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On Certain Discrete Inequalities and Their Continuous Analogs *

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The purpose of this paper is to find inequalities between the L^2-norms of a function and its kth and mth derivatives.

Key Words: Inequalities, norms, Wirtinger.

1. Introduction

In 1955 [2] [2] Fan, Taussky, and Todd discovered discrete analogs of certain integral inequalities involving functions and their derivatives. They considered inequalities of the Wirtinger type: $\int_0^{2\pi} x(t)^2dt \leq \int_0^{2\pi} x'(t)^2dt$, where x has period 2π and $\int_0^{2\pi} x(t)dt=0$. By taking limits they were able to derive continuous inequalities by matrix techniques and avoided the differential equations of the calculus of variations. We have attempted to generalize the techniques of [2] to polynomials in the derivatives of x. We have also considered analogs of inequalities of Műller [5] and Redheffer [7] and have used these inequalities to derive a necessary and sufficient condition on ordered pairs of numbers so that the first number is the square norm of the kth derivative of some periodic function and the second number is the square norm of the mth derivative of the same periodic function. This last result is the L_2 analog of a result of Kolmogoroff [4] on the uniform norm.

2. General Technique

The following is the basic technique of Fan, Taussky, and Todd [2] and will be used to derive most of our inequalities. Let S be a real symmetric $n \times n$ matrix with eigenvalues $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_n$ and let $v_1, v_2, \ldots, v_n$ be the corresponding linearly independent eigenvectors:

$$Sv_i = \lambda_i v_i (i=1,2, \ldots, n).$$

If P is a polynomial with real coefficients, then $P(S)$ is a real symmetric $n \times n$ matrix and we know that $P(S)$ has eigenvalues $P(\lambda_1), P(\lambda_2), \ldots, P(\lambda_n)$ with corresponding eigenvectors $v_1, v_2, \ldots, v_n$ so that $P(S)v_i = P(\lambda_i)v_i$ $(i=1,2, \ldots, n)$.

By a well-known property of symmetric matrices

$$\left[\min_{1 \leq i \leq n}\{P(\lambda_i)\}\right] \cdot (x, x) \leq (x, P(S)x) \leq \left[\max_{1 \leq i \leq n}\{P(\lambda_i)\}\right] \cdot (x, x)$$

for every n-vector x, where $(x, y) = x^Ty$ for n-vectors x and y.

Now we must have

$$\left[\inf_{\lambda_n \leq t \leq \lambda_1}\{P(t)\}\right] \leq \left[\min_{1 \leq n}\{P(\lambda_i)\}\right]$$

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[2] Figures in brackets indicate the literature references at the end of this paper.

and

$$\sup_{\lambda_n \leqslant t \leqslant \lambda_1}\{P(t)\} \leqslant \max_{1 \leqslant i \leqslant n}\{P(\lambda_i)\},$$

hence

$$\inf_{\lambda_n \leqslant t \leqslant \lambda_1}\{P(t)\} \cdot (x, x) \leqslant (x, P(S)x) \leqslant \sup_{\lambda_n \leqslant t \leqslant \lambda_1}\{P(t)\} \cdot (x, x)$$

for every n-vector x.

We note that equality is possible in the last expression if and only if $P(\lambda_i) = \sup_{\lambda_n \leqslant t \leqslant \lambda_1}\{P(t)\}$ for some i between 1 and n or $P(\lambda_i) = \inf_{\lambda_n \leqslant t \leqslant \lambda_1}\{P(t)\}$ for some i between 1 and n. If equality does occur then the set of extremal vectors must be the space spanned by the eigenvectors corresponding to λ_i.

For reasons of simplicity we will only consider vectors with real components and real symmetric matrices, in all cases the extension to complex vectors and Hermitian matrices will be apparent.

3. Periodic Boundary Conditions

Because of the absence of troublesome boundary conditions the periodic case is easiest to handle. We may periodically extend any n-vector $x = (x_1, x_2, \ldots, x_n)^T$ by setting $x_{i+rn} = x_i$ for $i = 1, 2, \ldots, n$ and r any integer.

DEFINITION: If x is a periodically extended n-vector then for $m = 0, 1, 2, \ldots$

$$x^{(m)} = (\Delta^m x_1, \Delta^m x_2, \ldots, \Delta^m x_n)^T,$$

where

$$\Delta^m x_i = \sum_{r=0}^{m} (-1)^{m-r} \binom{m}{r} x_{i-[m/2]+r} \qquad (i = 1, 2, \ldots, n).$$

We call $x^{(m)}$ the mth difference of the n-vector x.

It is clear that $x^{(r+s)} = (x^{(r)})^{(s)}$ for $r, s = 0, 1, 2, \ldots$.

The following lemma, which is crucial for our inequalities, was proved by Fan-Taussky-Todd [2] for $m = 0, 1, 2$.

LEMMA 1. *If* x *is a periodically extended* n*-vector, then* $(x^{(m)}, x^{(m)}) = (x, P^m x)$ $(m = 0, 1, 2, \ldots)$, *where* P *is the* $n \times n$ *symmetric circulant*

$$\begin{bmatrix} 2 & -1 & 0 & . & . & . & -1 \\ -1 & 2 & -1 & . & . & . & 0 \\ . & . & . & & & & . \\ . & . & . & & & & . \\ . & . & . & & & & . \\ -1 & 0 & . & . & . & -1 & 2 \end{bmatrix}$$

PROOF:

$$Px = \begin{bmatrix} -x_n + 2x_1 - x_2 \\ \cdot \\ \cdot \\ \cdot \\ -x_1 + 2x_n - x_{n-1} \end{bmatrix} = -x^{(2)}, \text{ hence } Px^{(j)} = -x^{(j+2)}.$$

By partial summation

$$(x, x^{(2)}) = \sum_{k=1}^{n} x_k(x_{k-1} - 2x_k + x_{k+1}) = -\sum_{k=1}^{n} (x_{k+1} - x_k)^2 = -(x^{(1)}, x^{(1)}),$$

yielding

$$(x^{(j)}, Px^{(j)}) = (x^{(j+1)}, x^{(j+1)}).$$

Thus if m is even we have

$$(x^{(m)}, x^{(m)}) = (P^{\frac{m}{2}}x, P^{\frac{m}{2}}x) = (x, P^m x),$$

while if m is odd

$$(x^{(m)}, x^{(m)}) = (x^{(m-1)}, Px^{(m-1)}) = (P^{\frac{m-1}{2}}x, P \cdot P^{\frac{m-1}{2}}x) = (x, P^m x).$$

Rutherford [8] has shown that P has eigenvalues $4 \sin^2 \left(\frac{k\pi}{n}\right)$ $(k = 1, 2, \ldots, n)$, hence $\lambda_n = 0$, $\lambda_{n-1} = \lambda_{n-2} = 4 \sin^2 \left(\frac{\pi}{n}\right), \ldots, \lambda_1 = 4 \sin^2 \left(\left[\frac{n}{2}\right] \frac{\pi}{n}\right)$. The eigenvector corresponding to λ_n is $(1, 1, \ldots, 1)^T$.

At this point we have two alternatives open to us. If F is a polynomial with real coefficients then we may use the method that we have described in section 2 or else we may also add the auxiliary condition $\sum_{k=1}^{n} x_i = 0$, which is equivalent to the requirement $(x, e) = 0$, where $e = (1, 1, \ldots, 1)^T$. Since e is the eigenvector corresponding to $\lambda_n = 0$, the auxiliary orthogonality condition implies that we need only take our maximum and minimum over $\{F(\lambda_1), F(\lambda_2), \ldots, F(\lambda_{n-1})\}$.

The following example should illustrate the differences in the two approaches.

Let $P(t) = t^m$ for $m \geqslant 1$. Since $P(t)$ is increasing for nonnegative t and since $0 = \lambda_n < \lambda_{n-1} \leqslant \ldots \leqslant \lambda_1$, we have $\min_{1 \leqslant i \leqslant n}\{P(\lambda_i)\} = 0$ and $\min_{1 \leqslant i \leqslant n-1}\{P(\lambda_i)\} = P(\lambda_{n-1}) = 4^m \sin^{2m}\left(\frac{\pi}{n}\right)$.

Thus if x is a periodically extended n-vector then we have the trivial inequality $(x^{(m)}, x^{(m)}) \geqslant 0(x, x) = 0$; however if we add the auxiliary condition $\sum_{i=1}^{n} x_i = 0$, then we obtain $(x^{(m)}, x^{(m)}) \geqslant 4^m \sin^{2m}\left(\frac{\pi}{n}\right)(x, x)$.

We note that the above inequalities are best possible and we include a discussion of the possibility of equality. For brevity this will be the only case where we discuss equality in the discrete case.

If $(x^{(m)}, x^{(m)}) = 0$, then x must be in the subspace spanned by the eigenvectors corresponding to $\lambda_n = 0$. Thus $x = a \cdot (1, 1, \ldots, 1)^T$, where a is a real number.

If $\sum_{i=1}^{n} x_i=0$ and $(x^{(m)}, x^{(m)})=4^m \sin^{2m}\left(\frac{\pi}{n}\right)\cdot(x, x)$, then x must be in the subspace spanned by the eigenvectors corresponding to $\lambda_{n-1}=\lambda_{n-2}=4\sin^2\left(\frac{\pi}{n}\right)$. It is known, [2], that this subspace is spanned by the vectors $u=(u_1, u_2, \ldots, u_n)^T$ and $w=(w_1, w_2, \ldots, w_n)^T$, where $u_j=\cos\left(\frac{2\pi j}{n}\right)$ and $w_j=\sin\left(\frac{2\pi j}{n}\right)$. Thus $x=a\cdot u+b\cdot w$, where a and b are real numbers.

Hence we have established the following extension of Theorem 10 of [2].

THEOREM 1. *If* x *is a periodically extended* n*-vector and if* $\sum_{i=1}^{n} x_i=0$, *then*

$$(x^{(m)}, x^{(m)}) \geqslant 4^m \sin^{2m}\left(\frac{\pi}{n}\right)\cdot(x, x).$$

Equality holds if and only if x *is the periodic extension of a vector of the form* $a\cdot u+b\cdot w$.

COROLLARY 1.1. *If* $x(t) \in C^m[a, b]$, $x(t)$ *has period* $b-a$, *and* $\int_a^b x(t)dt=0$, *then*

$$\int_a^b \{x^{(m)}(t)\}^2 dt \geqslant \left(\frac{2\pi}{b-a}\right)^{2m}\int_a^b \{x(t)\}^2 dt.$$

PROOF:

If we let $\Delta=\frac{b-a}{n+1}$ and $y_i=x(t_i)-\frac{1}{n}\sum_{j=1}^{n} x(t_j)$, where $t_i=a+i\Delta$, then

$$\frac{(y^{(m)}, y^{(m)})}{\Delta^{2m}}\cdot\Delta \geqslant \frac{4^m\sin^{2m}\left(\frac{\pi}{n}\right)}{\Delta^{2m}}\cdot(y, y)\Delta.$$

The result now follows when we let $n\to\infty$.

A careful inspection of the proof of Theorem 1 will show that we only used the periodic extension property of the n-vector x on $x_{1-\left[\frac{m}{2}\right]}, x_{2-\left[\frac{m}{2}\right]}, \ldots, x_0$ and $x_{n+1}, x_{n+2}, \ldots, x_{n+m-\left[\frac{m}{2}\right]}$. Hence we may weaken our requirement $x(t)$ has period $b-a$ to a condition on the end points, namely $x(a)=x(b)$, $x'(a)=x'(b)$, $\ldots$, $x^{(m-1)}(a)=x^{(m-1)}(b)$.

Halperin and Pitt [3], Müller [5], Nirenberg [6], and Redheffer [7] have developed inequalities of the form:

$$\int_a^b \{x'(t)\}^2 dt \leqslant \alpha\int_a^b \{x''(t)\}^2 dt+H(\alpha)\int_a^b \{x(t)\}^2 dt,$$

where $x(t)\in C^2[a, b]$, $\alpha>0$. Redheffer obtained the best possible value of $H(\alpha)$. We will now develop generalizations of the form:

$$\int_a^b \{x^{(k)}(t)\}^2 dt \leqslant \alpha\int_a^b \{x^{(m)}(t)\}^2 dt+H_{k,m}(\alpha)\int_a^b \{x(t)\}^2 dt,$$

$1\leqslant k<m$ under the further restriction that $x(a)=x(b)$, $x'(a)=x'(b)$, $\ldots$, $x^{(m-1)}(a)=x^{(m-1)}(b)$. These inequalities will be established by taking limits of the appropriate discrete inequalities.

Incidentally we observe that $H_{1,2}(\alpha)$ for our periodic case is much smaller than Redheffer's general value of $\frac{1}{\alpha}+\frac{12}{(b-a)^2}$.

If we let $P(t)=\theta^k t^k-\alpha\theta^m t^m$, then we wish to maximize $P(t)$ on $[\lambda_n, \lambda_1]=\left[0, 4\sin^2\left(\left[\frac{n}{2}\right]\frac{\pi}{n}\right)\right]$. Since $P'(t)=k\theta^k t^{k-1}-m\alpha\theta^m t^{m-1}$, if we assume that $\theta>0$ then the maximum value of $P(t)$ on $t\geqslant 0$ occurs at $t=\frac{1}{\theta}\left(\frac{k}{m\alpha}\right)^{1/m-k}$ Thus $\sup\limits_{\lambda_n\leqslant t\leqslant\lambda_1}\{P(t)\}\leqslant P\left(\frac{1}{\theta}\left\{\frac{k}{m\alpha}\right\}^{1/m-k}\right)=\left(\frac{k}{m\alpha}\right)^{k/m-k}\left(1-\frac{k}{m}\right)$.

An inspection of the graph of $P(t)$ will yield the following cases:

If $\lambda_1\geqslant\frac{1}{\theta}\left(\frac{k}{m\alpha}\right)^{1/m-k}$ then

$$\sup_{\lambda_n\leqslant t}\leqslant\lambda_1\{P(t)\}=P\left(\frac{1}{\theta}\left[\frac{k}{m\alpha}\right]^{1/m-k}\right)=\left(\frac{k}{m\alpha}\right)^{k/m-k}\left(1-\frac{k}{m}\right).$$

If $\lambda_1\leqslant\frac{1}{\theta}\left(\frac{k}{m\alpha}\right)^{1/m-k}$ then

$$\sup_{\lambda_n\leqslant t\leqslant\lambda_1}\{P(t)\}=P(\lambda_1)=P\left(4\sin^2\left(\left[\frac{n}{2}\right]\frac{\pi}{n}\right)\right)$$

$$=4^k\theta^k\sin^{2k}\left(\left[\frac{n}{2}\right]\frac{\pi}{n}\right)\left\{1-4^{m-k}\theta^{m-k}\alpha\sin^{2m-2k}\left(\left[\frac{n}{2}\right]\frac{\pi}{n}\right)\right\}.$$

If n is even then $P(\lambda_1)=4^k\theta^k\{1-4^{m-k}\theta^{m-k}\alpha\}$.

Hence we have the following result.

THEOREM 2. *If* x *is a periodically extended* n-*vector*, α *and* θ *are positive constants*, $1\leqslant k<m$ *then we have the following inequality:*

$$\theta^k(x^{(k)}, x^{(k)})\leqslant\alpha\theta^m(x^{(m)}, x^{(m)})+\left(\frac{k}{m\alpha}\right)^{k/m-k}(1-k/m)(x, x).$$

If $4\leqslant\frac{1}{\theta}\left(\frac{k}{m\alpha}\right)^{1/m-k}$, *then we may improve the above inequality to:*

$$\theta^k(x^{(k)}, x^{(k)})-\alpha\theta^m(x^{(m)}, x^{(m)})\leqslant 4^k\theta^k\{1-4^{m-k}\theta^{m-k}\alpha\}\cdot(x, x).$$

PROOF:

We note that

$$4\leqslant\frac{1}{\theta}\left(\frac{k}{m\alpha}\right)^{1/m-k}$$

certainly implies that

$$4\sin^2\left(\left[\frac{n}{2}\right]\frac{\pi}{n}\right)\leqslant\frac{1}{\theta}\left(\frac{k}{m\alpha}\right)^{1/m-k}$$

and $P(t)$ is increasing on the interval $\left[0, \frac{1}{\theta}\left(\frac{k}{m\alpha}\right)^{1/m-k}\right]$.

If we take $\theta=\left(\frac{n+1}{b-a}\right)^2$ and let $n\to\infty$ then we obtain a continuous analog of Theorem 2.

COROLLARY 2.1. *If* $x(t)\in C^m[a, b]$, $1\leq k<m$, $x(a)=x(b)$, $x'(a)=x'(b)$, . . ., $x^{(m-1)}(a)=x^{(m-1)}(b)$, *and if* $\alpha>0$ *then we have the following inequality:*

$$\int_a^b \{x^{(k)}(t)\}^2dt\leq\alpha\int_a^b \{x^{(m)}(t)\}^2dt+\left(\frac{k}{m\alpha}\right)^{k/m-k}(1-k/m)\int_a^b \{x(t)\}^2dt.$$

If we let $k=1$, $m=2$ *then our constant has the value* $\frac{1}{4\alpha}$ *as compared to the Redheffer value of* $\frac{1}{\alpha}+\frac{12}{(b-a)^2}$.

One may now inquire about the possibility of equality in Corollary 2.1; by our previous discussion we see that this is possible if and only if

$$\lim_{n\to\infty}[\max_{1\leq i\leq n}\{\theta^k\lambda_i^k-\alpha\theta^m\lambda_i^m\}]=\left(\frac{k}{m\alpha}\right)^{k/m-k}\left(1-\frac{k}{m}\right).$$

However a brief investigation shows that the above expression is not always satisfied, hence Corollary 2.1 is in general not best possible. A little more work will yield the best inequality.

We notice that the smaller eigenvalues of P approach zero as $1/n^2$ and the positive root of $\theta^kt^k-\alpha\theta^mt^m$ exhibits a similar behavior:

$$\lambda_n=0,\ \lambda_{n-1}=4\sin^2\left(\frac{\pi}{n}\right)\approx\frac{4\pi^2}{n^2},\ \lambda_{n-3}=4\sin^2\left(\frac{2\pi}{n}\right)\approx\frac{8\pi^2}{n^2},\ldots$$

$$\text{root}=\frac{(b-a)^2}{(n+1)^2}\cdot\left(\frac{1}{\alpha}\right)^{1/m-k}.$$

Hence if we set

$$L=\left[\frac{b-a}{2\pi}\cdot\left(\frac{1}{\sqrt{\alpha}}\right)^{1/m-k}\right],$$

where [ ] is the greatest integer function, then L is the limiting number of distinct eigenvalues of P which are greater than 0 and less than or equal to the root.

If $L=0$, which occurs when $\alpha>\left(\frac{b-a}{2\pi}\right)^{2m-2k}$, then

$$\lim_{n\to\infty}[\max_{1\leq i\leq n}\{\theta^k\lambda_i^k-\alpha\theta^m\lambda_i^m\}]=0,$$

yielding the best possible result:

COROLLARY 2.2. *Let* $x(t)$, k, m *be as in Corollary 2.1 and let* $\alpha>\left(\frac{b-a}{2\pi}\right)^{2m-2k}$, *then*

$$\int_a^b \{x^{(k)}(t)\}^2dt\leq\alpha\int_a^b \{x^{(m)}(t)\}^2dt.$$

If we add the auxiliary condition $\int_a^b x(t)dt = 0$, then

$$\lim_{n\to\infty} \max[1 \leqslant i \leqslant n-1\{\theta^k\lambda_i^k - \alpha\theta^m\lambda_i^m\}] = \left(\frac{2\pi}{b-a}\right)^{2k} - \alpha\left(\frac{2\pi}{b-a}\right)^{2m} \leqslant 0$$

and we obtain the best possible result:

COROLLARY 2.3. *Let* x(t), k, m, α *be as in Corollary 2.2 and let* x *also satisfy* $\int_a^b x(t)dt = 0$, *then*

$$\int_a^b \{x^{(k)}(t)\}^2 dt \leqslant \alpha \int_a^b \{x^{(m)}(t)\}^2 dt + \left\{\left(\frac{2\pi}{b-a}\right)^{2k} - \alpha\left(\frac{2\pi}{b-a}\right)^{2m}\right\} \cdot \int_a^b \{x(t)\}^2 dt.$$

If $L \geqslant 1$ then there are limiting eigenvalues between zero and the positive root. Hence if we set

$$J = \left[\frac{b-a}{2\pi} \cdot \left(\sqrt{\frac{k}{m\alpha}}\right)^{1/m-k}\right],$$

then J is the limiting number of distinct eigenvalues of P which are greater than 0 and less than or equal to the maximum of $\theta^k t^k - \alpha\theta^m t^m$.

Thus if

$$\left[\frac{b-a}{2\pi} \cdot \left(\sqrt{\frac{k}{m\alpha}}\right)^{1/m-k}\right] = \frac{b-a}{2\pi} \cdot \left(\sqrt{\frac{k}{m\alpha}}\right)^{1/m-k}$$

then

$$\lim_{n\to\infty} \max[1 \leqslant i \leqslant n\{\theta^k\lambda_i^k - \alpha\theta^m\lambda_i^m\}] = \left(\frac{k}{m\alpha}\right)^{k/m-k}(1-k/m)$$

and Corollary 2.1 is best possible. Otherwise we may replace

$$\left(\frac{k}{m\alpha}\right)^{k/m-k}(1-k/m) \text{ by } \max\left\{\left(\frac{2J\pi}{b-a}\right)^{2k} - \alpha\left(\frac{2J\pi}{b-a}\right)^{2m}, \left(\frac{2(J+1)\pi}{b-a}\right)^{2k} - \alpha\left(\frac{2(J+1)\pi}{b-a}\right)^{2m}\right\}.$$

We may summarize the above results by the following best possible inequalities.

COROLLARY 2.4. *If* $x(t) \in C^m[a, b], 1 \leqslant k < m$, $x(a) = x(b)$, $x'(a) = x'(b), \ldots, x^{(m-1)}(a) = x^{(m-1)}(b)$, *and if* $\alpha > 0$ *then*

$$\int_a^b \{x^{(k)}(t)\}^2 dt \leqslant \alpha \int_a^b \{x^{(m)}(t)\}^2 dt + H_{k,m}(\alpha) \int_a^b \{x(t)\}^2 dt,$$

where

$$H_{k,m}(\alpha) = 0 \text{ for } \alpha > \left(\frac{b-a}{2\pi}\right)^{2m-2k},$$

$$H_{k,m}(\alpha) = \left(\frac{k}{m\alpha}\right)^{k/m-k}(1-k/m) \text{ if } \frac{b-a}{2\pi} \cdot \left(\sqrt{\frac{k}{m\alpha}}\right)^{1/m-k}$$

is a positive integer, and otherwise

$$H_{k,m}(\alpha)=\max\left\{\left(\frac{2J\pi}{b-a}\right)^{2k}-\alpha\left(\frac{2J\pi}{b-a}\right)^{2m},\left(\frac{2(J+1)\pi}{b-a}\right)^{2k}-\alpha\left(\frac{2(J+1)\pi}{b-a}\right)^{2m}\right\}.$$

COROLLARY 2.5. *Let* x(t), k, m, α *be as in Corollary* 2.4 *and let* x *also satisfy* $\int_a^b x(t)dt=0$, *then*

$$\int_a^b \{x^{(k)}(t)\}^2dt\leq\alpha\int_a^b\{x^{(m)}(t)\}^2dt+G_{k,m}(\alpha)\int_a^b\{x(t)\}^2dt,$$

where

$$G_{k,m}(\alpha)=\left(\frac{2\pi}{b-a}\right)^{2k}-\alpha\left(\frac{2\pi}{b-a}\right)^{2m}\text{ for }\alpha>\left(\frac{b-a}{2\pi}\right)^{2m-2k},$$

and otherwise

$$G_{k,m}(\alpha)=H_{k,m}(\alpha).$$

It is not difficult to see that $H_{k,m}(\alpha)$ and $G_{k,m}(\alpha)$ are piecewise linear functions of α, which are monotonic decreasing, and that the jumps in the derivatives of $H_{k,m}$ and $G_{k,m}$ occur for those positive numbers α_i which yield

$$\left(\frac{2\pi i}{b-a}\right)^{2k}-\alpha_i\left(\frac{2\pi i}{b-a}\right)^{2m}=\left(\frac{2\pi i+2\pi}{b-a}\right)^{2k}-\alpha_i\left(\frac{2\pi i+2\pi}{b-a}\right)^{2m},$$

where i is a nonnegative integer. We have $\infty>\alpha_0>\alpha_1>\ldots>0$ and $\lim_{i\to\infty}\alpha_i=0$. Note that

$$\int_a^b\left\{\sin^{(k)}\left(\frac{2\pi it}{b-a}\right)\right\}^2dt=\alpha\int_a^b\left\{\sin^{(m)}\left(\frac{2\pi it}{b-a}\right)\right\}^2dt+H_{k,m}(\alpha)\int_a^b\left\{\sin\left(\frac{2\pi it}{b-a}\right)\right\}^2dt,$$

for $\alpha_i\leq\alpha\leq\alpha_{i-1}$. This accounts for the piecewise linearity of $H_{k,m}$ and $G_{k,m}$ and will be a great importance in section 4.

At this stage it is apparent that there is an unlimited number of possible inequalities, in fact every polynomial will give discrete inequalities and polynomials in θt yield continuous analogs.

It is known, see for example [1], that if $f(t)$ is represented by a power series with real coefficients and if A is a real symmetric matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ which lie strictly within the circle of convergence of the power series for $f(t)$, then $f(A)$ is real symmetric and has eigenvalues $f(\lambda_1), f(\lambda_2), \ldots, f(\lambda_n)$. This result enables us to extend our polynomial inequalities to discrete inequalities on analytic functions.

THEOREM 3. *If* x *is a periodically extended* n-*vector and if* $f(t)=\sum_{m=0}^{\infty}a_mt^m$, *where the* a_m *are real numbers and the series converges on a set containing* [0, 4] *in its interior—if* n *is odd then we need only require convergence on a set containing* $\left[0, 4\sin^2\left(\left[\frac{n}{2}\right]\frac{\pi}{n}\right)\right]$ *in its interior—then*

$$[\inf_{0\leq t\leq 4}\{f(t)\}](x,x)\leq\sum_{m=0}^{\infty}a_m(x^{(m)},x^{(m)})\leq[\sup_{0\leq t\leq 4}\{f(t)\}](x,x).$$

The following inequalities are examples of the application of Theorem 3.

COROLLARY 3.1. *If* x *is a periodically extended* n-*vector then*

$$\sum_{m=1}^{\infty} \frac{1}{m!} (x^{(m)}, x^{(m)}) \leq (e^4 - 1)(x, x).$$

COROLLARY 3.2. *If* x *is a periodically extended* n-*vector and* $\theta > 0$ *then*

$$-(x, x) \leq \sum_{m=0}^{\infty} (-1)^m \frac{\theta^{2m+1}}{(2m+1)!} (x^{(2m+1)}, x^{(2m+1)}) \leq (x, x).$$

PROOF:

$$\sin t = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} t^{2m+1} \text{ and} |\sin t| \leq 1.$$

COROLLARY 3.3. *If* $x(t) \in C^\infty[b-a]$ *and* x(t) *has period* b − a *then*

$$-\int_a^b \{x(t)\}^2 dt \leq \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} \int_a^b \{x^{(2m+1)}(t)\}^2 dt \leq \int_a^b \{x(t)\}^2 dt.$$

4. Inequalities of Kolmogoroff Type

In analysis one frequently wants to obtain inclusion regions for the norms of functions; i.e., if M is in the inclusion region, then there exists a function f, from a certain set of functions, with norm of f equal to M. Kolmogoroff [4] established a result of this type using the uniform norm over the function set $C^m[0, \infty]$. If we set $M_k(x) = \sup_{0 \leq t \leq \infty} \{|x^{(k)}(t)|\}$ $(k = 0, 1, 2, \ldots, m)$, then Kolmogoroff [4] gave a necessary and sufficient condition "in order that to a triple of positive numbers M_0, M_k, $M_m (0 < k < m)$ there should correspond a function $x(t)$ for which

$$M_0 = M_0(x),\ M_k = M_k(x),\ M_m = M_m(x)."$$

In this section we will establish a similar result on a triple of positive numbers relative to the square norm, where our set of functions will be $C^m[a, b] \cap \{x | x(a) = x(b), x'(a) = x'(b), \ldots, x^{(m-1)}(a) = x^{(m-1)}(b)\}$. We will also add the restriction $\int_a^b x(t)dt = 0$ in order to eliminate the constant function.

If $x(t)$ is an element of our function class, then we set

$$A_k(x) = \int_a^b \{x^{(k)}(t)\}^2 dt \qquad (k = 0, 1, 2, \ldots, m).$$

We may assume without loss of generality that we have normalized x so that $A_0(x) = 1$. We will establish the following result:

THEOREM 4. *In order that to a pair of numbers* a_k, a_m *there should correspond a function* x(t) $\in C^m[a, b] \cap \{x | x(a) = x(b), x'(a) = x'(b), \ldots, x^{(m-1)}(a) = x^{(m-1)}(b), \int_a^b x(t)dt = 0\}$ *for which*

$$A_0(x) = 1,\ A_k(x) = a_k,\ A_m(x) = a_m$$

it is necessary and sufficient that

$$a_k \geq \left(\frac{2\pi}{b-a}\right)^{2k}, \ a_m \geq \left(\frac{2\pi}{b-a}\right)^{2m}, \ \textit{and}$$

$$a_k - \alpha a_m \leq G_{k,m}(\alpha) \text{ for all } \alpha > 0.$$

PROOF of THEOREM 4:

The necessity of the conditions is a direct consequence of Corollaries 1.1 and 2.5.

The proof of the sufficiency will be divided into several cases and we will use the notation of section 3 throughout. We will denote the normalized sin $\left(\frac{2\pi it}{b-a}\right)$ by $S_i(t)$. We recall that on the interval $\alpha_i \leq \alpha \leq \alpha_{i-1}$ we have $A_k(S_i) - \alpha A_m(S_i) = G_{k,m}(\alpha)$.

CASE 1: $a_k - \alpha a_m = G_{k,m}(\alpha)$ for at least two values of α, say $\alpha = t_1$ and $\alpha = t_2$.

We immediately have that $a_k - \alpha a_m = G_{k,m}(\alpha)$ for all $\alpha \in [t_1, t_2]$. Since we know that the S_i are maximizing functions there is an integer i with $\alpha_i \leq t_1 < t_2 \leq \alpha_{i-1}$ and $a_k - \alpha a_m = G_{k,m}(\alpha) = A_k(S_i) - \alpha A_m(S_i)$ for $\alpha_i \leq \alpha \leq \alpha_{i-1}$. Thus $a_k = A_k(S_i)$, $a_m = A_m(S_i)$.

CASE 2: $a_k - \alpha a_m = G_{k,m}(\alpha)$ for only one value of α, say $\alpha = t_1$.

We immediately have that $t_1 = \alpha_1$ for exactly one integer i.

Thus

$$a_k - \alpha_i a_m = G_{k,m}(\alpha_i) = A_k(S_i) - \alpha_i A_m(S_i) = A_k(S_{i+1}) - \alpha_i A_m(S_{i+1}).$$

Also since $a_k - \alpha a_m \leq G_{k,m}(\alpha)$ for all $\alpha > 0$ we must have $A_m(S_i) < a_m < A_m(S_{i+1})$. Thus there exist b_i, b_{i+1} with $0 < b_i, b_{i+1} < 1$, $b_i^2 + b_{i+1}^2 = 1$, and $a_m = A_m(b_i S_i + b_{i+1} S_{i+1}) = b_i^2 A_m(S_i) + b_{i+1}^2 A_m(S_{i+1})$. [Note that the last equality follows from the orthogonality of S_i and all its derivatives to S_{i+1} and all its derivatives.] Now,

$$A_k(b_i S_i + b_{i+1} S_{i+1}) - \alpha_i a_m = A_k(b_i S_i + b_{i+1} S_{i+1}) - \alpha_i A_m(b_i S_i + b_{i+1} S_{i+1})$$

$$= b_i^2[A_k(S_i) - \alpha_i A_m(S_i)] + b_{i+1}^2[A_k(S_{i+1}) - \alpha_i A_m(S_{i+1})] = (b_i^2 + b_{i+1}^2)[a_k - \alpha_i a_m].$$

Hence $a_k = A_k(b_i S_i + b_{i+1} S_{i+1})$.

CASE 3: $a_k - \alpha a_m < G_{k,m}(\alpha)$ for all $\alpha > 0$ and $a_m \geq \left(\frac{4\pi}{b-a}\right)^{2m} = A_m(S_2)$.

We know that there is exactly one integer $i \geq 2$ with $\left(\frac{2i\pi}{b-a}\right)^{2m} \leq a_m < \left(\frac{2(i+2)\pi}{b-a}^{2m}\right)$or $A_m(S_i) \leq a_m < A_m(S_{i+1})$. If we repeat the construction of Case 2 we can find b_i and b_{i+1} with $a_m = A_m(b_i S_i + b_{i+1} S_{i+1})$ and since $a_k - \alpha_i a_m < G_{k,m}(\alpha_i) = A_k(b_i S_i + b_{i+1} S_{i+1}) - \alpha_i A_m(b_i S_i + b_{i+1} S_{i+1}) = A_k(b_i S_i + b_{i+1} S_{i+1}) - \alpha_i a_m$ we have

$$a_k < A_k(b_i S_i + b_{i+1} S_{i+1}).$$

By our conditions on a_k and a_m there is an integer $j > i+1$ and an $\bar{\alpha} > 0$ such that $A_k(S_j) - \bar{\alpha} A_m(S_j) = A_k(S_1) - \bar{\alpha} A_m(S_1) < a_k - \bar{\alpha} a_m$. Now $A_m(S_1) < A_m(S_i) \leq a_m < A_m(S_{i+1}) < A_m(S_j)$, hence there are b_1, b_j with $a_m = A_m(b_1 S_1 + b_j S_j)$ and since

$$a_k - \bar{\alpha} a_m > G_{k,m}(\bar{\alpha}) = A_k(b_1 S_1 + b_j S_j) - \bar{\alpha} A_m(b_1 S_1 + b_j S_j) = A_k(b_1 S_1 + b_j S_j) - \bar{\alpha} a_m$$

we have

$$a_k > A_k(b_1S_1 + b_jS_j).$$

By the orthogonality of $b_1S_1 + b_jS_j$ and $b_iS_i + b_{i+1}S_{i+1}$ there are constants c and d with $c^2 + d^2 = 1$ for normalization and

$$a_m = A_m(cb_1S_1 + CB_jS_j + db_iS_i + db_{i+1}S_{i+1}),$$

$$a_k = A_k(cb_1S_1 + cb_jS_j + db_iS_i + db_{i+1}S_{i+1}).$$

CASE 4: $a_k - \alpha a_m < G_{k,m}(\alpha)$ for all α and $a_m < \left(\frac{4\pi}{b-a}\right)^{2m} = A_m(S_2)$.

If we repeat the process of Cases 2 and 3 we can find b_1, b_2 such that $a_m = A_m(b_1S_1 + b_2S_2)$ and $a_k < A_k(b_1S_1 + b_2S_2)$.

We cannot simply repeat the process of Case 3 because $b_1S_1 + b_2S_2$ will not be orthogonal to $b_1'S_1 + b_2'S_2$. However we note that the normalized multiple of $\cos\left(\frac{2\pi t}{b-a}\right)$, which we denote by $C_1(t)$, also satisfies $A_k(C_1) - \alpha A_m(C_1)$ for $\alpha_1 \leq \alpha \leq \alpha_0$ and is orthogonal to all the S_i. Hence we can find c_1, c_j such that

$$a_m = A_m(c_1C_1 + c_jS_j) \text{ and } a_k > A_k(c_1C_1 + c_jS_j)$$

and we continue as in Case 3.

Thus we have proven Theorem 4.

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Publications of the National Bureau of Standards*

Selected Abstracts

A note on the numerical evaluation of thermal radiation characteristics of diffuse cylindrical and conical cavities, B. A. Peavy, *J. Res. NBS* **70C** *(Engr. and Instr.) No. 2, 139–147 (Apr.–June 1966).*
Presented are methods that avoid the need to employ an extrapolation technique in the region of the critical points for evaluation of the apparent emissivity of diffuse cylindrical and conical cavities. The methods involve appropriate substitutions in the integrands of integral equations that are used in analytical solutions for determining the thermal radiation characteristics of diffuse and conical cavities. Equations for either isothermal or nonisothermal surface temperature conditions are provided in a direct form for computations. Numerical results are presented for a general linear temperature distribution along the length of a cylindrical cavity. The method is equally applicable for the solution of other problems in integral equations where discontinuities are encountered.

Two-dimensional probability distribution in a turbulent field, F. N. Frenkiel and P. S. Klebanoff, *Phys. Fluids* **8,** *No. 12, 2291–2293 (Dec. 1965).*
Using high speed computing techniques measurements were made for the first time of a two-dimensional probability distribution in a turbulent field generated by a grid in a wind tunnel. The two-dimensional probability distribution of simultaneous turbulent velocities at two points separated transversely to the mean flow (in the initial stage of decay) is presented, and the departure from the customarily assumed Gaussian distribution is shown.

Propagation of electromagnetic pulses in terrestrial waveguides, J. R. Wait, *IEEE Trans. Ant. Prop.* **AP-13,** *No. 6, 904–918 (Nov. 1965).*
The propagation of electromagnetic waves over the earth's surface is considered under transient conditions. The source is taken to be a vertical electric dipole whose current moment is suddenly established. The build-up of the radiated field is calculated under various assumed conditions. It is shown, even in the absence of an ionospherically reflected wave, that the influence of earth curvature has a pronounced effect on the distortion of the original pulse shape. For great distances (i.e., $d > 2000$ km), it is found to be more convenient to regard the field as a sum of modes. Particular attention is given to the transient characteristics of the dominant mode as a function of the source waveform.

The analysis, synthesis, and description of biological images, L. E. Lipkin, W. C. Watt, and R. A. Kirsch, *Ann. N.Y. Acad. Sci.* **128,** *No. 3, 984–1012 (Jan. 31, 1966).*
There may be distinguished two approaches to characterizing the properties of biological images: the statistical approach traditionally used for mechanized image-processing, and the articular approach generally used among biologists. We discuss both approaches, but focus attention on the second, for we hold that an articular approach allows the expression of much that is effectively impossible to express in the form of numerical measurements; and we know of no inherent reason why a computer cannot deal with information of a nonquantitative nature.
The articulation of biological images necessarily takes place on two fronts: the imposition of an articular structure on the image itself; and the expression of this construal in English sentences. We will study both kinds of articulation, and we will suggest how they may be brought together, in the form of linked pictorial and linguistic *grammars.* The computer system which this paper envisions will be able to analyze a presented image with respect to a pictorial grammar, and to formulate and accept descriptions of that image, in English sentences, with respect to a linguistic grammar. It will be able to present pictorial instances of English descriptions, and in other ways to respond to English directives.

Status of linear relations among heats of transport, R. E. Howard and A. B. Lidiard, *J. Chem. Phys.* **43,** *No. 11, 4158–4160 (Dec. 1, 1965).*
It is shown that in situations where the fluxes of the various components of a system are not linearly independent, previously derived linear relations among the heats of transport contain no physical information, but serve only to complete the definition of these quantities. Other defining equations are possible but physical predictions are unaffected by the choice so presented. This resolves the paradox of the apparent inconsistency of these linear relations with kinetic theories of heats of transport.

Other NBS Publications

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A note on the numerical evaluation of thermal radiation characteristics of diffuse cylindrical and conical cavities. B. A. Peavy. (See above abstract.)

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