Minimum Number of Subsets to Distinguish Individual Elements

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Given a set $S$ of cardinality $m$, we determine the minimum cardinality $f(m)$ for a family $F$ of subsets of $S$ such that each $s \in S$ can be expressed as the intersection of some subfamily of $F$. The problem is solved in the following inverse form. For a given number $n$ of subsets of $S$, find $g(n)$: the maximum number of elements of $S$ which can be written as the intersection of some of these subsets. We show that $g(n)$ is the largest binomial coefficient for combinations of $n$ things.

Key Words: Classification design, combinatorics, set theory.

1. Introduction

Let $S$ be a finite set of given cardinality $|S| = m$. An element $s \in S$ will be said to be distinguished by a family $\mathcal{F}$ of subsets of $S$, if $\{s\}$ is the intersection of some subfamily of $\mathcal{F}$. In this note we solve the following combinatorial problem (conveyed by K. E. Kloss): What is the minimum possible cardinality $f(m)$ for a family which distinguishes all elements of $S$? (Trivially $f(m) \geq m$, since $f(m) = \min \{n: g(n) \geq m\}$ follows when we observe that $m \leq g(n) \iff f(m) \leq n$.)

2. Proof

Let $g(n)$ be the binomial coefficient on the right-hand side of (1). We first show that $h(n) \leq g(n)$. For this purpose, let $A$ be a set with cardinality $|A| = n$ and let

$$S^* = \{s_1, \ldots, s_{h(n)}\}$$

be the collection of all subsets of $A$ which have cardinality $[n/2]$. For $1 \leq i \leq n$, let

$$H_i = \{s \in S^*: i \in s\},$$

and put $\mathcal{H} = \{H_1, \ldots, H_n\}$. The possibility

$$r \in \bigcap \{H_i: s \in H_i\}, \{r \in S^* - \{s\},$$

is ruled out because $r$ cannot be a subset of $s$, so that some $iA$ must satisfy $i \in s - r$ and thus $s \in H_i, r \in S^* - H_i$. It follows that

$$\{s\} = \bigcap \{H_i: s \in H_i\},$$

i.e., each element of $S^*$ is distinguished by $\mathcal{H}$. This implies $h(n) \leq g(n)$.

The proof of (1) will be completed by showing that $g(n) \leq h(n)$. Let $\mathcal{F} = \{F_1, \ldots, F_n\}$ be a family of subsets of $\mathcal{S}$.
finite sets with union $S$. For each $s \in S$, let

$$F(s) = \{F_i : s \in F_i\}, \quad F_s = \cap \{F_i : s \in F_i\}, \quad T = \{s \in S : F_s = \{s\}\}.$$ 

Then $T$ consists of those elements of $S$ which are distinguished by $\mathcal{F}$ so that $|T| \leq h(n)$ is what must be proved.

A collection of sets will be called independent if no set-inclusions hold between any pair of members. For example, the collection $\{F_i : s \in T\}$ is an independent collection of subfamilies of an $n$ member family. Since this collection has $|T|$ members, it suffices to show that any independent family of subsets of an $n$ element set has at most $h(n)$ members. This can be shown using the well-known SDR theorem but we find it as easy to employ an elementary argument.

For an $n$ element set $A$, let $S_i$ denote the family of subsets of $A$ which have cardinality $i$, $0 \leq i \leq n$. Each $S_i$, and in particular $S_{\lceil n/2 \rceil} = S^*$, is an independent family. If $\{n/2\}$ is the smallest integer not less than $n/2$, then

$$|S_{\lfloor n/2 \rfloor}| = |S_{\lfloor n/2 \rfloor}| = h(n).$$

We shall show that any other independent family $P$ of subsets of $A$ can be mapped $1 - 1$ into $S_{\lfloor n/2 \rfloor}$ and thus conclude that

$$|P| \leq |S_{\lfloor n/2 \rfloor}|.$$  (5)

Suppose some member of $P$ has cardinality less than $\{n/2\}$. Let $P_j$ be the family of members of $P$ which have smallest cardinality, say $j$. Let $M$ be the family of members of $S_{j+1}$ which contain a member of $P_j$. Since $P$ is independent, $P' = M \cup (P-P_j)$ is also independent, and $P \cap M = \emptyset$. We will show below that

$$j < \left\lfloor \frac{n}{2} \right\rfloor \implies |P_j| \leq |M|,$$  (6)

and so $|P| \leq |P'|$. Then by induction on the minimum cardinality of any member of $P'$, $P'$, etc., we obtain an independent family $Q$ such that $|P| \leq |Q|$, $Q \cap S_i = \emptyset$ for $0 \leq i < \left\lfloor \frac{n}{2} \right\rfloor$, and $Q \cap S_i = P \cap S_i$ for $\{n/2\} < i \leq n$.

The structure of the family of all subsets of $A$ is the same relative to the relationships “is a subset of” and “is a superset of.” Hence a “mirror-image” of the preceding construction will produce from $Q$ an independent family $R$ such that $|Q| \leq |R|$, $R \cap S_i = \emptyset$ for $n > i > \{n/2\}$, and such that

$$R \cap S_i = Q \cap S_i = \emptyset \text{ for } |n/2| > i > 0.$$

In this fashion we arrive at the result $|P| \leq |R|$ and $R \subset S_{\lfloor n/2 \rfloor}$.

It only remains to show (6).

Let $K$ be the number of distinct pairs $(p, m)$ where $p \in P_j$, $m \in M$, and $p \subseteq m$. We have

$$K = (n-j)|P_j|$$  (7)

since any $p \in P_j$ can be extended in exactly $(n-j)$ ways to an $m \in M$. Also however,

$$K \leq (j+1)|M|,$$  (8)

since any $m \in M$ contains $j+1$ subsets of cardinality $j$ and thus contains at most $j+1$ members of $P_j$.

Where $0 \leq j < \{n/2\}$,

$$(j+1)/(n-j) \leq 1$$

and therefore combining (7) and (8) we have (6), and the proof is complete.