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Indefinite Integrals Involving Bessel Functions

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Expressions are derived for the indefinite integrals,

$$\int rf(r)\ C_0(\alpha r)dr$$
$$\int rf(r)\ C_0^2(\alpha r)dr$$
$$\int rf(r)\ C_0(\alpha r)\ C_0(\beta r)dr \qquad \alpha\neq\beta$$
$$\int rf(r)\ C_0(\alpha r)\ Z_0(\lambda r)dr$$

where $C_0(\alpha r)$ are zero order Bessel functions, $Z_0(\lambda r)$ are zero order modified Bessel functions and $f(r)$ is a polynomial in r. In general, the expressions given for the integrals are given in terms of prescribed functions of the Bessel functions, and the coefficients of these functions are determined from a finite series, the terms of which are found from recurrence relationships that involve only the polynomial $f(r)$. Coefficients of the terms of the finite series are given in tabular form for up to an eleventh degree polynomial.

Key Words: Bessel functions, indefinite integrals.

1. Introduction

In this paper four indefinite integrals are studied; namely,

$$\int rf(r)\ C_0(\alpha r)dr \tag{1}$$

$$\int rf(r)\ C_0^2(\alpha r)dr \tag{2}$$

$$\int rf(r)\ C_0(\alpha r)\ C_0(\beta r)dr \qquad \alpha\neq\beta \tag{3}$$

$$\int rf(r)\ C_0(\alpha r)\ Z_0(\lambda r)dr, \tag{4}$$

where $f(r)$ is a pth degree polynomial in r,

$$f(r)=\sum_{n=0}^{p} a_n r^n,$$

$C_0(\alpha r)=AJ_0(\alpha r)+BY_0(\alpha r)$, and $Z_0(\lambda r)=DI_0(\lambda r)+EK_0(\lambda r)$. $J_0(\alpha r)$ and $Y_0(\alpha r)$ are zero order Bessel functions of the first and second kind, respectively, and $I_0(\lambda r)$ and $K_0(\lambda r)$ are zero order modified Bessel functions of the first and second kind, respectively. Expressions for integrals (1), (2), and (3) can be found in Watson [1][1] and Luke [2] which must be solved individually for every term of the polynomial $f(r)$ by rather cumbersome recursion formulas involving the Bessel functions. The expressions given here for the integrals are given in terms of prescribed functions

[1] Figures in brackets indicate the literature references at the end of this paper.

of the Bessel functions, and the coefficients of these functions are determined from a finite series, the terms of which are found from recurrence relationships that involve only the polynomial $f(r)$. Butler and Pohlhausen [3] have provided solutions of (1), (2), and (3) as definite integrals that specifically apply to solutions where $J_0(\alpha)=J_0(\beta)=0$. A more general application is needed. A method, not hitherto available in the literature, is presented here to enable a user to determine these indefinite integrals involving Bessel functions. Further comment in evaluation of the method is given at the end of the paper.

An example of the need for solutions to the integrals (1–4) is illustrated in the study of physical problems involving the steady-state diffusion process in a solid of cylindrical geometry, where a particular partial differential equation to be satisfied is

$$\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}+\frac{\partial^2 v}{\partial z^2}=0, \tag{5}$$

v is the potential, and r and z are the radial and longitudinal dimensions, respectively. A solution of (5) for the cylinder of finite length and outer radius is of the form

$$v=C_0(\alpha r)\begin{Bmatrix}\sinh \alpha z\\ \cosh \alpha z\end{Bmatrix}+Z_0(\lambda r)\begin{Bmatrix}\sin \lambda z\\ \cos \lambda z\end{Bmatrix}, \tag{6}$$

where, for the solid cylinder $B=E=0$, or $C_0(\alpha r)=AJ_0(\alpha r)$ and $Z_0(\lambda r)=DI_0(\lambda r)$, and final solution of (6) is dependent upon the evaluation of A, B, D, and E from the boundary conditions on the curved and plane surfaces of the cylinder. For rather simple boundary conditions, the constants can readily be determined by use of the orthogonality properties of the Bessel and trigonometric functions.

More difficult boundary conditions on one or more of the plane and curved surfaces of the cylinder may take the general forms

$$\frac{dv}{dz}=H(r)\,[v-V(r)] \text{ on a plane surface} \tag{7}$$

$$\frac{dv}{dr}=h(z)\,[v-V(z)] \text{ on a curved surface} \tag{8}$$

where the proportionality factors $H(r)$ and $h(z)$, and the potentials of the ambient $V(r)$ and $V(z)$ are position dependent. To obtain values for the constants of (6) upon the substitution of (6) and its first derivatives in (7) and (8), it would be convenient to use relationships such as

$$\lambda \cos \lambda+h(0) \sin \lambda=0 \tag{9}$$

and

$$\alpha C_1(\alpha)-H(0)C_0(\alpha)=0 \tag{10}$$

and their related orthogonality properties, and positive roots. For the determination of the constants A and B of (6), expressions are necessary for the integrals (1–4) with appropriate limits of integration assigned.

The reason for the specific boundary conditions cited above is to further and to ameliorate the study of radiant heat transfer and other nonlinear boundary conditions over nonisopotential surfaces in conjunction with heat conduction and other diffusion processes.

2. Evaluation of $\int rf(r)\, C_0(\alpha r)\,dr$

From expressions for the integral (1) for the special cases $f(r)=r^m$, $m=1, 2, 3$, and 4, derived by integration by parts, it can be shown by induction that a general expression for (1) is of the form

$$\int rf(r)\, C_0(\alpha r)dr=\frac{r}{\alpha}\left[F(r)\, C_1(\alpha r)+\frac{F'(r)\, C_0(\alpha r)}{\alpha}\right]-\frac{F'(0)}{\alpha^2}\int C_0(\alpha r)dr \tag{1a}$$

where the polynomial $F(r)$ defined by this relation is to be determined in the subsequent analysis, and

$$C_1(\alpha r) = AJ_1(\alpha r) + BY_1(\alpha r),$$

$$\int C_0(\alpha r)dr = rC_0(\alpha r) + \frac{\pi}{2}\left[C_1(\alpha r)\, H_0(\alpha r) - C_0(\alpha r)\, H_1(\alpha r)\right]r \qquad \text{(1b)}$$

and H_0 and H_1 are Struve functions of the first kind. Differentiation of (1a) gives

$$F''(r) + \frac{F'(r) - F'(0)}{r} + \alpha^2 F(r) = \alpha^2 F(r). \qquad (11)$$

The differential equation (11) is satisfied by the following relationships:

$$F(r) = f(r) + \sum_{n=2} \frac{(-1)^n P_n(r)}{\alpha^{2n}}$$

where for $n=1$

$$P_1(r) = f''(r) + \frac{f'(r) - f'(0)}{r}$$

and for $n > 1$

$$P_n(r) = P''_{n-1}(r) + \frac{P'_{n-1}(r) - P'_{n-1}(0)}{r}.$$

In terms of the coefficients a_m of a pth degree polynomial

$$P_n = \sum_{m=2n}^{p} a_m r^{m-2n} R_{m,\,n}$$

$$R_{m,\,n} = \prod_{s=1}^{n} [m - 2s + 2]^2, \; n \leq \frac{m}{2}$$

Table 1 shows values of $R_{m,\,n}$ for values of m from 1 to 11. For the sake of example, let $f(r) = r^7$, then

$$F(r) = r^7 - \frac{49r^5}{\alpha^2} + \frac{1225r^3}{\alpha^4} - \frac{11{,}025r}{\alpha^6},$$

and

$$F'(r) = 7r^6 - \frac{245r^4}{\alpha^2} + \frac{3675r^2}{\alpha^4} - \frac{11{,}025}{\alpha^6}$$

are determined for substitution in (1a).

TABLE 1. *Values of* $R_{m,\,n}$

m \ n =	1	2	3	4	5
1	0				
2	4	0			
3	9	0			
4	16	64	0		
5	25	225	0		
6	36	576	2,304	0	
7	49	1225	11,025	0	
8	64	2304	36,864	147,456	0
9	81	3969	99,225	893,025	0
10	100	6400	230,400	3,686,400	14,745,600
11	121	9801	480,249	12,006,225	108,056,025

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An interesting sidelight is the substitution of $Z_0(\lambda r)$ for $C_0(\alpha r)$ in (1a), which gives

$$\int rf(r)Z_0(\lambda r)dr=\frac{r}{\lambda^2}[\lambda Z_0'(\lambda r)F_1(r)-Z_0(\lambda r)F_1'(r)]+\frac{F_1'(0)}{\lambda^2}\int Z_0(\lambda r)dr \tag{12}$$

where

$$F_1(r)=f(r)+\sum_{n=1}\frac{P_n(r)}{\lambda^{2n}}$$

and $P_n(r)$ is as defined for (1a).

3. Evaluation of $\int rf(r)C_0^2(\alpha r)dr$

An expression for (2) can be written in the form

$$\int rf(r)C_0^2(\alpha r)dr=\frac{r^2}{2}\left\{G(r)[C_0^2(\alpha r)+C_1^2(\alpha r)]+\frac{G'(r)}{\alpha}C_0(\alpha r)C_1(\alpha r)\right.$$

$$\left.+\frac{1}{2\alpha^2}\left[G''(r)+\frac{1}{r}G'(r)\right]C_0^2(\alpha r)\right\}-\frac{G'(0)}{4\alpha^2}\int C_0^2(\alpha r)dr \tag{2a}$$

where the polynomial $G(r)$ is to be determined in the subsequent analysis. The integral $\int J_0^2(\alpha r)dr$ is given in (23), for $k=1$. Differentiation of both sides of (2a) gives

$$rG'(r)+G(r)=f(r)-\frac{1}{4\alpha^2}\left[rG'''(r)+3G''(r)+\frac{G'(r)-G'(0)}{r}\right]. \tag{13}$$

The expression on the left can be written $rG'(r)+G(r)=\frac{d}{dr}[rG(r)]$, then integration gives

$$G(r)=\frac{1}{r}\int f(r)dr-\frac{1}{4\alpha^2 r}\int\left[rG'''(r)+3G''(r)+\frac{G'(r)-G'(0)}{r}\right]dr. \tag{14}$$

$G(r)$ is satisfied by

$$G(r)=\sum_{n=0}\frac{(-1)^n S_n(r)}{\alpha^{2n}} \tag{15}$$

where for $n=0$

$$S_0(r)=\frac{1}{r}\int f(r)dr$$

and for $n>0$

$$S_n(r)=\frac{1}{4r}\int\left[rS_{n-1}'''(r)+3S_{n-1}''(r)+\frac{S_{n-1}'(r)-S_{n-1}'(0)}{r}\right]dr.$$

Also, (14) is satisfied for $G(r)$ in terms of the coefficients a_m of a pth degree polynomial by

$$G(r)=\sum_{n=0}^{p/2}\frac{(-1)^n}{(2\alpha)^{2n}}\sum_{m=2n}^{p}a_m T_{m,n}r^{m-2n}, \tag{15a}$$

where

$$T_{m,0}=1/(m+1),\text{ and } T_{m,n}=\frac{1}{m+1}\prod_{s=1}^{n}\frac{(m-2s+2)^3}{(m-2s+1)}.$$

Values of $T_{m,n}$ are given in table 2 for m from 0 to 11. For sake of example, let $f(r)=a_0+a_1r+a_2r^2+a_3r^3+a_4r^4$, then from (15a) and table 2,

$$G(r)=a_0+\frac{a_1r}{2}+\frac{a_2r^2}{3}+\frac{a_3r^3}{4}+\frac{a_4r^4}{5}-\frac{\frac{8a_2}{3}+\frac{27a_3r}{8}+\frac{64a_4r^2}{15}}{(2\alpha)^2}+\frac{512a_4}{15(2\alpha)^4}$$

where the first and second differentials of $G(r)$ are necessary for substitution in (2a).

TABLE 2. *Values of* $T_{m,n}$

m \ $n=$	0	1	2	3	4	5
0	1					
1	1/2					
2	1/3	8/3				
3	1/4	27/8				
4	1/5	64/15	512/15			
5	1/6	125/24	1,125/16			
6	1/7	216/35	4,608/35	36,864/35		
7	1/8	343/48	42,875/192	385,875/128		
8	1/9	512/63	12,288/35	262,144/35	2,097,152/35	
9	1/10	729/80	83,349/160	2,083,725/128	56,260,575/256	
10	1/11	1000/99	512,000/693	2,457,600/77	52,428,800/77	419,430,400/77
11	1/12	1331/120	323,433/320	37,979,173/640	924,479,325/512	24,960,941,775/1024

4. Evaluation of $\int rf(r)C_0(\alpha r)C_0(\beta r)dr,\ \alpha\neq\beta$

An expression for (3) may take the form

$$\int rf(r)C_0(\alpha r)C_0(\beta_r)dr=N_1U_1(r)+N_2U_1'(r)-\gamma[N_3U_2(r)+N_4U_2'(r)]$$

$$-[\psi U_1'(0)-\gamma^2U_2'(0)]\int C_0(\alpha r)C_0(\beta r)dr+\gamma[U_1'(0)-\psi U_2'(0)]\int C_1(\alpha r)C_1(\beta r)dr \qquad (3a)$$

where

$$\delta=\alpha^2-\beta^2,\ \psi=(\alpha^2+\beta^2)/\delta^2,\ \gamma=2\alpha\beta/\delta^2,$$

$U_1(r)$ and $U_2(r)$ are polynomials to be determined in the subsequent analysis, and

$$N_1=\int rC_0(\alpha r)C_0(\beta r)dr=\frac{r}{\delta}[\alpha C_1(\alpha r)C_0(\beta r)-\beta C_0(\alpha r)C_1(\beta r)]$$

$$N_3=\int rC_1(\alpha r)C_1(\beta r)dr=\frac{r}{\delta}[\beta C_1(\alpha r)C_0(\beta r)-\alpha C_0(\alpha r)C_1(\beta r)]$$

$$N_2/r=\psi C_0(\alpha r)C_0(\beta r)+\gamma C_1(\alpha r)C_1(\beta r)$$

$$N_4/r=\psi C_1(\alpha r)C_1(\beta r)+\gamma C_0(\alpha r)C_0(\beta r)$$

The integrals $\int J_0(\alpha r)J_0(\beta r)dr$ and $\int J_1(\alpha r)J_1(\beta r)dr$ are found in section 6. Differentiation of both sides of (3a) gives the following equations:

$$U_1(r)+\psi\left[U_1''(r)+\frac{U_1'(r)-U_1'(0)}{r}\right]-\gamma^2\left[U_2''(r)+\frac{U_2'(r)-U_2'(0)}{r}\right]=f(r)$$

$$U_2(r)+\psi\left[U_2''(r)-\frac{U_2'(r)-U_2'(0)}{r}\right]-\left[U_1''(r)-\frac{U_1'(r)-U_1'(0)}{r}\right]=0. \tag{16}$$

Letting

$$U_1(r)=\sum_{k=0}\gamma^{2k}\sum_{n=0}(-1)^n\psi^n L_{k,\,n}(r)$$

and

$$U_2(r)=\sum_{k=0}\gamma^{2k}\sum_{n=0}(-1)^n\psi^n M_{k,\,n}(r),$$

then substitution in (16) gives

$$L_{0,\,0}(r)=f(r)$$

$$L_{k,\,n}(r)=L''_{k,\,n-1}(r)+\frac{L'_{k,\,n-1}(r)-L'_{k,\,n-1}(0)}{r}+M''_{k-1,\,n}(r)+\frac{M'_{k-1,\,n}(r)-M'_{k-1,\,n}(0)}{r}$$

$$M_{k,\,n}(r)=L''_{k,\,n}(r)-\frac{L'_{k,\,n}(r)-L'_{k,\,n}(0)}{r}+M''_{k,\,n-1}-\frac{M'_{k,\,n-1}(r)-M'_{k,\,n-1}(0)}{r} \tag{17}$$

where negative subscripts denote zero quantities. In terms of the coefficients a_m of a pth order polynomial, the quantities of (17) have been evaluated for m from zero to 11 in table 3.

TABLE 3

m	Coefficients of a_m for γ^0						Coefficients of a_m for γ^2				Coefficients of a_m for γ^4	
	r^m $L_{0,0}$	r^{m-2} $L_{0,1}$ $M_{0,0}$	r^{m-4} $L_{0,2}$ $M_{0,1}$	r^{m-6} $L_{0,3}$ $M_{0,2}$	r^{m-8} $L_{0,4}$ $M_{0,3}$	r^{m-10} $L_{0,5}$ $M_{0,4}$	r^{m-4} $L_{1,0}$	r^{m-6} $L_{1,1}$ $M_{1,0}$	r^{m-8} $L_{1,2}$ $M_{1,1}$	r^{m-10} $L_{1,3}$ $M_{1,2}$	r^{m-8} $L_{2,0}$	r^{m-10} $L_{2,1}$ $M_{2,0}$
0	1 0											
1	1 0											
2	1 0	4 0										
3	1 0	9 3										
4	1 0	16 8	64 0				32 0					
5	1 0	25 15	225 120				135 0					
6	1 0	36 24	576 480	2,304 0			384 0	3,456 0				
7	1 0	49 35	1,225 1,260	11,025 7,445			875 0	19,215 2,625				
8	1 0	64 48	2,304 2,688	36,864 39,936	147,456 0		1,728 0	70,656 13,824	442,368 0		55,296 0	
9	1 0	81 63	3,969 5,040	99,225 135,135	893,025 703,080		3,087 0	203,175 46,305	3,044,790 748,440		416,745 0	
10	1 0	100 80	6,400 8,640	230,000 360,960	3,686,400 4,730,880	14,745,600 0	5,120 0	495,360 122,880	13,701,120 4,945,920	73,728,000 0	1,966,080 0	27,648,00 0
11	1 0	121 99	9,801 13,860	480,249 828,135	12,006,225 19,625,760	108,056,025 94,895,955	8,019 0	1,072,071 280,665	47,505,150 20,291,040	604,178,190 203,388,570	7,016,625 0	245,768,985 21,049,875

$$U_1(r)=f(r)-\psi L_{0,1}+\psi^2 L_{0,2}-\ldots+\gamma^2[L_{1,0}-\psi L_{1,1}+\psi^2 L_{1,2}\ldots]+\gamma^4[L_{2,0}-\psi L-\psi L_{2,1}+\ldots] \quad (18)$$

$$U_2(r)=M_{0,0}-\psi M_{0,1}+\psi^2 M_{0,2}-\ldots+\gamma^2[M_{1,0}-\psi M_{1,1}+\ldots]+\gamma^4[M_{2,0}-\psi M_{2,1}+\ldots]. \quad (19)$$

From the table we get

$f(r)=r^2$ $\quad U_1(r)=r^2-4\psi$
$\quad U_2(r)=0$

$f(r)=r^3$ $\quad U_1(r)=r^3-9r\psi$
$\quad U_2(r)=3r$

$f(r)=r^4$ $\quad U_1(r)=r^4-16r^2\psi+64\psi^2+32\gamma^2$
$\quad U_2(r)=8r^2$

$f(r)=r^5$ $\quad U_1(r)=r^5-25r^3\psi+225r\psi^2+135r\gamma^2$
$\quad U_2(r)=15r^3-120r\psi$

$f(r)=r^6$ $\quad U_1(r)=r^6-36r^4\psi+576r^2\psi^2-2304\psi^4+384\gamma^2r^2-3456\gamma^2\psi$
$\quad U_2(r)=24r^4-480r^2\psi$

$f(r)=a_0+a_1r+a_2r^2+a_3r^3$ $\quad U_1(r)=a_0+a_1r+a_2r^2+a_3r^3-(4a_2+9a_3r)\psi$
$\quad U_2(r)=3a_3r.$

5. Evaluation of $\int rf(r)C_0(\alpha r)Z_0(\lambda r)\,dr$

An expression for (4) may take a form similar to (3a)

$$\int rf(r)C_0(\alpha r)Z_0(\lambda r)\,dr=W_1V_1(r)-W_2V_1'(r)+\gamma[W_3V_2(r)-W_4V'(r)]$$
$$+[\xi V_1'(0)+\omega^2V_2'(0)]\int C_0(\alpha r)Z_0(\lambda r)\,dr-\omega[V_1'(0)+\xi V_2'(0)]\int C_1(\alpha r)Z_1(\lambda r)\,dr \quad (4a)$$

where $\qquad \phi=\alpha^2+\lambda^2,\ \xi=(\lambda^2-\alpha^2)/\phi^2,\ \omega=2\alpha\lambda/\phi^2$

$$W_1=\int rC_0(\alpha r)Z_0(\lambda r)\,dr=\frac{r}{\phi}[\alpha C_1(\alpha r)Z_0(\lambda r)+\lambda C_0(\alpha r)Z_1(\lambda r)]$$

$$W_3=\int rC_1(\alpha r)Z_1(\lambda r)\,dr=\frac{r}{\phi}[\lambda C_1(\alpha r)Z_0(\lambda r)-\alpha C_0(\alpha r)Z_1(\lambda r)]$$

$$\frac{W_2}{r}=\xi C_0(\alpha r)Z_0(\lambda r)+\omega C_1(\alpha r)Z_1(\lambda r)$$

$$\frac{W_4}{r}=\xi C_1(\alpha r)Z_1(\lambda r)-\omega C_0(\alpha r)Z_0(\lambda r)$$

and $\qquad Z_1(\lambda r)=DI_1(\lambda r)-EK_1(\lambda r).$

An analysis similar to that of section (4) gives

$$V_1(r)=\sum_{k=0}(-1)^k\omega^{2k}\sum_{n=0}\xi^n L_{k,n}(r) \tag{20}$$

$$V_2(r)=\sum_{k=0}(-1)^k\omega^{2k}\sum_{n=0}\xi^n M_{k,n}(r) \tag{21}$$

where $L_{k,n}(r)$ and $M_{k,n}(r)$ are as defined in section 4 and table 3.

6. Evaluation of $\int J_0(\alpha r)J_0(\beta r)dr$ and $\int J_1(\alpha r)J_1(\beta r)dr$

The expression for the first integral may be written in the form

$$\int J_0(\alpha r)J_0(\beta r)dr=\int J_0(\alpha r)dr-\alpha^2\int rf(r)J_0(\alpha r)dr$$

where from (1a), the integral becomes

$$\int J_0(\alpha r)J_0(\beta r)dr=[1+F'(0)]\int J_0(\alpha r)dr-r[\alpha F(r)J_1(\alpha r)F'(r)J_0(\alpha r)]$$

and where

$$f(r)=k^2\left[\frac{1-J_0(\beta r)}{\beta^2 r}\right]=k^2\left(\frac{r}{2^2}-\frac{\beta^2r^3}{2^2\cdot 4^2}+\frac{\beta^4r^5}{2^2\cdot 4^2\cdot 6^2}-\ \cdot\ \cdot\ \cdot\right)$$

is an infinite series and $k=\beta/\alpha<1$. From section 2, $F(r)$ becomes

$$F(r)=-\sum_{n=1}^{\infty}\frac{(-1)^n\alpha^{2(n-1)}r^{2n-1}}{[1\cdot 3\cdot 5\cdot\cdot\cdot(2n-1)]^2}\sum_{m=n}^{\infty}\frac{(\frac{1}{2})_m^2k^{2m}}{(m!)^2}$$

where $(\frac{1}{2})_m=\Gamma(\frac{1}{2}+m)/\Gamma(\frac{1}{2})$.
The complete elliptic integral of the first kind is defined

$$K(k^2)=\int_0^{\pi/2}\frac{d\varphi}{[1-k^2\sin^2\varphi]^{1/2}}=\frac{\pi}{2}\sum_{m=0}^{\infty}\frac{(\frac{1}{2})_m^2k^{2m}}{(m!)^2}.$$

The summation on m in the above expression for $F(r)$ is then defined by

$$L_n(k^2)=\frac{2}{\pi}K(k^2)-\sum_{m=0}^{n-1}\frac{(\frac{1}{2})_m^2k^{2m}}{(m!)^2}.$$

The integral upon substitution becomes

$$\int J_0(\alpha r)J_0(\beta r)dr=\frac{2}{\pi}K(k^2)\int J_0(\alpha r)dr$$

$$+\sum_{n=1}^{\infty}\frac{(-1)^n\alpha^{2(n-1)}r^{2n-1}}{[1\cdot 3\cdot 5\ .\ .\ .\ (2n-1)]^2}L_n(k^2)\{\alpha rJ_1(\alpha r)+(2n-1)J_0(\alpha r)\} \tag{22}$$

where the integral $\int J_0(\alpha r)dr$ has been defined in section 2. The series is convergent for $k<1$. An expression for the integral $\int J_0(\beta r)Y_0(\alpha r)dr$ is found by substituting $Y_0(\alpha r)$, $Y_1(\alpha r)$ for $J_0(\alpha r)$ and $J_1(\alpha r)$, respectively, in (22). An expression for the integral $\int J_0(\alpha r)I_0(\beta r)dr$ is found from (22) by substitution $k^2=-k^2$, $k^{2m}=(-1)^m k^{2m}$.

The expression (22) is limited for numerical evaluation due to a large loss of significance for large arguments although for values of $k^2 \ll 1$ there is probably no restriction for numerical evaluation. Another more general expression for the integral can be obtained using the method of Butler and Pohlhausen [3, pp. 11–18]. An integral representation of the Bessel function is

$$J_0(kr)=\frac{2}{\pi}\int_0^1 \frac{\cos(krt)dt}{(1-t^2)^{1/2}}=\frac{2}{\pi}\int_{\cos^{-1}k}^{\pi/2}\frac{\cos(r\cos\theta)\sin\theta d\theta}{(k^2-\cos^2\theta)^{1/2}}$$

where the expression on the right is derived from the substitution, $kt=\cos\theta$. A definite integral upon appropriate substitution becomes

$$\int_0^y J_0(\alpha r)J_0(\beta r)dr=\frac{1}{\alpha}\int_0^z J_0(r)J_0(kr)dr$$

$$=\frac{2}{\pi\alpha}\int_{\cos^1 k}^{\pi/2}\frac{\sin\theta}{(k^2-\cos^2\theta)^{1/2}}\int_0^z J_0(r)\cos(r\cos\theta)drd\theta$$

$$=y[J_0^2(z)+2J_1^2(z)]+2yJ_1(z)\sum_{n=1}^{\infty}\frac{(-1)^nJ_{2n+1}(z)}{2n+1}\Psi_{2n+1}$$

$$+yJ_0(z)\sum_{n=1}^{\infty}(-1)^nJ_{2n}(z)\left[\frac{\Psi_{2n-1}}{2n-1}+\frac{\Psi_{2n+1}}{2n+1}\right] \tag{23}$$

where $z=\alpha y$, and

$$\Psi_{2n+1}=\frac{2}{\pi}\int_{\cos^1 k}^{\pi/2}\frac{\sin(2n+1)\theta d\theta}{(k^2-\cos^2\theta)^{1/2}}.$$

From [3, eq 55] values of Ψ_{2n+1} are determined by the recurrence formula

$$\Psi_{m+2}=\frac{2m(2k^2-1)}{m+1}\Psi_m-\frac{m-1}{m+1}\Psi_{m-2}$$

where $\Psi_1=1$, $\Psi_3=2k^2-1$, $\Psi_5=6k^4-6k^2+1$, etc. The integral for $J_0^2(\alpha r)dr$ is found from (23) by setting $k=1$, whereby $\Psi_{2n+1}=1$.

Numerical values for the definite integral

$$\int_0^1 J_0(\alpha r)J_0(\beta r)dr$$

have been determined from (22) and (23) and are shown plotted against k for various values of α in figure 1. For a few cases, the numerical values have also been obtained by numerical integration by Simpson's rule.

The expression for the second integral may be written by integration by parts in the form

$$\int J_1(\alpha r)J_1(\beta r)dr=-\frac{J_0(\alpha r)J_1(\beta r)}{\alpha}+k\int J_0(\alpha r)J_0(\beta r)dr-k\int\frac{J_1(\beta r)}{\beta r}J_0(\alpha r)dr.$$

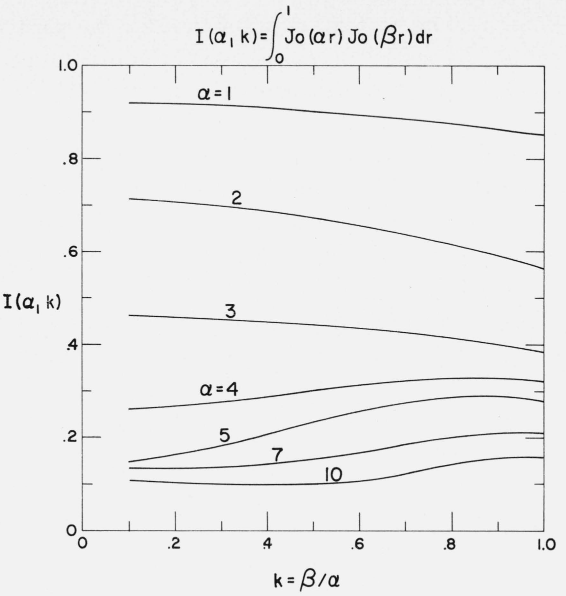

FIGURE 1

The first integral on the right has been solved previously ((22) or (23)), and the second integral from (1a) takes the form

$$\int \frac{J_1(\beta r)}{\beta r} J_0(\alpha r)\,dr$$

$$=\left[\frac{1}{2}+F'(0)\right]\int J_0(\alpha r)\,dr - r[\alpha F(r)J_1(\alpha r)+F'(r)J_0(\alpha r)]$$

$$\text{with } f(r)=k^2\frac{\beta r/2-J_1(\beta r)}{\beta^3 r}=k^2\left[\frac{r}{2^2\cdot 4}-\frac{\beta^2 r^3}{2^2\cdot 4^2\cdot 6}+\frac{\beta^4 r^5}{2^2\cdot 4^2\cdot 6^2\cdot 8}-\ \cdot\ \cdot\ \cdot\right].$$

From section 2, $F(r)$ becomes

$$F(r)=-\sum_{n=1}^{\infty}\frac{(-1)^n\alpha^{2(n-1)}r^{2n-1}}{[1\cdot 3\cdot 5\cdots(2n-1)]^2}\sum_{m=n}^{\infty}\frac{(\frac{1}{2})_m^2 k^{2m}}{(m!)^2(2m+2)}.$$

An integral of $K(k^2)$ is defined by

$$\int_0^k k'K(k'^2)\,dk'=E(k^2)-(1-k^2)K(k^2)=\frac{\pi}{2}\sum_{m=0}^{\infty}\frac{(\frac{1}{2})_m^2 k^{2m+2}}{(m!)^2(2m+2)}$$

where $E(k^2)$ is the complete elliptic integral of the second kind. The summation on m in the above expression for $F(r)$ is then defined by

$$Q_n(k^2)=\frac{2}{\pi k^2}[E(k^2)-(1-k^2)K(k^2)]-\sum_{m=0}^{n-1}\frac{(\frac{1}{2})_m^2 k^{2m}}{(m!)^2(2m+2)}.$$

The integral upon substitution becomes

$$\int J_1(\beta r)J_1(\alpha r)\,dr = -\frac{J_0(\alpha r)J_1(\beta r)}{\alpha} + \frac{2}{\pi k}[K(k^2)-E(k^2)]\int J_0(\alpha r)\,dr$$

$$+k\sum_{n=1}^{\infty}\frac{(-1)^n\alpha^{2(n-1)}r^{2n-1}}{[1\cdot 3\cdot 5\cdots(2n-1)]^2}R_n(k^2)\{\alpha rJ_1(\alpha r)+(2n-1)J_0(\alpha r)\} \tag{24}$$

where

$$R_n(k^2) = \frac{2}{\pi k^2}[K(k^2)-E(k^2)] - \sum_{m=0}^{n-1}\frac{(2m+1)\left(\frac{1}{2}\right)_m^2 k^{2m}}{(2m+2)(m!)^2}.$$

7. Discussion

An example of the usefulness of the method presented in this paper is given for the evaluation of the definite integral

$$\int_0^1 rf(r)J_0(\alpha r)\,dr \tag{25}$$

where $f(r)=a_0+a_2r^2+a_3r^3+a_4r^4$. The integral is of the form given in (1a), where the coefficients of the polynominal $F(r)$ are given in table 1. The resulting expressions for the upper limit are

$$F(1)=f(1)-\frac{4a_2+9a_3+16a_4}{\alpha^2}+\frac{64a_4}{\alpha^4}$$

$$F'(1)=f'(1)-\frac{9a_3+32a_4}{\alpha^2}, \text{ and}$$

$$F'(0)=-\frac{9a_3}{\alpha^2}.$$

From (1a), the definite integral (25) becomes

$$\int_0^1 rf(r)J_0(\alpha r)\,dr = \frac{F(1)J_1(\alpha)}{\alpha}+\frac{F'(1)J_0(x)}{\alpha^2}+\frac{9a_3}{\alpha^4}\int_0^1 J_0(\alpha r)\,dr,$$

where the integral on the right is evaluated using (1b). Given values for a_0, a_2, a_3, a_4, and α, and tables of Bessel and Struve functions, the value of the definite integral is readily computed by desk calculator. If subroutines are available for computing the Bessel and Struve functions, digital computer calculation is rapidly accomplished.

In this instance, the value of the definite integral is determined with need only for evaluation of $F(r)$, using table 1, and then of $F(1)$, $F'(1)$, $F'(0)$, and of four additional functions, namely, $J_0(\alpha)$, $J_1(\alpha)$, $H_0(\alpha)$ and $H_1(\alpha)$. The computational task is minor as compared with that of evaluating many subdivisions or steps of the original integrand for a numerical integration of equivalent accuracy.

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(Paper 71B2&3–207)

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Relative enthalpy of beryllium 1:1-aluminate, $BeO\cdot Al_2O_3$, from 273 to 1173 °K. Thermodynamic properties from 273 to 2150 °K. D. A. Ditmars and T. B. Douglas.
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