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Abscissas and Weights for Gaussian Quadrature for $N=2$ to 100, and $N=125$, 150, 175, and 200[1]

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The abscissas and weights for Gaussian Quadrature of order $N=2$ to 100, and $N=125$, 150, 175, and 200 are given. The abscissas are given to twenty-four places and the error is estimated to be no more than 1 unit in the last place. The weights are given to twenty-three places and the error is estimated to be no more than 1 unit in the last place.

Key Words: Gaussian quadrature, integral equations, numerical integration, zeros of Legendre polynomials.

1. Introduction

The advent of high speed digital computers has made the use of Gaussian Quadrature formulae of very high order a practical procedure. It can be programmed with a few instructions and the awkward abscissas are easily handled by the computer. It is a familiar technique which usually converges rapidly, and since the whole range of integration can be covered at once instead of making many subdivisions, it will usually be very efficient.

In addition to its use in numerical integration Gaussian Quadrature can be used in the numerical solution of integral equations and also in the evaluation of functions that can be written as integrals. The last four sets of very high order were computed [1][3] since "exact values of these quantities are also interesting in view of certain unsettled theoretical conjectures that have been made about distributions of the weights and abscissas." It was reported by Davis and Rabinowitz [2] that there was a "brisk demand" for the sets $N=64$, 80, and 96 even though they had not been published and there existed some doubt about their accuracy.

Higher order Gaussian Quadratures were found to be very efficient in the calculation of the inductance of rectangular conductors such as strip transmission lines at various frequencies [3]. It was found that if the width of the outer conductors were divided into n equal parts and the width of the inner conductors according to a Gaussian distribution, then the rate of convergence of the inductance function (with respect to n the number of subdivisions) was greatly improved. In order to obtain the limiting value of inductance (for an infinite number of subdivisions) a formula of the form $L_\infty = L_n + an^{-\alpha}$ was used for sufficiently large n. This equation is most easily solved for the unknowns L_∞, a, and α, if the inductance function L_n is calculated for four different n chosen such that $N_1/N_2 = N_3/N_4$. Therefore, specific high order Gaussian Quadratures were needed.

The only high order Gaussian Quadratures that were found in the literature were those mentioned in the references. A. H. Stroud is working on tables for $N=2$, 64, 96, 168, 256, 384, and 512 but he has not published them at the present time [4].

2. Method

The Gaussian Quadrature formula is given by

$$\int_{-1}^{1} F(X)dx \approx \sum_{k=1}^{n} H_k F(X_k), \tag{1}$$

[1] The complete tables, 77 pages, have recently been published as NBS Monograph 98, Abscissas and Weights for Gaussian Quadrature For $N=2$ to 100, and $N=125$, 150, 175, and 200. This monograph is available from the Superintendent of Documents, Government Printing Office, Washington, D.C. 20402. Price 55 cents.
[2] Radio Standards Engineering, NBS Laboratory, Boulder, Colo.
[3] Figures in brackets indicate the literature references at the end of this paper.

where the weighing coefficients H_k are given by

$$H_k = \frac{2}{nP_{n-1}(X_k)P_n'(X_k)}; \qquad (2)$$

$P_n(X)$ is Legendre's polynomial of degree n, $P_n'(X)$ is the first derivative of $P_n(X)$, and the $X_{k's}$ are the zeros of $P_n(X)$ [5].

From the equations

$$(1-X^2)P_n'(X) = nXP_n(X) + nP_{n-1}(X)$$

$$= (n+1)XP_n(X) - (n+1)P_{n+1}(X), \qquad (3)$$

and with X_k a zero of $P_n(X)$ in (3), the following relations can be obtained:

$$(1-X_k^2)P_n'(X_k) = nP_{n-1}(X_k) = -(n+1)P_{n+1}(X_k). \qquad (4)$$

Equation (2) can now be expressed as [6]

$$H_k = \frac{2}{(1-X_k^2)\,[P_n'(X_k)]^2} = \frac{2(1-X_k^2)}{(n+1)^2[P_{n+1}(X_k)]^2}. \qquad (5)$$

Since $P_n(X)$ is a symmetric function of X for even n and skew symmetric for odd n, only the positive roots between 0 and 1 have to be calculated [5]. The weights H_k will be the same for positive and negative X's of the same magnitude.

The roots of the Legendre Polynomials and their corresponding weighing factors were computed as follows. Upper and lower bounds were obtained for the kth root of $P_n(X)$, namely $X_{n,k}$, from the fact that the roots of $P_{n-1}(X)$ separate the roots of $P_n(X)$:

$$1 < X_{n,1} < X_{n-1,1} \qquad \text{For } k=1$$
$$X_{n-1,k-1} < X_{n,k} < X_{n-1,k} \qquad \text{For } k>1. \qquad (6)$$

When n is odd, $X=0$ is also a root of $P_n(X)$.

Setting the lower limit equal to a and the upper limit equal to b, the method of false position can be used to obtain an initial approximation c:

$$C = a + \frac{F(a)}{F(a)-F(b)}(b-a) = a + \frac{P_n(a)}{P_n(a)-P_n(b)}(b-a). \qquad (7)$$

In order to calculate $P_n(X)$ and $P_{n+1}(X)$ the following recurrence relation was used:

$$P_{n+1}(X) = \frac{2n+1}{n+1}XP_n(X) - \frac{n}{n+1}P_{n+1}(X). \qquad (8)$$

After an approximation has been obtained, the root $X_{n,k}$ can be obtained much faster by using the Newton-Raphson technique which converges quadrically [6]:

$$X_{n+1} = X_n - \frac{F(X_n)}{F'(X_n)}, \qquad (9)$$

where X_{n+1} is the next approximation to the root X_n. The value of $F(X_n)$ can be obtained from (4).

The last four sets ($n=125$, etc.) required a little different approach. The first root, $X_{n,1}$, was obtained for all n between 101 and 200. The fact that the roots of $P_n(X)$ became monotonically increasingly further apart was then used to locate succeeding roots:

$$1 - X_{n,1} < X_{n,1} - X_{n,2} < \ldots . < X_{n,k} - X_{n,k+1}. \qquad (10)$$

In order to find the $X_{n,k}$ root, the quantity $a = X_{n,k-1} - X_{n,k-2}$ was found. Then b was set equal to $X_{n,k-1} - a$ and a equal to $b-a$. The polynomials P_n(a) and P_n(b) were examined to see if they were of opposite sign. If so, the upper and lower bounds were found and the root was calculated as before. If they had the same sign, then a and b were both decreased by a and the above process repeated until P_n (a) and P_n (b) did indeed have opposite signs.

3. Errors

The calculations were done with a double precision computer routine. The routine was accurate to 1.9×10^{25} (or 2^{84}).

In order to check the accuracy of the abscissas the formula $\sum_1^n X_k^2 = \frac{n(n-1)}{2n-1}$ was used. The largest error found was 5 units in the 24th decimal place and the next largest was 2 units in the 24th place. More than half of the checks had zero error in the 24th place. Squaring a number that has an error in the kth place gives an error in the kth place that is approximately twice that of the original error. When checking the tables in references [1], [2], and [5] only one error in the abscissas was found. This was in reference [5] where an error of 0.5 units in the 21st place was discovered. The calculations had been carried out to 7.6×10^{22} and the abscissas given to 21 places. The 21st and 23d places are 851 but the value given in the table was 8 in the 21st place instead of being rounded up to 9.

In this report the abscissas have been given to 24 places and it has been concluded that there is an error of no more than 1 unit in the last place. It is also concluded that this occurrence will be very rare. There will be further comments on the abscissas after the weights have been examined.

The weights were checked by summing them between zero and one and comparing the results to unity. The largest error found was 1.48 units in the 23d place. The previously mentioned references were checked for errors in the weights and no error larger than 1 unit in the last place was found through $n=48$. In reference [5] a table of errors obtained by summing the weights in references [1] and [2] is given. Even though the error in reference [1] reaches 11 units in the last place, there is no error in any individual weight of more than 1 unit. It is interesting to note that the errors in [1] are always 1 unit too small and are caused by not rounding up. It was only when the first omitted digit was 9 that the rounding was done correctly and then only half the time.

It was found that the errors in the higher order tables of reference [5] (up to $n=64$) were larger than expected. The sums of some weights were in error 4 units in the last place, but almost all the error was concentrated in the first or second weight. For example, the value of the first weight for $n=64$ was found to be 4.7 units too small in the last place and the sum of the weights was 4 units too small.

Since the preceding observation questioned the accuracy of the 23d place of the first two weights of higher n in this report, the following analysis was made. It was assumed that the errors would follow the same behavior pattern in double precision as in single precision. Starting at $n=50$, all the higher orders were calculated in single precision and the sum of the weights obtained. Single precision was 6.87 $\times 10^{10}$ (or 2^{36}) which is less than 11 significant digits. It was assumed that there were 11 good digits and the error in the 9th place was examined. For n up to 100 the largest error in the sum was 3.0 units in the 9th place and 2.2 units for an individual weight. The largest error in the last four sets was 3.9 units in the 9th place of the sum but only 2.3 units for any individual term. It was observed that if the error in the sum was 1 unit or more in the 9th place, it was concentrated in the first two weights.

These results tend to show that the errors do follow a similar pattern and give an error bound on the weights. It is to be remembered that in single precision there were *less* than 11 significant digits and the error in the 9th place was examined while in double precision there were *more* than 25 significant digits and the error in the 23d place was examined. Hence, the error in the 9th place in single precision should be greater than the error in the 23d place in double precision. It was concluded, on the basis of the preceding discussion and the fact that the largest error in the sum in double precision was 1.48 units in the 23d place, that the individual weights were accurate to within 1 unit in the 23d place.

The sum of the squares of the abscissas were checked and the largest error was 4 units in the 10th place. This was a better check than the double precision because of a "digit" problem. The sum of the checks reaches a value of approximately 50 for higher n and hence it is impossible to obtain the sum accurately to more than 23 decimal places. When the single precision was used, the squaring and summing were done in double precision and hence the error in the 10th decimal place could be examined. It has been mentioned previously that squaring tends to double the error and also that there are less than 11 good digits in single precision while in double precision there are more than 25 good digits. Hence, the error is no more than 1 unit in the 24th place.

The three values of n given in reference [2] were checked and were accurate to within 1 unit in the last place. This is better than one would expect from the above analysis since that author was only calculating his abscissas to an accuracy of 5.1×10^{-22}, which is approximately 1 digit less than reference [1]. It is believed the better accuracy was obtained because of the rapid rate of convergence of the Newton-Raphson method and because triple precision was used. The convergence rate was in fact observed to be approximately "squared"; the errors were approximately 10^{-4}, 10^{-8}, 10^{-16} for successive iterations. Hence, when the error condition was satisfied, the roots were probably more accurate than expected and therefore the weights were more accurate. It is believed that the larger error of the first two weights is caused by the $1-X^2$ term in the numerator of the expression for the weights.

4. Table Errors

After the final form of the manuscript had been prepared, the abscissas and weights were punched onto cards from the manuscript. These values were then subjected to the same numerical checks as before and the errors checked to insure they were the same as before. The values were also compared against the values that were punched onto cards during the computation.

Tables. Examples of the tables for $N=2$ to 100, and $N=125$, 150, 175, and 200 follow. The complete set of tables will be published in a NBS monograph by Carl H. Love.

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N = 70

ABSCISSAS	WEIGHTS
0.99941 82859 73575 84205 6743	0.00149 27212 88844 51573 104
0.99693 62519 61680 15660 9813	0.00347 18948 93078 14325 500
0.99247 60552 11689 98109 8942	0.00544 71118 74217 21831 282
0.98604 55580 70398 65992 7101	0.00741 17693 63190 21036 211
0.97765 74059 57592 40039 3908	0.00936 17627 69699 02681 150
0.96732 82236 64986 43838 8618	0.01129 31846 49931 53764 963
0.95507 85091 14292 84264 0327	0.01320 21908 14676 74762 507
0.94093 25790 03815 35552 2816	0.01508 49878 65443 12768 230
0.92491 85168 97934 44027 2266	0.01693 78363 76302 93253 184
0.90706 81162 60922 84943 6353	0.01875 70570 93133 42341 545
0.88741 68168 63348 17112 4374	0.02053 90378 24326 45338 449
0.86600 36342 13858 62938 0035	0.02228 02404 52256 59583 389
0.84287 10819 98980 24231 8594	0.02397 72078 89100 29227 869
0.81806 50876 25441 18902 7232	0.02562 65709 08468 48279 899
0.79163 49010 07892 75810 7676	0.02722 50548 18664 41715 911
0.76363 29967 71899 56892 5095	0.02876 94859 55808 28066 131
0.73411 49700 60942 64130 7652	0.03025 67979 80154 23781 654
0.70313 94261 51528 59706 2947	0.03168 40379 61308 48173 465
0.67076 78640 94077 40564 6281	0.03304 83722 39372 42047 087
0.63706 45546 09778 09627 8860	0.03434 70920 49906 53756 855
0.60209 64124 85355 48733 6767	0.03557 76189 01292 38053 277
0.56593 28637 18808 28637 2959	0.03673 75096 93672 69534 804
0.52864 57076 79711 12726 5081	0.03782 44615 69222 81719 727
0.49030 89745 57636 58926 9778	0.03883 63164 84073 40397 900
0.45099 87783 81647 86573 1640	0.03977 10654 92776 56747 785
0.41079 31659 02630 58937 1263	0.04062 68527 36789 61635 123
0.36977 19616 38461 89583 9405	0.04140 19791 29045 20863 823
0.32801 66093 89643 25784 6132	0.04209 49057 27284 40602 098
0.28561 00105 40037 86169 1665	0.04270 42567 89449 77776 997
0.24263 63594 63740 64578 3579	0.04322 88225 05068 69978 940
0.19918 09763 64857 66415 1404	0.04366 75613 97201 44025 255
0.15533 01378 82070 24730 9006	0.04401 96023 90183 45875 736
0.11117 09057 94298 69373 5752	0.04428 42465 39055 40677 580
0.06679 09541 67551 32400 3719	0.04446 09684 17246 37082 356
0.02227 83952 86140 30969 3493	0.04454 94171 59754 66720 217

N = 75

ABSCISSAS	WEIGHTS
0.99949 27755 36035 45648 3467	0.00130 15917 17375 85599 389
0.99732 84289 62231 80330 9066	0.00302 76710 14606 04129 123
0.99343 85365 78892 70437 7047	0.00475 10691 85015 27396 590
0.98782 89667 67524 52014 1267	0.00646 64649 07037 53840 196
0.98050 93245 97416 60577 1693	0.00817 07107 07327 82640 372
0.97149 22563 43063 82336 3814	0.00986 08249 16114 01839 205
0.96079 33637 41892 31415 5448	0.01153 38733 28304 49596 681
0.94843 11650 79287 37792 1920	0.01318 69567 62824 80211 961
0.93442 70599 64107 15683 6194	0.01481 72122 89814 46852 014
0.91880 52912 28393 99127 1240	0.01642 18171 19024 64004 360
0.90159 29025 48446 33409 3964	0.01799 79931 25645 05063 795
0.88281 96914 47895 54216 0092	0.01954 30115 20127 88937 957
0.86251 81576 26883 42546 1262	0.02105 41975 12282 84223 645
0.84072 34466 54958 05009 7012	0.02252 89349 13865 77645 055
0.81747 32891 03541 31484 3395	0.02396 46706 53716 95917 477
0.79280 79352 13563 51036 0414	0.02535 89191 90216 37909 421
0.76677 00852 06641 92552 4095	0.02670 92668 10120 85177 235
0.73940 48153 58032 52649 6514	0.02801 33758 04780 54082 526
0.71075 94999 58041 12676 9559	0.02926 89885 15725 98680 503
0.68088 37292 96269 59804 4384	0.03047 39312 42214 53920 314
0.64982 92238 10262 23258 6887	0.03162 61180 03749 64805 603
0.61764 97445 46925 96445 0186	0.03272 35541 50934 22052 152
0.58440 10000 91577 88031 9607	0.03376 43398 18334 09264 696
0.55014 05501 25645 55150 3015	0.03474 66732 13330 40653 510
0.51492 77057 79916 64052 7992	0.03566 88537 35240 45308 912
0.47882 34269 55803 15701 5809	0.03652 92849 19290 33900 685
0.44189 02167 92348 26653 7336	0.03732 64772 00332 09016 731
0.40419 20134 61653 61319 0807	0.03805 90504 91513 60313 563
0.36579 40794 80035 92532 6018	0.03872 57365 73432 57584 147
0.32676 28887 26526 30235 8400	0.03932 53812 89635 16252 077
0.28716 60113 64297 24694 2702	0.03985 69465 44656 35257 597
0.24707 19968 64234 75690 4240	0.04031 95121 01141 57755 817
0.20655 02553 33159 60711 4475	0.04071 22771 72937 33029 876
0.16567 09373 52137 81838 5950	0.04103 45618 11392 10667 622
0.12450 48125 32900 26162 5085	0.04128 58080 82467 18908 346
0.08312 31470 02610 99169 7497	0.04146 55810 32619 09213 525
0.04159 75800 29079 45597 4891	0.04157 35694 41781 27878 300
0.00000 00000 00000 00000 0000	0.04160 95863 62141 40938 047

N = 80

ABSCISSAS	WEIGHTS
0.99955 38226 51630 62988 0080	0.00114 49500 03186 94153 455
0.99764 98643 98237 68889 9494	0.00266 35335 89512 68166 929
0.99422 75409 65688 27789 2064	0.00418 03131 24694 89523 674
0.98929 13024 99755 53102 6503	0.00569 09224 51403 19864 927
0.98284 85727 38629 07041 8288	0.00719 29047 68117 31275 268
0.97490 91405 85727 79338 5645	0.00868 39452 69260 85842 641
0.96548 50890 43799 25145 2273	0.01016 17660 41103 06452 083
0.95459 07663 43634 90549 3482	0.01162 41141 20797 82691 647
0.94224 27613 09872 67475 2266	0.01306 87615 92401 33929 379
0.92845 98771 72445 79595 3046	0.01449 35080 40509 07611 696
0.91326 31025 71757 65416 4734	0.01589 61835 83725 68804 490
0.89667 55794 38770 68319 4324	0.01727 46520 56269 30635 858
0.87872 25676 78213 82870 3773	0.01862 68142 08299 03142 874
0.85943 14066 63111 09697 7192	0.01995 06108 78141 99892 889
0.83883 14735 80255 27561 6623	0.02124 40261 15782 00638 871
0.81695 41386 81463 47037 1125	0.02250 50902 46332 46192 622
0.79383 27175 04605 44994 8639	0.02373 18828 65930 10129 319
0.76950 24201 35041 37386 5616	0.02492 25357 64115 49110 512
0.74400 02975 83597 27231 6541	0.02607 52357 67565 11790 297
0.71736 51853 62099 88025 4068	0.02718 82275 00486 38067 442
0.68963 76443 42027 60077 1208	0.02825 98160 57276 86239 675
0.66085 98989 86119 80173 5967	0.02928 83695 83267 84769 277
0.63107 57730 46871 96624 7928	0.03027 23217 59557 98066 122
0.60033 06228 29751 74315 4746	0.03121 01741 88114 70164 244
0.56867 12681 22709 78472 5486	0.03210 04986 73487 77314 806
0.53614 59208 97131 93201 9857	0.03294 19393 97645 40138 284
0.50280 41118 88784 98759 3673	0.03373 32149 84611 52281 668
0.46869 66151 70544 47703 6078	0.03447 31204 51753 92879 436
0.43387 53708 31756 09306 2387	0.03516 05290 44747 59349 553
0.39839 34058 81969 22702 4380	0.03579 43939 53416 05460 286
0.36230 47534 99487 31561 9043	0.03637 37499 05835 97804 396
0.32566 43707 47701 91461 9113	0.03689 77146 38276 00883 915
0.28852 80548 84511 85310 9139	0.03736 54902 38730 49002 671
0.25095 23583 92272 12049 3159	0.03777 63643 62001 39748 978
0.21299 45028 57666 13257 2389	0.03812 97113 14477 63834 421
0.17471 22918 32646 81255 9339	0.03842 49930 06959 42318 521
0.13616 40228 09143 88655 9241	0.03866 17597 74076 46332 708
0.09740 83984 41584 59906 3278	0.03883 96510 59051 96893 177
0.05850 44371 52420 66862 8993	0.03895 83959 62769 53119 863
0.01951 13832 56793 99765 4351	0.03901 78136 56306 65481 128

N = 85

ABSCISSAS	WEIGHTS
0.99960 44773 57478 45432 6304	0.00101 49719 08967 74369 537
0.99791 66011 98116 93315 0309	0.00236 13317 04285 02089 677
0.99488 23742 95616 27845 3367	0.00370 65001 25759 31670 687
0.99050 52177 16415 96378 7124	0.00504 68384 26924 44272 545
0.98479 09576 85580 49732 7164	0.00638 03985 87897 51509 869
0.97774 72884 12243 39193 3334	0.00770 53559 60382 75707 990
0.96938 37119 23678 32800 5136	0.00901 99154 39993 63127 897
0.95971 15159 57188 36239 6347	0.01032 23002 30524 24589 382
0.94874 37562 54578 95340 4649	0.01161 07512 86703 89800 962
0.93649 52381 16430 69968 6698	0.01288 35288 56498 08429 051
0.92298 24960 96090 04558 8769	0.01413 89145 48400 83293 056
0.90822 37715 39091 88489 6825	0.01537 52135 42389 62687 441
0.89223 89878 91355 90158 1714	0.01659 07568 31154 67007 521
0.87504 97237 69097 08763 8653	0.01778 39034 51398 17090 774
0.85667 91838 09956 18909 5811	0.01895 30426 88182 84044 681
0.83715 21673 37082 65019 8542	0.02009 65962 43575 42174 179
0.81649 50348 74791 61472 0573	0.02121 30203 64089 37967 242
0.79473 56725 59116 50544 7149	0.02230 08079 22839 37418 946
0.77190 34544 90293 07866 0311	0.02335 84904 52989 89189 770
0.74802 92030 77434 07395 6270	0.02438 46401 29435 68314 242
0.72314 51474 28601 02138 4583	0.02537 78716 95866 08847 737
0.69728 48798 42249 54765 0964	0.02633 68443 34514 35982 173
0.67048 33104 58662 66070 2672	0.02726 02634 76011 16478 577
0.64277 66201 32514 71484 3400	0.02814 68825 46865 07584 638
0.61420 22115 90136 99307 1297	0.02899 55046 52190 15208 987
0.58479 86589 37388 08571 0292	0.02980 49841 91395 88737 561
0.55460 56555 86269 43315 8991	0.03057 42284 04649 99572 392
0.52366 39606 70567 87478 8045	0.03130 21988 48020 87044 840
0.49201 53440 22851 31047 7142	0.03198 79127 95304 67445 977
0.45970 25297 87088 72076 1470	0.03263 04445 64642 17818 904
0.42676 91387 43009 70172 1995	0.03322 89267 68132 76976 253
0.39325 96294 20059 19065 0215	0.03378 25514 82757 53033 131
0.35921 92380 80438 04875 6645	0.03429 05713 41029 84670 822
0.32469 39176 52247 55745 4017	0.03475 23005 39900 63752 925
0.28973 02756 95173 70353 8710	0.03516 71157 66555 78824 981
0.25437 55114 82453 52249 7731	0.03553 44570 39855 69908 199
0.21867 73522 84059 01571 0243	0.03585 38284 66280 81255 692
0.18268 39889 37112 67387 6281	0.03612 47989 09362 46037 475
0.14644 40107 90510 98921 0902	0.03634 70025 71695 20376 676
0.11000 63401 11577 23735 2680	0.03652 01394 88744 88485 748
0.07342 01660 43291 12561 9508	0.03664 39759 33785 70248 641
0.03673 48782 01249 66352 0469	0.03671 83447 33419 61622 215
0.00000 00000 00000 00000 0000	0.03674 31454 93252 10660 021

N = 90

ABSCISSAS	WEIGHTS
0.99964 69712 86638 43746 3248	0.00090 59323 71214 83309 373
0.99814 03799 38568 15356 1306	0.00210 77787 74526 32989 148
0.99543 18120 58344 66392 6755	0.00330 88672 43336 01819 543
0.99152 39288 11062 78612 9147	0.00450 61236 13674 97786 414
0.98642 13650 57832 84873 4254	0.00569 79815 60747 35260 085
0.98013 02513 45148 38545 8953	0.00688 29832 08463 28431 473
0.97265 81620 90193 13999 7465	0.00805 96949 44620 01565 867
0.96401 40981 71505 48339 3667	0.00922 66969 57741 99094 032
0.95420 84738 81500 33616 0720	0.01038 25823 09893 21461 381
0.94325 31036 45357 76815 3575	0.01152 59578 89148 05885 059
0.93116 11875 00432 00700 5847	0.01265 54458 37168 12886 888
0.91794 72950 66586 38337 2356	0.01376 96851 12337 09343 075
0.90362 73479 31302 69386 9986	0.01486 73330 88043 32405 038
0.88821 86004 34745 98129 8376	0.01594 70671 51006 63901 321
0.87173 96188 62903 43447 4028	0.01700 75862 85222 67570 940
0.85421 02590 67071 88228 6021	0.01804 76126 34460 23616 405
0.83565 16425 33377 04556 4199	0.01906 58930 39137 31842 532
0.81608 61309 29481 05644 3754	0.02006 12005 44639 59596 453
0.79553 72991 58248 13486 3601	0.02103 23358 78722 56311 706
0.77402 99069 50334 24680 6958	0.02197 81288 95934 13383 869
0.75158 98690 29638 46817 8415	0.02289 74399 87163 18463 499
0.72824 42238 87390 36362 5800	0.02378 91614 52528 72321 010
0.70402 11012 02391 14355 5469	0.02465 22188 35904 85293 597
0.67894 96879 46597 14618 1269	0.02548 55722 19443 22848 447
0.65306 01932 16842 19196 1262	0.02628 82174 76514 58736 160
0.62638 38118 35045 12676 2609	0.02705 91874 81547 95852 161
0.59895 26867 60742 18588 8769	0.02779 75532 75302 27515 804
0.57079 98703 61220 97870 5362	0.02850 24251 84161 41631 876
0.54195 92845 85913 42618 9305	0.02917 29538 92100 74248 656
0.51246 56800 93027 97098 8463	0.02980 83314 64031 27548 715
0.48235 45943 77665 69252 0121	0.03040 77923 19286 95269 039
0.45166 23089 51869 36757 6379	0.03097 06141 54080 92094 594
0.42042 58056 28197 75610 1396	0.03149 61188 11818 63607 696
0.38868 27219 59498 20677 6509	0.03198 36731 00218 57603 946
0.35647 13058 88567 84613 1189	0.03243 26895 54255 61691 179
0.32383 03696 62345 96651 1395	0.03284 26271 44007 50457 863
0.29079 92430 66166 65154 9602	0.03321 29919 26551 31651 404
0.25741 77260 34420 12992 0888	0.03354 33376 41124 27668 293
0.22372 60406 94722 85926 8958	0.03383 32662 46831 68725 793
0.18976 47829 03379 01902 0874	0.03408 24284 02253 99546 361
0.15557 48733 30529 11951 1405	0.03429 05238 86375 04193 170
0.12119 75081 53924 08296 8749	0.03445 73019 60324 25617 460
0.08667 41094 20734 77008 5237	0.03458 25616 69496 89141 805
0.05204 62751 37206 94905 9279	0.03466 61520 85688 24018 827
0.01735 57291 46299 65246 1298	0.03470 79724 88950 05792 046

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