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E-Transforms*

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The following transform pair is established:

$$g(x)=\int_0^\infty (xy)^k[Ep;\ \alpha_r:\ q;\ \rho_t:\ (xy)^{\pm n}]f(y)dy;$$

$$f(x)=n^2\int_0^\infty (xy)^{-k}E\left[\begin{matrix}0 \\ \\ 1;\mp\frac{1}{n}\end{matrix}\Bigg/\begin{matrix}q;\ 1-\rho_t\mp\frac{1}{n}:\ (xy)^{\pm n} \\ \\ p+1;\ 1-\alpha_r\mp\frac{1}{n},\ 1\end{matrix}\right]g(y)dy,$$

where n is any positive integer and E is MacRobert's function and the generalized MacRobert's function, respectively.

Special choices of the parameters in the last transform lead in turn to the derivation of Hankel transform, Y-transform, K-transform, Fourier transform, Laplace transform and other integral transforms with tables to illustrate these new transforms.

Key Words: Fourier-transform, functional transforms, generalized MacRobert's function, Hankel transform, K-transform, Laplace transform, MacRobert's function, Y-transform.

1. Introduction

In this paper we discuss a new class of integral transforms and their inversion formulas. The kernel in the transform is MacRobert's E-function whose properties and definitions are to be found in [1] [2] pp. 348–358, and which will be discussed further later on. In the inversion formula, the kernel is likewise the generalized E-function of MacRobert (see [1], p. 419). Known special cases of our results are the Hankel transform pair ([2], p. 3):

$$g(x)=\int_0^\infty J_\nu(xy)(xy)^{1/2}f(y)dy \tag{1}$$

$$f(x)=\int_0^\infty J_\nu(xy)(xy)^{1/2}g(y)dy; \tag{2}$$

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the Y-transform pair ([2]; p. 93):

$$g(x)=\int_0^\infty Y_\nu(xy)(xy)^{1/2}f(y)dy, \tag{3}$$

$$f(x)=\int_0^\infty H_\nu(xy)(xy)^{1/2}g(y)dy; \tag{4}$$

and the K-transform pairs ([2], p. 121):

$$g(x)=\int_0^\infty K_\nu(xy)(xy)^{1/2}f(y)dy, \tag{5}$$

$$f(x)=\frac{1}{\pi i}\int_{c-i\infty}^{c+i\infty} I_\nu(xy)(xy)^{1/2}g(y)dy. \tag{6}$$

When $\nu=\pm\frac{1}{2}$, the Hankel transform and the K-transform reduce to the Fourier and Laplace transform, respectively. So it is not necessary to reproduce them here as particular cases of our results.

Section 2 contains a treatment of the E-function and preliminary results, and section 3 contains the derivation of our main results. Section 4 contains the derivation of the K-transform pair ((5) and (6)) from the Hankel transform pair ((1) and (2)); while several other particular cases are presented in section 5.

It may be noted that the parameters and the variables are such that the functions involved exist and the integrals are convergent.

2. Subsidiary Formulas

The Mellin transform pair ([3], p. 7):

$$g(s)=\int_0^\infty x^{s-1}f(x)dx, \tag{7}$$

$$f(x)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s}g(s)ds, \tag{8}$$

will be utilized.

The E-function is defined by the Mellin-Barnes contour integral (see [1], p. 374).

$$E(p;\ \alpha_r : q;\ \rho_t : z)=\frac{1}{2\pi i}\int \Gamma(\xi)\frac{\prod_{r=1}^{p}\Gamma(\alpha_r-\xi)}{\prod_{t=1}^{q}\Gamma(\rho_t-\xi)}z^{\xi}d\xi, \tag{9}$$

where the integral is taken along the η-axis with loops, if necessary, to ensure that the pole at the origin lies to the left and the poles at $\alpha_1, \alpha_2, \ldots, \alpha_p$ lie to the right of the contour. Zero and negative values of the α's and ρ's are excluded. When $p<q+1$ the contour is bent to the left at both ends. Convergence is secured if $|\arg z|<\frac{1}{2}(p-q+1)$ if $p>q+1$ and $|z|>1$ if $p=q+1$.

The generalized E-function is defined by (see [1], p. 419)

$$E\left(\begin{matrix} p; \alpha_r \\ q; \rho_s \end{matrix} \middle/ \begin{matrix} m; \rho_{q+s} \\ l+1; \alpha_{p+r}, 1 \end{matrix} \quad : z \right) = \frac{1}{2\pi i} \int \frac{\prod_{r=1}^{p} \Gamma(\alpha_r - \xi) \prod_{s=1}^{m} \Gamma(\xi - \rho_{q+s} + 1)}{\prod_{s=1}^{q} \Gamma(\rho_s - \xi) \prod_{r=1}^{l} \Gamma(\xi - \alpha_{p+r} + 1)} z^{\xi} d\xi$$

$$= \pi^{m-l-1} \sum_{s=1}^{m} \frac{\prod_{r=l}^{l} \sin(\rho_{q+s} - \alpha_{p+r})\pi}{\prod_{t=1}^{m}{}' \sin(\rho_{q+s} - \rho_{q+t})\pi} z^{\rho_{q+s} - 1}$$

$$\times E\left(\begin{matrix} p+l; \alpha_r - \rho_{q+s} + 1 & & : \omega z \\ \rho_1 - \rho_{q+s} + 1, \ldots & * & \ldots, \rho_{q+m} - \rho_{q+s} + 1 \end{matrix}\right), \tag{10}$$

where l, m are the integers, ω is equal to $e^{\pm i\pi}$ or 1 according as $l+m$ is even or odd and the contour passes along the η-axis from $-\infty$ to ∞ with loops, if necessary, to ensure that the poles of the integrand at the origin and at $\rho_{q+1}-1$, $\rho_{q+2}-1$, . . ., $\rho_{q+m}-1$ lie to the left and the poles at α_1, α_2, . . ., α_p to the right of the contour, when necessary the contour is bent to the left or to the right at both ends until it is parallel to the ξ-axis. The E-function is also defined thus:

If $p \leqslant q$; then

$$E(p;\ \alpha_r : q;\ \rho_s : z) = \frac{\Gamma(\alpha_1) \ldots \Gamma(\alpha_p)}{\Gamma(\rho_1) \ldots \Gamma(\rho_q)} pFq\left(\begin{matrix} p;\ \alpha_r; -\frac{1}{z} \\ q;\ \rho_s \end{matrix}\right), \tag{11}$$

while if $p \geqslant q+1$; then

$$E(p;\ \alpha_r : q;\ \rho_s : z) = \sum_{r=1}^{p} \prod_{s=1}^{p}{}' \Gamma(\alpha_s - \alpha_r) \left\{\prod_{t=1}^{q} \Gamma(\rho_t - \alpha_r)\right\}^{-1} \Gamma(\alpha_r) z^{\alpha_r}$$

$$F\left(\begin{matrix} \alpha_r, \alpha_r - \rho_1 + 1, \ldots, \alpha_r - \rho_q + 1; (-1)^{p-q} \\ \alpha_r - \alpha_1 + 1, \ldots * \ldots, \alpha_r - \alpha_p + 1 \end{matrix}\right). \tag{12}$$

From (11) and (12) it is clear that the E-function is immediately related to the generalized hypergeometric function and reduces to simple expressions in the ordinary or Gauss hypergeometric function when $p=2$, $q=1$. For $p=q=1$ it is also evident that the E-function reduces to the confluent hypergeometric function or Kummer's function. The case $p=2$, $q=0$ yields the relations (see [1] p. 351)

$$\cos n\pi E(\tfrac{1}{2}+n, \tfrac{1}{2}-n : 2z) = \sqrt{(2\pi z)}\, e^{z} K_n(z), \tag{13}$$

and

$$E(\tfrac{1}{2}-k+m, \tfrac{1}{2}-k-m :: z) = \Gamma(\tfrac{1}{2}-k-m)\Gamma(\tfrac{1}{2}-k+m) z^{-k} e^{z/2} W_{k,m}(z), \tag{14}$$

where $K_n(z)$ and $W_{k,m}(z)$ are the modified Bessel function of the second kind and Whittaker function, respectively. When $p=q=0$ the E-function is just $e^{-1/z}$. When $p=0$, $q=1$, then we obtain the

Bessel function of the first kind. Thus we have

$$E(:\nu+1:x)=x^{\nu/2}J_\nu(2x^{-1/2}). \tag{15}$$

The case $p=1$, $q=0$ gives

$$E(\alpha::x)=\Gamma(\alpha)\left(1+\frac{1}{x}\right)^{-\alpha}. \tag{16}$$

More parameters in the E-function lead to the equivalence of the E-function with products of Hankel functions, with Lommel functions, Bessel functions and with product of Whittaker functions. The following are some examples:

$$x^\mu H_\nu^{(1)}(x)H_\nu^{(2)}(x)=2\,(\cos\nu\pi)\pi^{-5/2}x^{\mu-1}E(\tfrac{1}{2}+\nu,\tfrac{1}{2}-\nu,\tfrac{1}{2}::x), \tag{17}$$

$$S_{\mu,\nu}(x)=2^{\mu-1}\left\{\Gamma\left(\frac{1}{2}-\frac{\mu}{2}-\frac{\nu}{2}\right)\Gamma\left(\frac{1}{2}-\frac{\mu}{2}+\frac{\nu}{2}\right)\right\}^{-1}\left(\frac{x}{2}\right)^{\mu-1}E\left(1,\frac{1}{2}-\frac{\mu}{2}+\frac{\nu}{2},\frac{1}{2}-\frac{\mu}{2}-\frac{\nu}{2}::\frac{x^2}{4}\right) \tag{18}$$

$$\begin{aligned}W_{k,m}(2ix)W_{k,m}(-2ix)=\pi^{-1/2}\left(\frac{x}{2}\right)^{2k}&\left\{\Gamma\left(\frac{1}{2}-k+m\right)\Gamma\left(\frac{1}{2}-k-m\right)\right\}^{-1}\\&\times E\left(\frac{1}{2}-k+m,\frac{1}{2}-k-m,\frac{1}{2}-k,1-k:1-2k:\frac{x^2}{4}\right),\end{aligned} \tag{19}$$

$$J_\nu(x)J_{-\nu}(x)=\pi^{-1/2}E\left(\frac{1}{2}:1-\nu,1+\nu:\frac{1}{x^2}\right), \tag{20}$$

$$J_\nu^2(x)=\pi^{-1/2}E\left(\frac{1}{2}+\nu:1+\nu,1+2\nu:\frac{1}{x^2}\right), \tag{21}$$

$$H_\nu(x)=E\left(\begin{matrix}1;\frac{1}{2}+\frac{1}{2}\nu\\0;\;.\;.\;.\end{matrix}\Bigg/\begin{matrix}1;\frac{1}{2}+\frac{1}{2}\nu & :\frac{x^2}{4}\\3;\frac{1}{2}\nu,-\frac{1}{2}\nu,1 & \end{matrix}\right)=\left(\frac{x^2}{4}\right)^{\frac{\nu+1}{2}}E\left(\begin{matrix}1:\frac{4}{x^2}\\\frac{3}{2},\frac{3}{2}+\nu\end{matrix}\right) \tag{22}$$

and

$$Y_\nu(x)=E\left(\begin{matrix}0;\;.\;.\;.\\1;\frac{3}{2}+\frac{\nu}{2}\end{matrix}\Bigg/\begin{matrix}2;1+\frac{\nu}{2},1-\frac{\nu}{2}:\frac{4}{x^2}\\2,\frac{1}{2}\nu+\frac{3}{2},1\end{matrix}\right). \tag{23}$$

It is easily seen from (9) and (10) that the generalized E-function reduces to the ordinary E-function when $l=0$, $m=1$ and $\rho_{q+1}=1$.

3. The Main Results

The main theorem to be proved is

If n is any positive integer, $q \neq 0$ and if

$$g(x)=\int_0^\infty (xy)^k E[p;\ \alpha_r : q;\ \rho_t : (xy)^{\pm n}] f(y)\, dy, \tag{24}$$

then

$$f(x)=n^2 \int_0^\infty (xy)^{-k} E\left[\begin{matrix} 0; \\ 1; \mp\frac{1}{n} \end{matrix} \Bigg/ \begin{matrix} q;\ 1-\rho_t \mp \frac{1}{n} : (xy)^{\pm n} \\ p+1;\ 1-\alpha_r \mp \frac{1}{n},\ 1 \end{matrix}\right] g(y)\, dy, \tag{25}$$

where the kernel in the inversion formula (25) is given by (10) and the integrals are convergent.

We wish to solve the integral equation

$$g(x)=\int_0^\infty (xy)^k E[p;\ \alpha_r : q;\ \rho_s : (xy)^n] f(y)\, dy. \tag{26}$$

Using the notation of [3], p. 315, we have

$$G(s)=\int_0^\infty x^{s-1} g(x)\, dx,$$

or

$$G(1-s)=\int_0^\infty x^{-s} g(x)\, dx. \tag{27}$$

Also

$$\mathscr{R}(s)=\int_0^\infty u^{s+k-1} E[p;\ \alpha_r : q;\ \rho_t : u^n]\, du =\frac{1}{n}\int_0^\infty y^{\frac{s}{n}+\frac{k}{n}-1} E[p;\ \alpha_r : q;\ \rho_t : y]\, dy$$

$$=\Gamma\left(-\frac{s}{n}-\frac{k}{n}\right) \frac{\prod_{r=1}^{p} \Gamma\left(\alpha_r+\frac{k}{n}+\frac{s}{n}\right)}{\prod_{t=1}^{q} \Gamma\left(\rho_t+\frac{k}{n}+\frac{s}{n}\right)},$$

by (7), (8) and (9). Thus

$$n\mathscr{R}(1-s)=\Gamma\left(-\frac{1}{n}-\frac{k}{n}+\frac{s}{n}\right) \frac{\prod_{r=1}^{p} \Gamma\left(\alpha_r+\frac{k}{n}+\frac{1}{n}-\frac{s}{n}\right)}{\prod_{t=1}^{q} \Gamma\left(\rho_t+\frac{k}{n}+\frac{1}{n}-\frac{s}{n}\right)}, \tag{28}$$

and so by [3], p. 316, (27) and (28)

$$f(x)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{G(1-s)}{\mathscr{R}(1-s)}x^{-s}ds=\frac{n}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\prod_{t=1}^{q}\Gamma\left(\rho_t+\frac{k}{n}+\frac{1}{n}-\frac{s}{n}\right)}{\prod_{t=1}^{p}\Gamma\left(\alpha_r+\frac{k}{n}+\frac{1}{n}-\frac{s}{n}\right)}\cdot\frac{x^{-s}}{\Gamma\left(-\frac{1}{n}-\frac{k}{n}+\frac{s}{n}\right)}\times\int_0^\infty y^{-s}g(y)\,dy\,ds$$

$$=n\int_0^\infty g(y)\,dy\int_{c-i\infty}^{c+i\infty}\frac{\prod_{t=1}^{q}\Gamma\left(\rho_t+\frac{k}{n}+\frac{1}{n}-\frac{s}{n}\right)}{\Gamma\left(-\frac{1}{n}-\frac{k}{n}+\frac{s}{n}\right)\prod_{t=1}^{p}\Gamma\left(\alpha_r+\frac{k}{n}+\frac{1}{n}-\frac{s}{n}\right)}(xy)^{-s}ds$$

$$=n^2\int_0^\infty (xy)^{-k}g(y)\,dy\int_{c-i\infty}^{c+i\infty}\frac{\prod_{t=1}^{q}\Gamma\left(\rho_t+\frac{1}{n}-\xi\right)}{\Gamma\left(-\frac{1}{n}+\xi\right)\prod_{t=1}^{p}\Gamma\left(\alpha_r+\frac{1}{n}-\xi\right)}(xy)^{-n\xi}d\xi$$

$$=n^2\int_0^\infty (xy)^{-k}g(y)\,dy\int_{c-i\infty}^{c+i\infty}\frac{\prod_{t=1}^{q}\Gamma\left(\rho_t+\frac{1}{n}+\xi\right)}{\Gamma\left(-\frac{1}{n}-\xi\right)\prod_{r=1}^{p}\Gamma\left(\alpha_r+\frac{1}{n}+\xi\right)}(xy)^{n\xi}d\xi$$

$$=n^2\int_0^\infty (xy)^{-k}g(y)E\left[\begin{matrix}0; & q;\ 1-\rho_t-\frac{1}{n} \\ 1;\ -\frac{1}{n} & p+1;\ 1-\alpha_r-\frac{1}{n},\ 1\end{matrix}:(xy)^n\right]dy$$

by (10).

Thus the first part of the main transform pair (24) and (25) is proved. The case with $-n$ instead of n can be dealt in a similar way.

4. Derivations of Hankel and K Transforms

When $n=2$, the transform pair (24), (25) becomes

$$g(x)=\int_0^\infty (xy)^k E(p;\ \alpha_r:q;\ \rho_s:x^{-2}y^{-2})f(y)\,dy, \tag{29}$$

$$f(x)=4\int_0^\infty (xy)^{-k}E\left(\begin{matrix}q;\ 1-\rho_t+\frac{1}{2}:(xy)^2 \\ 1;\ \frac{1}{2}\ p+1;\ 1-\alpha_r+\frac{1}{2},\ 1\end{matrix}\right)g(y)\,dy. \tag{30}$$

In (29) and (30), take $p=0$, $q=1$, with $\rho_1=\nu+1$, apply (15) and so obtain a generalization of Hankel transform namely:

$$g(x)=\int_0^\infty (xy)^{k-\nu}J_\nu(xy)f(y)\,dy, \tag{31}$$

$$f(x)=\int_0^\infty (xy)^{-k+\nu+1}J_\nu(xy)g(y)\,dy. \tag{32}$$

When $k=\nu+\frac{1}{2}$, then (31) and (32) reduce to the Hankel transform pair (1) and (2).

To derive the K-transform (5) and (6), we have the integral equation

$$g(y)=\int_0^\infty K_\nu(xy)(xy)^{1/2}f(x)\,dx.$$

Thus

$$e^{-\nu\pi i}g(y)=\int_0^\infty e^{-\nu\pi i}K_\nu(xy)(xy)^{1/2}f(x)\,dx,$$

$$g(ye^{\pi i})=\int_0^\infty K_\nu(xye^{\pi i})e^{\frac{i\pi}{2}}f(x)\,dx.$$

Subtract the last two equations using the relation

$$\pi i I_\nu(z)=e^{-\nu\pi i}K_\nu(z)-K_\nu(ze^{\pi i}),$$

so getting

$$\int_0^\infty I_\nu(xy)(xy)^{1/2}f(x)\,dx=e^{-\frac{\pi i}{2}}\frac{(e^{-\nu\pi i}g(y)-g(ye^{\pi i}))}{\pi i}. \tag{33}$$

In (33) write $e^{\frac{1}{2}\pi i}y$ for y and get after using the relation

$$I_\nu(ze^{\pm\frac{1}{2}\pi i})=e^{\pm\frac{1}{2}\nu\pi i}J_\nu(z),$$

$$e^{\frac{\pi i}{4}-\frac{\nu\pi i}{2}}g(ye^{-\frac{\pi i}{2}})-e^{-\frac{\pi i}{4}+\frac{\nu\pi i}{2}}g(ye^{\frac{\pi i}{2}})=\pi i\int_0^\infty J_\nu(xy)(xy)^{1/2}f(x)\,dx. \tag{34}$$

Apply Hankel transform to the last equation and get

$$\pi i f(x)=\int_0^\infty J_\nu(xy)(xy)^{1/2}\{e^{\frac{\pi i}{4}-\frac{\nu\pi i}{2}}g(ye^{-\frac{\pi i}{2}})-e^{-\frac{\pi i}{4}+\frac{\nu\pi i}{2}}g(ye^{\frac{\pi i}{2}})\}\,dy.$$

Here write $e^{-\frac{\pi i}{2}}y$ for y, apply (34) and get

$$\begin{aligned}\pi i f(x)&=\int_0^{i\infty} I_\nu(xy)(xy)^{1/2}\{e^{-\frac{\pi i}{2}-\nu\pi i}g(ye^{-\frac{i\pi}{2}})+g(y)\}\,dy\\
&=\int_{-i\infty}^0 I_\nu(xy)(xy)^{1/2}g(y)\,dy+\int_0^{i\infty} I_\nu(xy)(xy)^{1/2}g(y)\,dy\\
&=\int_{-i\infty}^{+i\infty} I_\nu(xy)(xy)^{1/2}g(y)\,dy\\
&=\int_{c-i\infty}^{c+i\infty} I_\nu(xy)(xy)^{1/2}g(y)\,dy,\end{aligned}$$

provided that there are no poles of the integrand between the two contours. Thus

$$f(x)=\frac{1}{\pi i}\int_{c-i\infty}^{c+i\infty} I_\nu(xy)(xy)^{1/2}g(y)\,dy,$$

and the K-transform pair is obtained.

To derive the Y-transform, take in (24) $n=-2$, $p=1$, $q=2$ with $\alpha_1=1$, $\rho_1=\frac{3}{2}$, $\rho_2=\frac{3}{2}+\nu$, apply (22), (23) and so obtain a generalization of the Y-transform pair namely:

$$g(x)=\int_0^\infty (xy)^{k-\nu-1}H_\nu(x)f(y)\,dy, \tag{35}$$

then

$$f(x)=\int_0^\infty (xy)^{-k+\nu+2}Y_\nu(x)g(y)\,dy, \tag{36}$$

which reduces to the Y-transform pair when $k=\nu+\frac{3}{2}$.

5. Applications: Generalized Hankel Transform

In (24) and (25) write $n=2m$ $(m=1, 2, 3, \ldots)$, take the $(-)$ sign and apply (16) so obtaining another generalization of the Hankel transform pair namely

$$g(x)=\int_0^\infty (xy)^{k-\nu m}J_\nu[2x^my^m]f(y)\,dy, \tag{37}$$

$$f(x)=4m^2\int_0^\infty (xy)^{m\nu+2m-1-k}J_\nu[2x^my^m]g(y)\,dy, \tag{38}$$

where m is any positive integer and the integrals are convergent. When $m=1$, $k=\nu+\frac{1}{2}$ it reduces to Hankel transform pair (1) and (2). When $k=m\nu+m-\frac{1}{2}$, the last transform pair becomes:

$$g(x)=\int_0^\infty (xy)^{m-\frac{1}{2}}J_\nu[2x^my^m]f(y)\,dy, \tag{39}$$

$$f(x)=4m^2\int_0^\infty (xy)^{m-\frac{1}{2}}J_\nu[2x^my^m]g(y)\,dy, \tag{40}$$

where m is any positive integer. It may be noted that (39) and (40) can be deduced from (1) and (2) by a change of variable.

If $k=0$, then (31) and (32) give another form of Hankel transform namely:

$$g(x)=x^{\frac{\nu}{2}}\int_0^\infty y^{-\frac{\nu}{2}}J_\nu(2\sqrt{xy})f(y)\,dy, \tag{41}$$

$$f(x)=x^{\frac{\nu}{2}}\int_0^\infty y^{-\frac{\nu}{2}}J_\nu(2\sqrt{xy})g(y)\,dy. \tag{42}$$

Again Hankel transform pair (1) and (2) can be written in the form

$$g(x) = \int_0^\infty y J_\nu(xy) f(y)\, dy, \tag{43}$$

$$f(x) = \int_0^\infty y J_\nu(xy) g(y)\, dy. \tag{44}$$

Equations (43) and (44) can be extended to the case for n variables thus

$$g(x_1, x_2, \ldots, x_n) = \int_0^\infty \cdots \int_0^\infty \left[\prod_{r=1}^{n} y_r J_{\nu_r}(x_r y_r)\right] f(y_1, y_2, \ldots, y_n)\, dy_1 dy_2 \ldots dy_n, \tag{45}$$

$$f(x_1, x_2, \ldots, x_n) = \int_0^\infty \cdots \int_0^\infty \left[\prod_{r=1}^{n} y_r J_{\nu_r}(x_r y_r)\right] g(y_1, y_2, \ldots, y_n)\, dy_1 dy_2 \ldots dy_n. \tag{46}$$

Other particular cases of the transform pair (24) and (25) are the following new-integral transforms:
EXAMPLE 1. *The KJ-transforms:* From (24), (25) and the formula

$$E\left[\begin{matrix} 0; \\ 1; \frac{1}{4} \end{matrix} \middle/ \begin{matrix} 3: \frac{1}{4}-\nu, \frac{1}{4}-\frac{1}{2}\nu, -\frac{1}{4}-\frac{1}{2}\nu: \frac{1}{x4} \\ 1; 1 \end{matrix}\right] = (2\pi)^{1/2} x^{2\nu+3} K_\nu(2\sqrt{2}x) J_\nu(2\sqrt{2}x); \tag{47}$$

we arrive at the transform-pair:

$$g(x) = \frac{1}{\Gamma(1+\nu)\Gamma\left(1+\frac{1}{2}\nu\right)\Gamma\left(\frac{3}{2}+\frac{1}{2}\nu\right)} \int_0^\infty (xy)^k {}_0F_3,\left(1+\nu, 1+\frac{1}{2}\nu, \frac{3}{2}+\frac{1}{2}\nu; -x^4y^4\right) f(y)\, dy, \tag{48}$$

$$f(x) = 16\sqrt{2\pi} \int_0^\infty (xy)^{-k+2\nu+3} K_\nu(2\sqrt{2}xy) J_\nu(2\sqrt{2}xy) g(y)\, dy. \tag{49}$$

The following table illustrates some examples of the last transformation.

TABLE 1

$f(x)$	$g(x)$
e^{-x}	$\frac{\Gamma(k+1)}{\Gamma(1+\nu)\Gamma\left(1+\frac{1}{2}\nu\right)\Gamma\left(\frac{3}{2}+\frac{1}{2}\nu\right)} x^k$ ${}_4F_3\left(\begin{matrix} \frac{k+1}{4}, \frac{k+2}{4}, \frac{k+3}{4}, \frac{k+4}{4}: -4^4x^4 \\ 1+\nu, 1+\frac{1}{2}\nu, \frac{3}{2}+\frac{1}{2}\nu \end{matrix}\right)$
$R(k) > -1$	$R(\nu) > -1, \quad R(k-4\nu) > 0, \quad R(k-2\nu-2) > 0.$

TABLE 1—Continued

$f(x)$	$g(x)$
$K_n(x)$	$\displaystyle 2^{k-1}\frac{\Gamma\left(\frac{k+n+1}{4}\right)\Gamma\left(\frac{k+n+3}{4}\right)\Gamma\left(\frac{k-n+1}{4}\right)\Gamma\left(\frac{k-n+3}{4}\right)}{(2\pi)\Gamma(1+\nu)\Gamma\left(1+\frac{1}{2}\nu\right)\Gamma\left(\frac{3}{2}+\frac{1}{2}\nu\right)}x^k \; {}_4F_3\left(\begin{matrix}\frac{k+n+1}{4},\ \frac{k+n+3}{4},\ \frac{k-n+1}{4},\ \frac{k-n+3}{4};\ -4^4x^4\\ 1+\nu,\ 1+\frac{1}{2}\nu,\ \frac{3}{2}+\frac{1}{2}\nu\end{matrix}\right)$
$R(k\pm n+1)>0$	$R(\nu)>-1,\quad R(k\pm n-4\nu)>0,\quad R(k\pm n-2\nu-2)>0.$
$e^{-x}E(\gamma,\delta::x)$	$\displaystyle \frac{\Gamma(\gamma)\Gamma(\delta)\Gamma(k+\gamma+1)\Gamma(k+\delta+1)}{\Gamma(k+\gamma+\delta+1)\Gamma(1+\nu)\Gamma\left(1+\frac{1}{2}\nu\right)\Gamma\left(\frac{3}{2}+\frac{1}{2}\nu\right)}x^k \; {}_8F_7\left(\begin{matrix}\frac{k+\gamma+1}{4},\ \frac{k+\gamma+2}{4},\ \frac{k+\gamma+3}{4},\ \frac{k+\gamma+4}{4},\\ \frac{k+\gamma+\delta+1}{4},\ \frac{k+\gamma+\delta+2}{4},\ \frac{k+\gamma+\delta+3}{4}\end{matrix}\right.\left.\begin{matrix}\frac{k+\delta+1}{4},\ \frac{k+\delta+2}{4},\ \frac{k+\delta+3}{4},\ \frac{k+\delta+4}{4};\ -4^4x^4\\ \frac{k+\gamma+\delta+4}{4},\ 1+\nu,\ 1+\frac{1}{2}\nu,\ \frac{3}{2}+\frac{1}{2}\nu\end{matrix}\right)$
$R(k+\gamma+1)>0,\ R(k+\delta+1)>0$	$R(\nu)>-1,\quad R(k+\gamma-4\nu)>0,\quad R(k+\delta-4\nu)>0,$ $R(k+\gamma-2\nu-2)>0,\quad R(k+\delta-2\nu-2)>0.$
$\displaystyle \frac{\sqrt{\pi x}}{\cos n\pi}e^{-\frac{x}{2}}K_n\left(\frac{x}{2}\right)$	$\displaystyle \frac{\Gamma\left(\frac{1}{2}+n\right)\Gamma\left(\frac{1}{2}-n\right)\Gamma\left(k+n+\frac{3}{2}\right)\Gamma\left(k-n+\frac{3}{2}\right)}{\Gamma(k+2)\Gamma(1+\nu)\Gamma\left(\frac{1}{2}\nu+1\right)\Gamma\left(\frac{1}{2}\nu+\frac{3}{2}\right)}x^k \; {}_8F_7\left(\begin{matrix}\frac{2k+2n+3}{8},\ \frac{2k+2n+5}{8},\ \frac{2k+2n+7}{8},\ \frac{2k+2n+9}{8},\\ \frac{k+2}{4},\ \frac{k+3}{4},\ \frac{k+4}{4},\ \frac{k+5}{4},\ \frac{\nu+3}{2},\end{matrix}\right.\left.\begin{matrix}\frac{2k-2n+3}{8},\ \frac{2k-2n+5}{8},\ \frac{2k-2n+7}{8},\ \frac{2k-2n+9}{8};\ -4^4x^4\\ 1+\nu,\ 1+\frac{1}{2}\nu\end{matrix}\right)$
$R\left(k+\frac{3}{2}\pm n\right)>0$	$R(\nu)>-1,\ R\left(k+\frac{1}{2}\pm n-4\nu\right)>0,$ $R\left(k-\frac{3}{2}\pm n-2\nu\right)>0.$

$f(x)$	$g(x)$
$\Gamma\left(\frac{1}{2}-k'+m\right)\Gamma\left(\frac{1}{2}-k'-m\right)$ $x^{-k'}e^{-\frac{x}{2}}W_{k',m}(x)$ $R\left(\frac{3}{2}+k-k'\pm m\right)>0$	$\frac{\Gamma\left(\frac{3}{2}+k-k'+m\right)\Gamma\left(\frac{3}{2}+k-k'-m\right)}{\Gamma(k+2-2k')\Gamma(1+\nu)\Gamma\left(1+\frac{1}{2}\nu\right)\Gamma\left(\frac{3}{2}+\frac{1}{2}\nu\right)}x^k$ ${}_8F_7\left(\frac{2k-2k'+2m+3}{8}, \frac{2k-2k'+2m+5}{8}, \frac{2k-2k'+2m+7}{8}, \frac{2k+2k'+2m+9}{8}, \frac{k-2k'+2}{4}, \frac{k-2k'+3}{4}, \frac{k-2k'+4}{4}, \frac{k-2k'+5}{4}, \frac{2k-2k'-2m+3}{8}, \frac{2k-2k'-2m+5}{8}, \frac{2k-2k'-2m+7}{8}, \frac{2k-2k'-2m+9}{8}; -4^4x^4\right)$ $1+\nu,\ 1+\frac{1}{2}\nu,\ \frac{3}{2}+\frac{1}{2}\nu$ $R(\nu)>-1,\ R\left(\frac{1}{2}+k-k'\pm m-4\nu\right)>0,$ $R\left(k-k'\pm m-2\nu-\frac{3}{2}\right)>0.$
$E\left(-\frac{k+1}{4}+\nu+1, -\frac{k+1}{4}+\frac{\nu}{2}+1, -\frac{k+1}{4}+\frac{\nu}{2}+\frac{3}{2} :: x^4\right)$ $R(\nu)>-1,\ R(k)<-1$	$2^{-1}x^{\frac{k}{2}-\frac{1}{2}}K_{\frac{k+1}{4}}(2x^2);$ $R(\nu)>-1,\ R(7+8\nu\pm k)>0$ $R(7+4\nu\pm k)>0.$
$x^{4\nu-k+3}(1+x^4)^{-\frac{3}{2}-\nu}$ $R(\nu)>-1$	$\frac{1}{4}x^{k-\nu}J_\nu(4x)$ $R(\nu)>-1.$

EXAMPLE 2. *The $J_\nu Y_\nu$-transform:* From (24), (25) and the formula

$$x^\nu J_\nu(\sqrt{x})Y_\nu(\sqrt{x}) = -\pi^{-1/2}E\left(\begin{array}{c|c} 0 & 2;\ 1-2\nu,\ 1-\nu;\ \frac{1}{x} \\ 1;\ 1 & 2;\ \frac{1}{2}-\nu,\ 1 \end{array}\right); \tag{50}$$

we arrive at the transform pair

$$g(x)=\frac{\Gamma\left(\frac{3}{2}+\nu\right)}{\Gamma(1+2\nu)\Gamma(1+\nu)}\int_0^\infty (xy)^k {}_1F_2\left(\begin{matrix}\frac{3}{2}+\nu;\ -xy\\ 1+2\nu,\ 1+\nu\end{matrix}\right) f(y)\,dy, \tag{51}$$

$$f(x)=-\pi^{1/2}\int_0^\infty (xy)^{-k+\nu} J_\nu(\sqrt{x})\,Y_\nu(\sqrt{x})\,g(y)\,dy. \tag{52}$$

Some examples of this transform are given in the following table.

TABLE 2

$f(x)$	$g(x)$
e^{-x}	$x^k\dfrac{\Gamma(1+k)\Gamma\left(\frac{3}{2}+\nu\right)}{\Gamma(1+2\nu)\Gamma(1+\nu)}\ {}_2F_2\left(\begin{matrix}1+k,\ \frac{3}{2}+\nu;\ -x\\ 1+2\nu,\ 1+\nu\end{matrix}\right);$
$R(k+1)>0$	$R(k)>\nu-\frac{1}{2},\qquad R(1+2\nu)>0.$
$e^{-x}E(\gamma,\delta::x)$	$x^k\dfrac{\Gamma(\gamma)\Gamma(\delta)\Gamma(\gamma+k+1)\Gamma(\delta+k+1)\Gamma\left(\frac{3}{2}+\nu\right)}{\Gamma(1+2\nu)\Gamma(1+\nu)\Gamma(\gamma+\delta+k+1)}\ {}_3F_2\left(\begin{matrix}\gamma+k+1,\ \delta+k+1,\ \frac{3}{2}+\nu;\ -x\\ 1+2\nu,\ 1+\nu,\ \gamma+\delta+k+1\end{matrix}\right);$
	$R(1+2\nu)>0,\qquad R(\nu-\gamma-k)<\frac{1}{2},$
$R(\gamma+k+1)>0,\qquad R(\delta+k+1)>0$	$R(\nu-\delta-k)<\frac{1}{2}.$
$\dfrac{\sqrt{\pi x}}{\cos n\pi}e^{-\frac{x}{2}}K_n\left(\frac{x}{2}\right)$	$x^k\dfrac{\pi\Gamma\left(\frac{3}{2}+k+n\right)\Gamma\left(\frac{3}{2}+k-n\right)\Gamma\left(\frac{3}{2}+\nu\right)}{\cos n\pi\,\Gamma(1+2\nu)\Gamma(1+\nu)\Gamma(2+k)}\ {}_3F_3\left(\begin{matrix}\frac{3}{2}+k+n,\ \frac{3}{2}+k-n,\ \frac{3}{2}+\nu;\ -x\\ 1+2\nu,\ 1+\nu,\ 2+k\end{matrix}\right);$
$R\left(\frac{3}{2}+k\pm n\right)>0$	$R(1+2\nu)>0,\qquad R(1+k-\nu\pm n)>0.$

TABLE 2—Continued

$f(x)$	$g(x)$
$\Gamma\left(\frac{1}{2}-k'+m\right)\Gamma\left(\frac{1}{2}-k'-m\right)x^{-k'}e^{-\frac{x}{2}}$ $W_{k',m}(x)$ $R\left(\frac{3}{2}+k-k'\pm m\right)>0$	$x^{k}\dfrac{\Gamma\left(\frac{1}{2}-k'+m\right)\Gamma\left(\frac{1}{2}-k'-m\right)\Gamma\left(\frac{3}{2}+\nu\right)}{\Gamma(1+2\nu)\Gamma(1+\nu)\Gamma(2-2k'+k)}$ $\Gamma\left(\frac{3}{2}-k'+k+m\right)\Gamma\left(\frac{3}{2}-k'+k-m\right)$ ${}_3F_3\left(\begin{matrix}\frac{3}{2}-k'+k+m,\ \frac{3}{2}-k'+k-m,\ \frac{3}{2}+\nu;\ -x\\ \\ 1+2\nu,\ 1+\nu,\ 2-2k'+k\end{matrix}\right);$ $R(1+2\nu)>0,\qquad R(1+k-k'-\nu\pm m)>0.$
$K_{2n}(2\sqrt{x})$ $R(1+k\pm n)>0$	$x^{k}\dfrac{\Gamma(1+k+n)\Gamma(1+k-n)\Gamma\left(\frac{3}{2}+\nu\right)}{\Gamma(1+2\nu)\Gamma(1+\nu)}$ ${}_3F_2\left(\begin{matrix}1+k+n,\ 1+k-n,\ \frac{3}{2}+\nu;\ -x\\ \\ 1+2\nu,\ 1+\nu\end{matrix}\right);$ $R(1+2\nu)>0,\qquad R\left(\frac{1}{2}+k-\nu\pm n\right)>0.$
$\Gamma(2\nu-k)\Gamma(\nu-k)\left\{\Gamma\left(\frac{1}{2}+\nu-k\right)\right\}^{-1}$ ${}_2F_1\left(\begin{matrix}2\nu-\kappa,\ \nu-k;\ -\frac{1}{x}\\ \\ \frac{1}{2}+\nu-k\end{matrix}\right)$ $R\left(\nu+\frac{1}{2}-k\right)>0,\qquad R(2\nu+1)>0$	$2x^{\frac{k}{2}-\frac{1}{2}}K_{k+1}(2\sqrt{x});$ $R(2\nu+1)>0,\qquad R(2\nu-k+1)>0.$

EXAMPLE 3. *The* $\{J^2_{-\nu}-J^2_{\nu}\}$ *transform:* From (24), (25), and the formula

$$E\left(\begin{matrix}0\\ \\ 1;\ 1\end{matrix}\left|\begin{matrix}2;\ 1+\nu,\ 1-\nu:\frac{1}{x}\\ \\ 2;\ \frac{1}{2},\ 1\end{matrix}\right.\right)=\frac{\sqrt{\pi}}{2\sin\nu\pi}\{J^2_{-\nu}(\sqrt{x})-J^2_{\nu}(\sqrt{x})\}; \tag{53}$$

we arrive at the transform pair

$$g(x)=\frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma(1+\nu)\Gamma(1-\nu)}\int_0^\infty (xy)^k{}_1F_2\left(\begin{matrix}\frac{3}{2};-xy\\ 1+\nu,\ 1-\nu\end{matrix}\right)f(y)\,dy \tag{54}$$

$$f(x)=\frac{\sqrt{\pi}}{2\sin\nu\pi}\int_0^\infty (xy)^{-k}\{J^2_{-\nu}(\sqrt{x})-J^2_\nu(\sqrt{x})\}g(y)\,dy. \tag{55}$$

Particular cases of this transform are given in the following table.

TABLE 3

$f(x)$	$g(x)$
e^{-x};	$x^k\dfrac{\Gamma\left(\frac{3}{2}\right)\Gamma(1+k)}{\Gamma(1+\nu)\Gamma(1-\nu)}\;{}_2F_2\left(\begin{matrix}\frac{3}{2},\ 1+k;\ -x\\ 1+\nu,\ 1-\nu\end{matrix}\right);$
$R\left(k+\frac{1}{2}\right)>0$	$R\left(k+\frac{1}{2}\right)>0,\quad R(1\pm\nu)>0$
$K_{2n}(2\sqrt{x})$;	$x^k\dfrac{\Gamma\left(\frac{3}{2}\right)\Gamma(1+k+n)\Gamma(1+k-n)}{\Gamma(1+\nu)\Gamma(1-\nu)}\;{}_3F_2\left(\begin{matrix}\frac{3}{2},\ 1+k+n,\ 1+k-n;\ -x\\ 1+\nu,\ 1-\nu\end{matrix}\right);$
$R\left(\frac{1}{2}+k\pm n\right)>0$	$R(1\pm\nu)>0,\quad R\left(\frac{1}{2}+k\pm n\right)>0.$
$e^{-x}E(\gamma,\delta::x)$;	$x^k\dfrac{\Gamma(\gamma+k+1)\Gamma(\delta+k+1)\Gamma\left(\frac{3}{2}\right)\Gamma(\gamma)\Gamma(\delta)}{\Gamma(\gamma+k+\delta+1)\Gamma(1+\nu)\Gamma(1-\nu)}\;{}_3F_3\left(\begin{matrix}\frac{3}{2},\gamma+k+1,\ \delta+k+1;\ -x\\ 1+\nu,\ 1-\nu,\ \gamma+\delta+k+1\end{matrix}\right);$
$R\left(\gamma+k+\frac{1}{2}\right)>0,\ R\left(\delta+k+\frac{1}{2}\right)>0$	$R(1\pm\nu)>0,\quad R\left(\gamma+k+\frac{1}{2}\right)>0,\quad R\left(\delta+k+\frac{1}{2}\right)>0.$

TABLE 3 — Continued

$f(x)$	$g(x)$
$\dfrac{\sqrt{\pi x}}{\cos n\pi} e^{-\frac{x}{2}} K_n\left(\dfrac{x}{2}\right);$	$x^k \dfrac{\pi\Gamma\left(\frac{3}{2}+k+n\right)\Gamma\left(\frac{3}{2}+k-n\right)\Gamma\left(\frac{3}{2}\right)}{\cos n\pi\Gamma(2+k)\Gamma(1+\nu)\Gamma(1-\nu)}$ ${}_3F_3\left(\begin{matrix}\frac{3}{2}, \frac{3}{2}+k+n, \frac{3}{2}+k-n; -x \\ 2+k, 1+\nu, 1-\nu\end{matrix}\right);$
$R(1+k\pm n)>0$	$R(1\pm\nu)>0, \quad R(1+k\pm n)>0.$
$\Gamma\left(\frac{1}{2}-k'+m\right)\Gamma\left(\frac{1}{2}-k'-m\right)x^{-k'}$ $e^{-\frac{x}{2}} W_{k',m}(x);$	$x^k \dfrac{\Gamma\left(\frac{1}{2}-k'+m\right)\Gamma\left(\frac{1}{2}-k'-m\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma(2-2k'+k)\Gamma(1+\nu)\Gamma(1-\nu)}$ $\Gamma\left(\frac{3}{2}-k'+k+m\right)\Gamma\left(\frac{3}{2}-k'+k-m\right)$ ${}_3F_3\left(\begin{matrix}\frac{3}{2}-k'+k+m, \frac{3}{2}-k'+k-m, \frac{3}{2}; -x \\ 2-2k'+k, 1+\nu, 1-\nu\end{matrix}\right);$
$R(1+k-k'\pm m)>0$	$R(1\pm\nu)>0, \quad R(1+k-k'\pm m)>0.$
$\dfrac{\Gamma(\nu-k)\Gamma(-\nu-k)}{\Gamma\left(\frac{1}{2}-k\right)}$ ${}_2F_2\left(\begin{matrix}\nu-k, -\nu-k; -\frac{1}{x} \\ \frac{1}{2}-k\end{matrix}\right);$	$2x^{\frac{k}{2}-\frac{1}{2}} K_{k+1}(2\sqrt{x});$
$R\left(k-\frac{1}{2}\right)<0$	$R(1\pm\nu)>0.$

6. References

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