JOURNAL OF RESEARCH of the National Bureau of Standards — B. Mathematics and Mathematical Physics
Vol. 71B, No. 1, January–March 1967

The Coefficients of the Powers of a Polynomial

Morris Newman

Institute for Basic Standards, National Bureau of Standards, Washington, D.C. 20234

(December 5, 1966)

It is shown that if $f(z)$ is a polynomial with no zeroes inside the unit circle and if r is any positive number, then the coefficients of $f^r(z)$ tend to zero like n^{-1-r}, and this is best possible.

Key Words: Binomial coefficients, bounds, polynomials.

The general question which prompted this note is the following: Suppose that $f(z)$ is analytic and zero-free inside the unit circle, and normalized so that $f(0)=1$. Let r be real and positive and choose (for definiteness) that determination of $f^r(z)$ for which $f^r(0)=1$. Then what can be said about the order of magnitude of the coefficients of $f^r(z)$? In particular, when will they converge to zero? In such a general setting no simple answer is to be expected and examples illustrating almost any sort of behavior may be given; but the case treated here, namely that when $f(z)$ is a polynomial, is reasonably simple. In fact we shall prove the following theorem:

THEOREM 1. *Let* f(z) *be a polynomial of degree* p *which is zero-free inside the unit circle and such that* f(0) = 1. *Let* r *be positive and suppose that*

$$f^r(z)=1+\sum_{n=1}^{\infty} a_r(n)z^n$$

Then

$$a_r(n)=0(n^{-1-r}), \qquad n\rightarrow\infty, \tag{1}$$

and consequently

$$\lim_{n\rightarrow\infty} a_r(n)=0.$$

The constant implied by (1) depends on r and p, of course.

Precise inequalities rather than 0-estimates can be obtained at the expense of some additional complication, but we do not pursue this point here.

The statement is "best possible" in the sense that $a_r(n)$ is of true order of magnitude n^{-1-r} for certain polynomials $f(z)$; for example for $f(z)=1+z$.

The statement is false for negative r. For example if $f(z)=(1-z)^p$ and $r=-2/p$ then $a_r(n)=n+1$.

We require the following lemmas.

LEMMA 1. *Let* r *be arbitrary. Then*

$$\lim_{n\rightarrow\infty} (-1)^n\binom{r}{n} n^{1+r}=\frac{1}{\Gamma(-r)}, \tag{2}$$

and so

$$\binom{r}{n}=0(n^{-1-r}), \qquad n\rightarrow\infty.$$

Formula (2) is just Euler's definition of the Γ-function. Notice that if r is a positive integer then $\binom{r}{n}$ vanishes for all sufficiently large n, which agrees with the fact that $\Gamma(s)$ has poles at $s=0, -1, -2, \ldots$.

LEMMA 2. *Let* r *be positive. For* p = 1, 2, 3, . . . *define* $t_p(n)$ *by*

$$\sum_{n=0}^{\infty} t_p(n)z^n=\left\{1+\sum_{n=1}^{\infty} n^{-1-r}z^n\right\}^p. \tag{3}$$

Then

$$t_p(n)=0(n^{-1-r}).$$

PROOF. Differentiating (3) and comparing coefficients of corresponding powers of z in the result we obtain the recurrence formula

$$nt_p(n)=p\sum_{k=1}^{n} k^{-r}t_{p-1}(n-k). \tag{4}$$

242-010 O-67-2

The proof is by induction on p. For $p=1$, $t_p(n)=n^{-1-r}$, $n \geq 1$ and so the lemma is certainly true then. Assume the truth of the lemma for $p-1$, $p \geq 2$. Write (4) as

$$nt_p(n)=p \sum_{1 \leq k \leq \frac{n-1}{2}} k^{-r} t_{p-1}(n-k) + p \sum_{\frac{n-1}{2} < k \leq n-1} k^{-r} t_{p-1}(n-k) + pn^{-r}.$$

Then

$$\sum_{1 \leq k \leq \frac{n-1}{2}} k^{-r} t_{p-1}(n-k) = 0\left\{ \sum_{1 \leq k \leq \frac{n-1}{2}} k^{-r}(n-k)^{-1-r} \right\}$$

$$= 0\left\{ n^{-1-r} \sum_{1 \leq k \leq \frac{n-1}{2}} k^{-r} \right\} = 0(n^{-r})$$

(using the crude estimate $k^r \geq 1$), and

$$\sum_{\frac{n-1}{2} < k \leq n-1} k^{-r} t_{p-1}(n-k) = 0\left\{ \sum_{\frac{n-1}{2} < k \leq n-1} k^{-r}(n-k)^{-1-r} \right\}$$

$$= 0\left\{ n^{-r} \sum_{\frac{n-1}{2} < k \leq n-1} (n-k)^{-1-r} \right\} = 0(n^{-r}),$$

since the series $\sum_{k=1}^{\infty} k^{-1-r}$ converges. Thus

$$nt_p(n) = 0(n^{-r}),$$

$$t_p(n) = 0(n^{-1-r})$$

and the lemma is proved for p. This concludes the proof of the lemma.

We turn now to the proof of theorem 1. We may write

$$f(z)=(1+\alpha_1 z)(1+\alpha_2 z) \;.\;.\;.\; (1+\alpha_p z),$$

where, because of the assumptions made about $f(z)$, the numbers α_i are all of modulus not exceeding 1. We have

$$f^r(z) = 1 + \sum_{n=1}^{\infty} a_r(n) z^n$$

$$= \prod_{i=1}^{p} (1+\alpha_i z)^r$$

$$= \prod_{i=1}^{p} \left\{ \sum_{n_i=0}^{\infty} \binom{r}{n_i} \alpha_i^{n_i} z^{n_i} \right\} \ll \left\{ \sum_{n=0}^{\infty} \left|\binom{r}{n}\right| z^n \right\}^p$$

$$\ll K \left\{ 1 + \sum_{n=1}^{\infty} \frac{z^n}{n^{1+r}} \right\}^p$$

$$= K \sum_{n=0}^{\infty} t_p(n) z^n,$$

where K is some suitable positive constant (lemma 1).[1] Hence

$$|a_r(n)| \leq K t_p(n),$$

and lemma 2 implies the truth of the theorem for all positive r.

It is clear that the critical case is when $f(z)$ has all its roots on the unit circle, since otherwise much stronger inequalities for the coefficients $a_r(n)$ will hold. In fact, we can prove

THEOREM 2. *Let* g(z) *be a polynomial of degree* p *such that* g(0) = 1, *and let* T *be the distance from the origin to the nearest zero of* g(z). *Let* r *be positive,*

$$g^r(z) = 1 + \sum_{n=1}^{\infty} b_r(n) z^n.$$

Then

$$b_r(n) = 0(n^{-1-r} T^{-n}), \qquad n \to \infty.$$

PROOF. It is only necessary to apply theorem 1 to the polynomial $g(Tz)=f(z)$ which now has no zeroes inside the unit circle and satisfies $f(0)=1$.

[1] We write $\sum a_n z^n \ll \sum b_n z^n$ to mean that $b_n \geq 0$ and $|a_n| \leq b_n$.

(Paper 71B1–190)