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Three Observations on Nonnegative Matrices*

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Some results on nonnegative matrices are proved, of which the following is representative: Let $A=(a_{ij})$ be a nonnegative row stochastic matrix. If $\lambda \neq 1$ is an eigenvalue of A, then

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$$|\lambda| \leq \min \left(1-\sum_j \min_i a_{ij}, \sum_j \max_i a_{ij}-1\right).$$

Key Words: Bounds, eigenvalues, nonnegative matrices.

1. Introduction

In this note, the following results are proved about nonnegative matrices $A=(a_{ij})$ of order n.

THEOREM A: *If* A *is symmetric, and* $c_i = \sum_j a_{ij}$, *then*

$$\sum_{i,j} (A_m)_{i,j} \leq \sum_i c_i^m, \qquad m=1, 2, \ldots .$$

This proves a conjecture of London [3], who has already proved Theorem A for small m and all n, and small n and all m.

THEOREM B: *If* A *is stochastic* $\left(i.e., \sum_j a_{ij}=1\right)$, *and* $\lambda \neq 1$ *is an eigenvalue of* A, *then*

$$|\lambda| \leq \min \left(1-\sum_j \min_i a_{ij}, \sum_j \max_i a_{ij}-1\right).$$

Goldberg ([1],[1] Theorem 1) has shown that

$$|A| \leqslant \left(1-\sum_j \min_i a_{ij}\right)^{n-1}.$$

Since $|A|$ is the product of the eigenvalues of A, and 1 is an eigenvalue of A, Goldberg's theorem is a consequence of our inequality.

THEOREM C: *If* A *is stochastic and nonsingular,* $B=A^{-1}$, *then*

$$\max_{i,j} |b_{ij}| = \max_{y \,|\, y\neq 0,\ y'A \geq 0} \frac{\max_j |y_i|}{\sum_j y_i}.$$

Goldberg ([1], Theorem 2) has shown that $|\det A|$ is at most the reciprocal of the right-hand side of the above equation, and Theorem C is an effort to amplify Goldberg's theorem by characterizing the right-hand side in terms of elements of A^{-1}.

2. Proof of Theorem A

We denote by the cardinality of a set S by $|S|$, and the set $\{1, \ldots, n\}$ by N. For any $S \subset N$, u_S is the vector $(u_1, \ldots, u_n)$ with $u_j=1$ if $j \in S$, $u_j=0$ if $j \notin S$. A nonnegative matrix A is substochastic if $u_N'A=u_N'$ and $Au_N \leq u_N$. A subpermutation matrix is a substochastic matrix in which every entry is 0 or 1. If x and y are nonnegative vectors, then $x \ll y$ means

(2.1) $$\max_{|S|=k} (u_S, x) \leq \max_{|S|=k} (u_S, y) \qquad k=1, \ldots, n.$$

The following are known (see [3]).

(2.2) If x and y are nonnegative vectors, $x \ll y$ if and only if there exists a substochastic matrix A such that $x=Ay$.

(2.3) The convex hull of the subpermutation matrices is the set of substochastic matrices.

If $a=(a_1, \ldots, a_n)$ and $b=(b_1, \ldots, b_n)$ are nonnegative vectors with

(2.4) $a_1 \geq \ldots \geq a_n$ and $b_1 \geq \ldots \geq b_n$, ab is the vector $(a_1 b_1, \ldots, a_n b_n)$.

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[1] Figures in brackets indicate the literature references at the end of this paper.

Also, we define Ξ_a to be the set of nonnegative matrices A such that

(2.5) $$Au_N \ll a \text{ and } (u_N'A)' \ll a.$$

Note that Ξ_{u_N} is the set of substochastic matrices.

LEMMA 2.1: *If* a *and* b *satisfy* (2.4), $X\epsilon\Xi_a$, $Y\epsilon\Xi_b$, *then* $XY\epsilon\Xi_{ab}$.

PROOF: By (2.1) and (2.5), we need only show, for $S \subset N$, that

(2.6) $$u_S'XYu_N \leq a_1b_1 + \ldots + a_{|S|}b_{|S|},$$

and $$u_N'XYu_S \leq a_1b_1 + \ldots + a_{|S|}b_{|S|}.$$

We prove only (2.6), the other inequality following by symmetry. Since X is nonnegative, $(u_S'X)' \leq (u_N'X)' \ll a$. It follows that

(2.7) $$\max_{|T|=h} u_T'(u_S'X)' \leq a_1 + \ldots + a_h, \quad h=1, \ldots, |S|.$$

Also, since $Xu_N \ll a$, it follows that

(2.8) $$\max_{|T|=h} u_T'(u_S'X)' \leq a_1 + \ldots + a_{|S|}, \quad h=|S|, \ldots, n.$$

Therefore, if $a^{|S|}$ *denotes the vector* $(a_1, \ldots, a_{|S|}, 0, \ldots, 0)$, *we have from* (2.7) *and* (2.8) *that*

(2.9) $$(u_S'X)' \ll a^{|S|}.$$

Using (2.2) and (2.3), we see that $(u_S'X)'$ is in the convex hull of all vectors formed by applying subpermutation matrices to $a^{|S|}$. Since $Yu_N \ll b$, Yu_N is in the convex hull of all vectors formed by applying subpermutation matrices to b. But the left side of (2.6) is the inner product of vectors in these respective convex hulls, and is at most the maximum inner product obtained from a vertex of one hull and a vertex of the other hull. From the well-known inequality ([2], p. 268), this is at most $\sum_{i=1}^{|S|} a_ib_i$, which is (2.6).

COROLLARY: *If* $X\epsilon\Xi_a$, $X^m\epsilon\Xi_{a^m}$, $m=1, 2, \ldots$

Theorem A is obviously a special case of the corollary.

3. Proof of Theorem B

If $\lambda \neq 1$, and $\lambda v' = v'A$, then $(v, u_N)=0$, since $Au_N = 1u_N$. Let $c_j = \min_i a_{ij}$. Consider the matrix $A_c = (a_{ij} - c_j)$. Since $(v, u_N)=0$, $v'A_c = v'A$. Hence,

(3.1) $$\lambda v' = v'A = v'A_c.$$

Taking absolute values in (3.1), we have

(3.2) $$|\lambda|\,|v_j| \leq \sum_i |v_i|(a_{ij} - c_j).$$

But A_c is a nonnegative matrix, and it is well-known (see [5]) that if a nonnegative nonzero vector x and a nonnegative number μ satisfy

$$\mu x_j \leq \sum_i x_ib_{ij}$$

for a nonnegative matrix B, then μ is at most the largest eigenvalue of B. Applying this to (3.2), and observing that $1 - \Sigma c_j$ is the largest eigenvalue of A_c, we get the first inequality of Theorem B. The second is proved in an analogous manner.

4. Proof of Theorem C

Let y be any real vector such that $y'A \geq 0$, and $z' = y'A$. Then, with $B = A^{-1}$,

(4.1) $$\max_{y \,|\, \substack{y\neq 0 \\ y'A \geq 0}} \frac{\max_j |y_j|}{\sum_j y_j} = \max_{\substack{z\geq 0 \\ z\neq 0}} \frac{\max_j |(z'B)_j|}{z'Bu},$$

where $u=(1, \ldots, 1)$. But $Au=u$ implies $Bu=u$, and numerator and denominator of the right side of (4.1) are homogeneous in z. Hence,

(4.2) $$\max_{\substack{z\geq 0 \\ z\neq 0}} \frac{\max_j |(z'B)_j|}{z'Bu} = \max_{\substack{z\geq 0 \\ \sum z_j = 1}} \max_j |(z'B)_j|.$$

Let i_0, j_0 be such that $|b_{i_0, j_0}| = \max_{i,j} |b_{i,j}|$. Then $|(z'B)_j| \leq \sum_i |z_ib_{ij}| \leq \sum_i z_i|b_{ij}| \leq \max_i |b_{ij}| \leq |b_{i_0,j_0}|$, since $z_i \geq 0$, $\sum z_i = 1$. Consequently, the right-hand side of (4.2) is at most $|b_{i_0, j_0}|$. But this number is achieved if z is the vector with 1 in position i_0, and is 0 everywhere else. Consequently, the right-hand side of (4.2) is $|b_{i_0, j_0}|$, which combines with (4.1) to prove Theorem C. Noting that

$$1 \geq |\text{ cofactor of } i_0, j_0 \text{ in } A| = |b_{i_0, j_0}| \cdot |\det A|,$$

one can deduce an alternative proof of [1], Theorem 2.

Incidentally, it is manifest that essentially the same arguments also prove:

$$\max_{i,j} b_{ij} = \max_{y \,\middle|\, \substack{y'A \geqslant 0 \\ y \neq 0}} \frac{\max_j y_j}{\sum_j y_j}$$

and

$$\min_{i,j} b_{ij} = \min_{y \,\middle|\, \substack{y'A \geqslant 0 \\ y \neq 0}} \frac{\min_j y_j}{\sum_j y_j}.$$

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