $\tilde{\alpha}_j \to \delta_j$, $\phi(\tilde{x}_j) \to \gamma_j$ will induce an isomorphism σ: $\pi_1(T_2) \to \pi_1(T)$. Since $\nu_2\phi(\tilde{x}_j) = 1$, it follows that kernel $\nu_2 = \langle \phi(\tilde{x}_1), \ldots, \phi(\tilde{x}_n)\rangle$. Thus kernel $i_{2\#} = \langle h_\#(\tilde{x}_1), \ldots, h_\#(\tilde{x}_n)\rangle$. Similarly, kernel $i_{1\#} = \langle h_\#^{-1}(\alpha_1), \ldots, h_\#^{-1}(\tilde{\alpha}_n)\rangle$. Hence σ establishes an isomorphism between kernel $i_{2\#}$ and kernel $j_{2\#} = \langle \gamma_1, \ldots, \gamma_n \rangle$. Now if σ is not induced by a homeomorphism, there is an inner automorphism τ of $\pi_1(T)$ so that $\tau\sigma$ is induced by a homeomorphism (Nielsen [5], Zieschang [13]). $\tau\sigma$ still carries kernel $i_{2\#}$ onto kernel $j_{2\#}$ and $\tau\sigma h_\#$ still carries kernel $i_{1\#}$ onto kernel $j_{1\#}$. Hence, by the handlebody theorem (see appendix), the homeomorphism k: $T_2 \rightarrow T$ inducing $\tau\sigma$ has the properties (i) k extends to a homeomorphism k_2: $H_2 \rightarrow Q_2$.

(ii) kh extends to a homeomorphism

k_1: $H_1 \rightarrow Q_1$.

Thus, M^3 is a 3-sphere.

This completes the proof.

REMARK: This result seems strongly related to results of Papakyriakopoulos ([9]). It is possible that topological methods may need to be introduced in order to demonstrate Conjecture 1. (Lickorish's topological classification of the isotopy classes of autohomeomorphisms of the closed orientable 2-manifolds suggests itself. See [2].) There seems to be a connection between finding a simple loop L in T_2, such that (i) L is not contractible in T_2 (ii) L is contractible in H_2 (iii) $h^{-1}(L)$ is contractible in H_1, and finding the $\tilde{\alpha}_j$ and $\tilde{x}_j$ having the properties required by Conjecture 1. The existence of such an L implies that M^3 is a 3-sphere. See [9], p. 252.

4. The Converse Result

We shall now prove a *weakened* converse to Theorem 2.

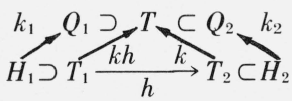
THEOREM 3: *Poincaré's conjecture implies Conjecture 1 if the following hypothesis is true:*

CONJECTURE 2: *Let* N *be a 3-sphere. Let* Y_1, Y_2, Z_1, Z_2 *be handlebodies of genus* n. *Suppose* $N = Y_1 \cup Y_2 = Z_1 \cup Z_2$; *where* $Y_1 \cap Y_2 = U$ *and* $Z_1 \cap Z_2 = V$, U *and* V *tori of genus* n. *Then there is an autohomeomorphism* f of N *such that* $f(U) = V$. (See Papakyriakopoulos [8], p. 330.)

PROOF: We note that any isomorphism μ satisfying the hypotheses of Conjecture 1 can be considered in the topological context of section 2 (whose notation we continue to use). This means that we take μ as inducing an isomorphism ψ of the fundamental groups of H_1 and H_2. ψ differs from an isomorphism $h_\#$ (induced by some homeomorphism h) by an inner automorphism of $\pi_1(H_2)$. Hence we can choose ϕ: $G_1 \rightarrow G_2$ inducing $h_\#$ and differing from μ by an inner automorphism of G_2.

Suppose that Poincaré's conjecture is true. Then hypothesis (3) of Conjecture 1 implies that M^3 considered in section 2 is simply connected and thus is a 3-sphere. If in addition Conjecture 2 is true, there is a homeomorphism k: $T_2 \rightarrow T$ having the properties (i) k extends to a homeomorphism $k_2 : H_2 \rightarrow Q_2$ (ii) kh extends to a homeomorphism $k_1 : H_1 \rightarrow Q_1$.

Hence we have the commutative diagram of inclusions and homeomorphisms:



Take $\tilde{\alpha}_r = k^{-1}(\delta_r)$ and $h(\tilde{x}_r) = k^{-1}(\gamma_r)$; $(r = 1, \ldots, n)$. Then conclusions (5) and (6) follow trivially.

Now $k_\#^{-1}$ carries kernel $j_{2\#} = \langle \gamma_1, \ldots, \gamma_n \rangle$ onto kernal $i_{2\#}$. Thus $i_{2\#} h_\#(\tilde{x}_r) = 1$, implying that $\nu_2\phi(\tilde{x}_r) = 1$. On the other hand, $\gamma_1 \cup \ldots \cup \gamma_n$ is a deformation retract of Q_1; so $\tilde{x}_1 \cup \ldots \cup \tilde{x}_n$ is a deformation retract of H_1. Hence $(\nu_1(\tilde{x}_1), \ldots, \nu_1(\tilde{x}_n)) = (X_1, \ldots, X_n)$. Similarly $\nu_1\phi^{-1}(\tilde{\alpha}_r) = 1$ and $(\nu_2(\tilde{\alpha}_1), \ldots, \nu_2(\tilde{\alpha}_n)) = (A_1, \ldots, A_n)$. Hence, conclusion (4) is also satisfied.

This completes the proof.

REMARK: Conjecture 2 appears to be almost as difficult to prove as Poincaré's conjecture itself. However, it should be apparent from the proof of Theorem 2 that Conjecture 1 implies Conjecture 2 *as well as* the Poincaré conjecture.

5. Appendix

We prove here the handlebody theorem (see introduction).

THEOREM 1: *Consider* T *and* T′, *spheres with* n *handles bounding handlebodies* Q *and* Q′. *Let* h *be a homeomorphism*, h: $T \rightarrow T'$. *A necessary and sufficient condition for* h *to have an extension to a homeomorphism*, $\tilde{h}$: $Q \rightarrow Q'$, *is that* $h_\#$ *maps kernel* $\nu_\#$: $\pi_1(T) \rightarrow \pi_1(Q)$ *onto kernel* $\eta_\#$: $\pi_1(T) \rightarrow \pi_1(Q')$ *where* $\nu_\#$ *and* $\eta_\#$ *are induced by inclusions.*

PROOF:

Necessity.

This follows immediately from the commutative diagram:

$$\begin{array}{ccc} \pi_1(T) & \xrightarrow{h_\#} & \pi_1(T') \\ \downarrow \nu_\# & & \downarrow \eta_\# \\ \pi_1(Q) & \xrightarrow[\tilde{h}_\#]{} & \pi_1(Q') \end{array}$$

Sufficiency.

Choose a set of generators $\{x_1, \ldots, x_n, y_1, \ldots, y_n\}$ for $\pi_1(T)$ such that:

(1) x_r, y_r are simple; $(r = 1, \ldots, n)$.

(2) $x_1 \cup \ldots \cup x_n$ is a deformation retract of Q.

(3) y_r is contractible in Q; $(r = 1, \ldots, n)$.

(4) $\pi_1(T) = (x_1, \ldots, x_n, y_1, \ldots, y_n : \prod_{i=1}^{n} [x_i, y_i])$.

Consider the case $n = 1$. Let p be a homeomorphism of $S^1 \times I$ into T such that $p(S \times \{0\}) = y_1$. Let $\bar{y}_1$ denote $p(S \times \{1\})$. Both y_1 and $\bar{y}_1$ are simple and contractible in Q. Hence, by Dehn's lemma (Papakyriakopoulos [6], Shapiro and Whitehead [11]), they bound nonsingular disks D and $\bar{D}$, respectively, in Q. We can choose these so that $D \cap T = y_1$, $\bar{D} \cap T = \bar{y}_1$. Now by Papakyriakopoulos ([9], p. 263, 6.2), if we cut along D, we obtain a 3-cell. It follows easily that we can choose $\bar{D}$ so that $D \cap \bar{D} = \phi$; from this it follows that $Q - T - D - \bar{D}$ consists of two disjoint open 3-cells. By hypothesis, $h(y_1)$ is contractible in Q', and it follows easily that $h(\bar{y}_1)$ is also. Hence, they bound nonsingular disks D' and $\bar{D}'$ in Q', which by analogy with the preceding, we can take so that $D' \cap \bar{D}' = \phi$, $D' \cap T' = h(y_1)$, $\bar{D}' \cap T' = h(\bar{y}_1)$. Again, by the same argument as before, $Q' - T' - D' - \bar{D}'$ consists of two disjoint open

3-cells. Clearly h extends to a homeomorphism $\bar{h}: T \cup D \cup \bar{D} \to T' \cup D' \cup \bar{D}'$. Since a homeomorphism on the boundary of a 3-cell extends to a homeomorphism of the 3-cell itself, we have the required result for $n=1$.

Assume by induction that we have proved the result for $n=1, \ldots, m$. Let $n=m+1$.

$L=[x_1, y_1]$ is contractible in Q, and L has a simple representative, which we also denote by L. By Dehn's lemma, L bounds a disk D in Q (which we can take so that $D \cap T = L$). L is not contractible in T, but is homologous to zero in T; hence $T-L$ is disconnected. By Papakyriakopoulos ([9], corollary 6.3, p. 263), Q is the union of two handlebodies, H_1 and H_2, intersecting in D, of genus p and q respectively, $p+q=m+1$ $(p, q \neq 0)$.

By hypothesis, $h(L)$ bounds a disk D' in Q', which we can take to be nonsingular, $D' \cap T' = h(L)$. By reasoning similar to the above, Q' is the union of two handlebodies, H_1' and H_2', intersecting in D', of genus p' and q' respectively.

Let the boundaries of H_i and H_i' be T_i and T_i'. Then, since h is a homeomorphism and since the genus of a 2-dimensional surface is unchanged by removing a disk, we can take the genus of T_i equal to the genus of T_i', i.e., $p=p'$ and $q=q'$.

h can be extended to a homeomorphism $\bar{h}: T \cup D \to T' \cup D'$. $\bar{h}$ splits into homeomorphisms $h_i: T_i \to T_i'$ agreeing on the intersection. Letting $\nu_i: T_i \to H_i$ and $\eta_i: T_i' \to H_i'$ denote inclusions, it is easy to see that $h_{i\#}$ carries kernel $\nu_{i\#}$ onto kernel $\eta_{i\#}$. Hence, by the inductive hypothesis, each h_i extends to a homeomorphism. This gives the required extension of h.

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