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On the Approximation of Functions of Several Variables[1]

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The purpose of this note is to point out how a certain type of approximation to functions of one real variable gives rise to similar approximations to functions of several variables. Information on the rapidity of convergence in the one dimensional case, yields at once corresponding information for the multidimensional case.

Key Word: Approximations, convergence, multidimensional, polynomials, functions, variables, Hermite-Fejér, Bernstein, Chebyshev.

1. The purpose of this note is to point out how a certain type of approximation to functions of one real variable, gives rise to similar approximations to functions of several variables. In fact, we show how information on the rapidity of convergence of the approximation in the one dimensional case, yields at once corresponding information for the multidimensional case.

2. Let $-\infty < a < b < \infty$, and for $n=1, 2, \ldots$, let $c_0^{(n)}, c_1^{(n)}, \ldots, c_n^{(n)}$ be points of $[a, b]$, and $K_0^{(n)}(x), K_1^{(n)}(x), \ldots, K_n^{(n)}(x)$ polynomials which are ≥ 0 throughout $[a, b]$, and such that

$$\sum_{j=0}^{n} K_j^{(n)}(x) \equiv 1.$$

We set, finally, for every real function f, continuous in $[a, b]$,

$$P_n(f, x) \equiv \sum_{j=0}^{n} f(c_j^{(n)}) K_j^{(n)}(x) \qquad (n=1, 2, \ldots). \tag{1}$$

3. The purpose of constructing such polynomials $P_n(f, x)$ is to obtain polynomial approximations to f. Here are two examples.

I. Let $a=0, b=1$, and for $n=1, 2, \ldots$, let $c_j^{(n)}=j/n$, $K_j^{(n)}(x) \equiv \binom{n}{j} x^j (1-x)^{n-j}$ $(j=0, 1, \ldots, n)$. Then $\Sigma_{j=0}^{n} K_j^{(n)}(x) \equiv 1$ $(n=1, 2, \ldots)$. If f is any real function, continuous in $[0, 1]$, then for $n=1. 2, \ldots$.

$$P_n(f, x) \equiv \Sigma_{j=0}^{n} f(c_j^{(n)}) K_j^{(n)}(x) \equiv \Sigma_{j=0}^{n} f(j/n) \binom{n}{j} x^j (1-x)^{n-j} \tag{2}$$

is the Bernstein polynomial [1] of order n of f. If, furthermore, $\omega(\delta)$ is the modulus of continuity of f in [0, 1], then ([7, 10]) the polynomial (2) satisfies

$$\max_{0 \leq x \leq 1} |f(x) - P_n(f, x)| \leq (5/4) \omega(n^{-1/2}) \qquad (\mathrm{n}=1, 2, \ldots).$$

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II. Let $a=-1$, $b=1$, and for $n=1, 2, \ldots$, let $c_j^{(n)}=\cos\left[\dfrac{2j+1}{2(n+1)}\pi\right]$,

$$K_j^{(n)}(x) \equiv (1-c_j^{(n)}x)[T_{n+1}(x)/\{(n+1)(x-c_j^{(n)})\}]^2 \qquad (j=0, 1, \ldots, n),$$

where $T_{n+1}(x) \equiv 2^n\Pi_{k=0}^n(x-c_k^{(n)})$ is the $(n+1)$th degree Chebyshev polynomial of the first kind satisfying $T_{n+1}(\cos\theta) \equiv \cos[(n+1)\theta]$. If f is any real function continuous in $[-1, 1]$, then

$$P_n(f, x) \equiv \Sigma_{j=0}^n f(c_j^{(n)})K_j^{(n)}(x) \equiv \Sigma_{j=0}^n f(c_j^{(n)})(1-c_j^{(n)}x)[T_{n+1}(x)/\{(n+1)(x-c_j^{(n)})\}]^2$$

$(n=1, 2, \ldots)$ are the well-known [3] Hermite-Fejér polynomials converging uniformly to f in $[-1, 1]$. Again $\Sigma_{j=0}^n K_j^{(n)}(x) \equiv 1$ $(n=1, 2, \ldots)$. If f is a real function, satisfying throughout $[-1,1]$

$$|f(v)-f(u)| \leq \lambda|v-u|$$

where λ is a positive constant, then [11] for $n=1, 2, \ldots$

$$\max_{-1\leq x\leq 1} |f(x)-P_n(f, x)| < 4\lambda\pi(n+1)^{-1}[\alpha+\log(n+1)]$$

where $\alpha=\frac{1}{2}+C-\log 2=0.384\ldots$, C being Euler's constant. Furthermore, if f is a real function, continuous in $[-1, 1]$, and if $\omega(\delta)$ is the modulus of continuity of f there, then for $n=1,2,\ldots$ we have [11]

$$\max_{-1\leq x\leq 1} |f(x)-P_n(f, x)| \leq [2+4\pi+\eta_n]\omega\left(\frac{\log(n+1)}{n+1}\right),$$

where η_n depends on n only and $\eta_n \to 0$ as $n\to\infty$.

4. THEOREM 1. *Assume the hypotheses and notation of section 2. Assume also that if* f *is a real function satisfying for some positive constant* λ, *throughout* $[a, b]$, $|f(v)-f(u)| \leq \lambda|v-u|$, *then for* $n=1, 2, \ldots$

$$\max_{a\leq x\leq b} |f(x)-P_n(f, x)| < a_{\lambda, n}$$

where $a_{\lambda, n}$ *depends on* λ *and* n *only. Let* $f(x_1, x_2 \ldots, x_p)$ $(p \geq 2)$ *be a real function, defined on the cube* C: $a \leq x_k \leq b$, $k=1, 2, \ldots, p$. *Suppose that for* $r=1, 2, \ldots, p$, λ_r *is a positive number such that throughout C*

$$|f(x_1, x_2, \ldots, x_{r-1}, v, x_{r+1}, \ldots, x_p)-f(x_1, x_2, \ldots, x_{r-1}, u, x_{r+1}, \ldots, x_p)| \leq \lambda_r|v-u|. \quad (3)$$

Let $n_1, n_2, \ldots, n_p$ *be arbitrary positive integers, and set*

$$P_{n_1, n_2, \ldots, n_p}(f, x_1, x_2, \ldots, x_p) \equiv \sum_{h_1=0}^{n_1} \cdots \sum_{h_p=0}^{n_p} f(c_{h_1}^{(n_1)}, \ldots, c_{h_p}^{(n_p)})K_{h_1}^{(n_1)}(x_1) \ldots K_{h_p}^{(n_p)}(x_p). \quad (4)$$

Then throughout C:

$$|f(x_1, x_2, \ldots x_p)-P_{n_1, n_2, \ldots, n_p}(f, x_1, x_2, \ldots, x_p)| < \sum_{r=1}^{p} a_{\lambda_r, n_r}.$$

PROOF. Observe that $\sum_{h_1=0}^{n_1} \dots \sum_{h_p=0}^{n_p} K_{h_1}^{(n_1)}(x_1) \dots K_{h_p}^{(n_p)}(x_p) \equiv 1.$

Hence, throughout C,

$$
\begin{aligned}
& f(x_1, \dots, x_p) - P_{n_1, n_2, \dots, n_p}(f, x_1, x_2, \dots, x_p) \\
&= \sum_{h_1=0}^{n_1} \dots \sum_{h_p=0}^{n_p} [f(x_1, \dots, x_p) - f(c_{h_1}^{(n_1)}, \dots, c_{h_p}^{(n_p)})] K_{h_1}^{(n_1)}(x_1) \dots K_{h_p}^{(n_p)}(x_p) \\
&= \sum_{h_1=0}^{n_1} \dots \sum_{h_p=0}^{n_p} \left\{ \sum_{r=1}^{p} [f(c_{h_1}^{(n_1)}, \dots, c_{h_{r-1}}^{(n_{r-1})}, x_r, \dots, x_p) - f(c_{h_1}^{(n_1)}, \dots, c_{h_r}^{(n_r)}, x_{r+1}, \dots, x_p)] \right\} \\
&\qquad \cdot \prod_{s=1}^{p} K_{h_s}^{(n_s)}(x_s) \\
&= \sum_{r=1}^{p} \sum_{\substack{h_q=0,1,\dots,n_q \\ q=1,2,\dots p,\, q\neq r}} \left\{ \sum_{h_r=0}^{n_r} [f(c_{h_1}^{(n_1)}, \dots, c_{h_{r-1}}^{(n_{r-1})}, x_r, \dots, x_p) - f(c_{h_1}^{(n_1)}, \dots, c_{h_r}^{(n_r)}, x_{r+1}, \dots, x_p)] \right. \\
&\qquad \left. \cdot K_{h_r}^{(n_r)}(x_r) \right\} \prod_{\substack{s=1 \\ s\neq r}}^{p} K_{h_s}^{(n_s)}(x_s).
\end{aligned}
$$

$(f(c_{h_1}^{(n_1)}, \dots, c_{h_{r-1}}^{(n_{r-1})}, x_r, \dots, x_p)$ means $f(x_1, \dots, x_p)$ if $r=1$, and

$(f(c_{h_1}^{(n_1)}, \dots, c_{h_r}^{(n_r)}, x_{r+1}, \dots, x_p)$ means $f(c_{h_1}^{(n_1)}, \dots, c_{h_p}^{(n_p)})$ if $r=p$).

Thus, throughout C,

$$
\begin{aligned}
& |f(x_1, \dots, x_p) - P_{n_1, n_2, \dots, n_p}(f, x_1, \dots, x_p)| \\
&\leq \sum_{r=1}^{p} \sum_{\substack{h_q=0,1,\dots,n_q \\ q=1,2,\dots,p,\, q\neq r}} \left| f(c_{h_1}^{(n_1)}, \dots, c_{h_{r-1}}^{(n_{r-1})}, x_r, \dots, x_p) - \sum_{h_r=0}^{n_r} f(c_{h_1}^{(n_1)}, \dots, c_{h_r}^{(n_r)}, x_{r+1}, \dots, x_p) \right. \\
&\qquad \left. \cdot K_{h_r}^{(n_r)}(x_r) \right| \prod_{\substack{s=1 \\ s\neq r}}^{p} K_{h_s}^{(n_s)}(x_s) \\
&< \sum_{r=1}^{p} \sum_{\substack{h_q=0,1,\dots,n_q \\ q=1,2,\dots,p,\, q\neq r}} a_{\lambda_r, n_r} \prod_{\substack{s=1 \\ s\neq r}}^{p} K_{h_s}^{(n_s)}(x_s) = \sum_{r=1}^{p} a_{\lambda_r, n_r}.
\end{aligned}
$$

5. EXAMPLE [11]. Let $f(x_1, x_2, \dots, x_p)$ $(p \geq 2)$ be a real function defined on the cube C: $-1 \leq x_k \leq 1$, $k=1,2, \dots, p$. Suppose that for $r=1,2, \dots, p$ we have, for some positive constant λ_r and throughout C, the inequality (3). For $j=0,1, \dots, n$; $n=1,2, \dots$, let $c_j^{(n)}$ and $K_j^{(n)}(x)$ be as in II, section 3. If $n_1, n_2, \dots, n_p$ are positive integers and $P_{n_1, n_2, \dots, n_p}(f, x_1, \dots, x_p)$ is defined by (4), then by Theorem 1 we have, throughout C,

$$|f(x_1, \dots, x_p) - P_{n_1, \dots, n_p}(f, x_1, \dots, x_p)| < \sum_{r=1}^{p} 4\lambda_r \pi (n_r+1)^{-1}[\alpha + \log (n_r+1)].$$

6. Similarly to Theorem 1, one can prove the following

THEOREM 2. *Assume the hypotheses and notation of Section 2. Assume also that for* $n=1,2,\ldots,$ α_n, $\beta_n(\beta_n \leq b-a)$ *are numbers such that if* f *is a real function, continuous in* [a, b], *with modulus of continuity* $\omega(\delta)$ *there, then*

$$\max_{a \leq x \leq b} |f(x)-P_n(f, x)| \leq \alpha_n \omega(\beta_n).$$

Let $f(x_1, x_2, \ldots, x_p)$ $(p \geq 2)$ *be a real function, continuous in the cube* $C: a \leq x_k \leq b, k=1,2,\ldots, p$. *For every* $\delta \epsilon [0, b-a]$ *and every* $r(=1,2,\ldots, p)$, *let*

$$\omega_r(\delta)=\max|f(x_1,\ldots, x_{r-1}, v, x_{r+1},\ldots, x_p)-f(x_1,\ldots, x_{r-1}, u, x_{r+1},\ldots, x_p)| \tag{5}$$

where the x_j, u, *and* v *vary in* [a, b] *with* $0 \leq v-u \leq \delta$. *Let* $n_1, n_2, \ldots, n_p$ *be arbitrary positive integers. Then with the notation* (4), *we have throughout* C,

$$|f(x_1,\ldots, x_p)-P_{n_1, n_2, \ldots, n_p}(f, x_1,\ldots, x_p)| \leq \sum_{r=1}^{p} \alpha_{n_r}\omega_r(\beta_{n_r}). \tag{6}$$

7. EXAMPLE. Let $f(x_1, \ldots, x_p)$ $(p \geq 2)$ be a real function, continuous in the cube C: $0 \leq x_k \leq 1$, $k=1,2,\ldots, p$. For every $\delta \epsilon [0,1]$ and every $r(=1,2,\ldots, p)$ let $\omega_r(\delta)$ be as in Theorem 2 (with $a=0$, $b=1$). Let $n_1, \ldots, n_p$ be arbitrary positive integers. Then by Theorem 2, and by I, section 3, we have throughout C:

$$\left|f(x_1,\ldots, x_p)-\sum_{h_1=0}^{n_1}\cdots\sum_{h_p=0}^{n_p} f\left(\frac{h_1}{n_1},\ldots,\frac{h_p}{n_p}\right)\binom{n_1}{h_1}\cdots\binom{n_p}{h_p} x_1^{h_1}\cdots x_p^{h_p}(1-x_1)^{n_1-h_1}\cdots(1-x_p)^{n_p-h_p}\right| \leq \sum_{r=1}^{p}(5/4)\omega_r(n_r^{-1/2}).$$

8. We consider now an analog of the situation considered in Theorems 1 and 2, sums being replaced by integrals.

9. Let $-\infty < a < b < \infty$, and for $n=1,2,\ldots$ let $K_n(x, t)$ be a real function which is ≥ 0 and continuous in the square $a \leq x \leq b$, $a \leq t \leq b$, and which satisfies for every $x \epsilon [a, b]$,

$$\int_a^b K_n(x, t)dt=1.$$

For every real function f, continuous in $[a, b]$, set

$$P_n(f, x) \equiv \int_a^b f(t)K_n(x, t)dt \qquad (n=1,2,\ldots). \tag{7}$$

10. Such $P_n(f, x)$ are constructed again, like their counterparts (1), in order to obtain approximations to f. Here are a few examples of such $P_n(f, x)$ prominent in Analysis.

I. Let $a=-\pi$, $b=\pi$ and for every real x, t, let

$$K_n(x, t)=\frac{1}{\pi}\left[-\frac{1}{2}+\sum_{j=0}^{n-1}\frac{n-j}{n}\cos\{j(t-x)\}\right] \qquad (n=1,2,\ldots).$$

If $t-x$ is not an integral multiple of 2π, then

$$K_n(x, t)=\sin^2\left\{\frac{n}{2}(t-x)\right\}\Big/\left[2n\pi \sin^2\left\{\frac{1}{2}(t-x)\right\}\right] \qquad (n=1,2,\ldots).$$

Also the properties in the first sentence of section 9 hold. For every real function f continuous in $(-\infty, \infty)$ and of period 2π, and for $n=1,2, \ldots, P_n(f, x)$ of (7) is in the present case the arithmetic mean

$$\sigma_n(f, x) \equiv n^{-1} \sum_{j=0}^{n-1} S_j(x),$$

where

$$S_0(x) \equiv a_0/2, \qquad S_j(x) \equiv (a_0/2) + \sum_{k=0}^{j} [a_k \cos(kx) + b_k \sin(kx)] \qquad (j=1,2, \ldots),$$

$$(a_0/2) + \sum_{k=0}^{\infty} [a_k \cos(kx) + b_k \sin(kx)]$$

being the Fourier series of f, and by a classical theorem of Fejér [2] $\sigma_n(f, x)$ converges uniformly to $f(x)$ in $(-\infty, \infty)$. Furthermore, suppose that a real function f(of period 2π) satisfies throughout the real line, for some constant λ,

$$|f(v) - f(u)| \leq \lambda |v-u|.$$

Then by a theorem of S. N. Bernstein ([6], p. 61; [9], p. 162) for every $n > 1$, $P_n(f, x) \equiv \sigma_n(f, x)$ satisfies

$$\max_{-\infty < x < \infty} |f(x) - P_n(f, x)| \leq C_0 \lambda \log n/n,$$

C_0 being an absolute constant.

II. Let $a = -\pi$, $b = \pi$, and for every real x, t, let

$$K_n(x, t) = \frac{3}{2\pi n(2n^2+1)} \left[-n + 2 \sum_{j=0}^{n-1} (n-j) \cos \{j(t-x)\} \right]^2 \qquad (n=1,2, \ldots).$$

If $t-x$ is not an integral multiple of 2π, then

$$K_n(x, t) = \frac{3}{2\pi n(2n^2+1)} \left[\frac{\sin \left\{ \frac{n}{2}(t-x) \right\}}{\sin \left(\frac{t-x}{2} \right)} \right]^4 \qquad (n=1,2, \ldots).$$

The properties in the first sentence of section 9 hold. For every real function f, continuous in $(-\infty, \infty)$ and of period 2π, and for $n=1,2, \ldots, P_n(f, x)$ of (7) is now a trigonometric polynomial introduced by Jackson ([4, 5, 6]). If f is a real function of period 2π, satisfying for every real u, v

$$|f(v) - f(u)| \leq \lambda |v-u|$$

λ being a constant, then by a classical theorem of Jackson ([4, 5, 6]) this particular $P_n(f, x)$ satisfies

$$\max_{-\infty < x < \infty} |f(x) - P_n(f, x)| \leq (c\lambda/n) \qquad (n=1, 2, \ldots),$$

c being an absolute constant.

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III. Let $a=-\pi$, $b=\pi$, and for every real x, t, let

$$K_n(x, t)=\frac{(2n)!!}{2\pi(2n-1)!!}\cos^{2n}\left(\frac{t-x}{2}\right) \qquad (n=1, 2, \ldots).$$

Here, $m!!$ denotes $2\cdot 4\cdot 6\cdot \ldots m$ for every positive even m, and $1\cdot 3\cdot 5\cdot \ldots m$ for every positive odd m. Again the properties in the first sentence of section 9 hold. For every real function f, continuous and of period 2π in $(-\infty, \infty)$ and for $n=1,2, \ldots, P_n(f, x)$ of (7) is in the present case a trigonometric polynomial introduced by de la Valée-Poussin [12]. If $\omega(\delta)\,(0 \leq \delta < \infty)$ is the modulus of continuity of such an f, then ([8], [9]) for $n=1,2, \ldots$

$$\max_{-\infty<x<\infty} |f(x)-P_n(f, x)| \leq 3\omega(n^{-1/2}).$$

11. An analog of Theorem 1 is

THEOREM 3. *Assume the hypotheses and notation of section 9. Assume also that if* f *is a real function satisfying for some positive constant* λ, *throughout* [a, b],

$$|f(v)-f(u)| \leq \lambda |v-u|,$$

then for $\quad n=1,2, \ldots \qquad \max_{a \leq x \leq b} |f(x)-P_n(f, x)| < a_{\lambda, n}$

where $a_{\lambda, n}$ *depends on* λ *and* n *only. Let* $f(x_1, x_2, \ldots, x_p)$ $(p \geq 2)$ *be a real function defined in the cube* C: $a \leq x_k \leq b$, $k=1,2, \ldots, p$. *Suppose that for* $r=1,2, \ldots, p$, λ_r *is a positive number such that throughout* C

$$|f(x_1, x_2, \ldots, x_{r-1}, v, x_{r+1}, \ldots, x_p)-f(x_1, x_2, \ldots, x_{r-1}, u, x_{r+1}, \ldots, x_p)| \leq \lambda_r |v-u|. \tag{8}$$

Let $n_1, n_2, \ldots, n_p$ *be arbitrary positive integers, and set*

$$P_{n_1, n_2, \ldots, n_p}(f, x_1, \ldots, x_p) \equiv \int_a^b \ldots \int_a^b f(t_1, \ldots, t_p) K_{n_1}(x_1, t_1) \ldots K_{n_p}(x_p, t_p)dt_1 \ldots dt_p. \tag{9}$$

Then throughout C:

$$|f(x_1, \ldots, x_p)-P_{n_1, n_2, \ldots, n_p}(f, x_1, \ldots, x_p)| < \sum_{r=1}^{p} a_{\lambda_r, n_r}.$$

PROOF. Observe that throughout C,

$$\int_a^b \ldots \int_a^b K_{n_1}(x_1, t_1) \ldots K_{n_p}(x_p, t_p)dt_1 \ldots dt_p=1.$$

Hence, throughout C,

$$\begin{aligned}
&f(x_1, \ldots x_p)-P_{n_1, n_2, \ldots, n_p}(f, x_1, \ldots, x_p)\\
&\quad=\int_a^b \ldots \int_a^b \Big[f(x_1, \ldots x_p)-f(t_1, \ldots, t_p)\Big] K_{n_1}(x_1, t_1) \ldots K_{n_p}(x_p, t_p)dt_1 \ldots dt_p\\
&\quad=\int_a^b \ldots \int_a^b \Big[\sum_{r=1}^{p} f(t_1, \ldots, t_{r-1}, x_r, \ldots, x_p)-f(t_1, \ldots, t_r, x_{r+1}, \ldots, x_p)\Big] K_{n_1}(x_1, t_1)\\
&\qquad\qquad \ldots K_{n_p}(x_p, t_p)dt_1 \ldots dt_p
\end{aligned}$$

$$=\sum_{r=1}^{p}\int_a^b \cdots \int_a^b \left\{\int_a^b [f(t_1, \ldots, t_{r-1}, x_r, \ldots, x_p) - f(t_1, \ldots, t_r, x_{r+1}, \ldots, x_p)]K_{n_r}(x_r, t_r)dt_r\right\} \prod_{\substack{s=1\\ s\neq r}}^{p} K_{n_s}(x_s, t_s)dt_1 \ldots dt_{r-1}dt_{r+1} \ldots dt_p.$$

($f(t_1, \ldots, t_{r-1}, x_r, \ldots, x_p)$ means $f(x_1, \ldots, x_p)$ if $r=1$, and $f(t_1, \ldots, t_r, x_{r+1}, \ldots, x_p)$ means $f(t_1, \ldots, t_p)$ if $r=p$). Thus, throughout C,

$$|f(x_1, \ldots, x_p) - P_{n_1, n_2, \ldots, n_p}(f, x_1, \ldots, x_p)| \leq \sum_{r=1}^{p}\int_a^b \cdots \int_a^b \left| f(t_1, \ldots, t_{r-1}, x_r, \ldots, x_p) - \int_a^b f(t_1, \ldots, t_r, x_{r+1}, \ldots, x_p)K_{n_r}(x_r, t_r)dt_r \right| \prod_{\substack{s=1\\ s\neq r}}^{p} K_{n_s}(x_s, t_s)dt_1 \ldots dt_{r-1}dt_{r+1} \ldots dt_p$$

$$< \sum_{r=1}^{p}\int_a^b \cdots \int_a^b a_{\lambda_r, n_r} \prod_{\substack{s=1\\ s\neq r}}^{p} K_{n_s}(x_s, t_s)dt_1 \ldots dt_{r-1}dt_{r+1} \ldots dt_p = \sum_{r=1}^{p} a_{\lambda_r, n_r}.$$

Similarly, one can prove the following analog of Theorem 2:

THEOREM 4. *Assume the hypotheses and notation of section 9. Assume also that for* $n=1,2, \ldots$, α_n, $\beta_n(\beta_n \leq b-a)$ *are numbers such that if* f *is a real function, continuous in* [a, b], *with modulus of continuity* $\omega(\delta)$ *there, then*

$$\max_{a\leq x\leq b} |f(x) - P_n(f, x)| \leq \alpha_n \omega(\beta_n).$$

Let $f(x_1, \ldots, x_p)$ $(p \geq 2)$ *be a real function, continuous in the cube* C: $a \leq x_k \leq b$, $k=1,2, \ldots, p$. *For every* $\delta \epsilon [0, b-a]$ *and every* $r(=1,2, \ldots, p)$ *let* $\omega_r(\delta)$ *be as in Theorem 2. Let* $n_1, \ldots, n_p$ *be arbitrary positive integers. Then with the notation* (9), *we have* (6) *throughout* C.

12. The last two theorems can obviously be modified in the following way. For $n=1,2, \ldots$ let $K_n(x, t)$ be a real function which throughout the plane $-\infty < x < \infty$, $-\infty < t < \infty$ is ≥ 0, continuous, and of period 2π with respect to x and to t, and such that for every real x,

$$\int_{-\pi}^{\pi} K_n(x, t)dt = 1.$$

For every real function f, continuous and of period 2π in $(-\infty, \infty)$, set

$$P_n(f, x) \equiv \int_{-\pi}^{\pi} f(t)K_n(x, t)dt \qquad (n=1,2, \ldots).$$

Let $f(x_1, \ldots, x_p)$ $(p \geq 2)$ be a real function, continuous and of period 2π with respect to each x_j in the (real) Euclidean p-space E_p. Let $n_1, n_2, \ldots, n_p$ be positive integers, and set

$$P_{n_1, \ldots, n_p}(f, x_1, \ldots, x_p) \equiv \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} f(t_1, \ldots, t_p)K_{n_1}(x_1, t_1) \ldots K_{n_p}(x_p, t_p)dt_1 \ldots dt_p. \tag{10}$$

A. Suppose that if f is a real function of period 2π, satisfying for some constant λ, throughout $(-\infty, \infty)$, $|f(v)-f(u)| \leq \lambda|v-u|$, then for $n=1,2, \ldots$

$$\max_{-\infty<x<\infty} |f(x)-P_n(f, x)| \leq a_{\lambda, n}$$

where $a_{\lambda, n}$ depends on λ and n only. Suppose that for $r=1,2, \ldots, p$, λ_r is a number such that (8) holds everywhere in E_p. Then throughout E_p we have

$$|f(x_1, \ldots, x_p)-P_{n_1, \ldots, n_p}(f, x_1, \ldots, x_p)| \leq \sum_{r=1}^{p} a_{\lambda_r, n_r}.$$

For example, if for $n=1,2, \ldots, K_n(x, t)$ is as in example II of section 10, then throughout E_p we have

$$\left|f(x_1, \ldots, x_p)-P_{n_1, \ldots, n_p}(f, x_1, \ldots, x_p)\right| \leq c \sum_{r=1}^{p} \lambda_r/n_r,$$

c being the absolute constant mentioned there.

B. Assume that for $n=1,2, \ldots, \alpha_n$ and β_n are numbers such that if f is a real function, continuous and of period 2π in $(-\infty, \infty)$, with modulus of continuity $\omega(\delta)(0 \leq \delta < \infty)$, then

$$\max_{-\infty<x<\infty} |f(x)-P_n(f, x)| \leq \alpha_n \omega(\beta_n).$$

For every $\delta \geq 0$ and for $r=1,2, \ldots, p$ let $\omega_r(\delta)$ be given by (5) where now the x_j, u and v vary in $(-\infty, \infty)$, subject to $0 \leq v-u \leq \delta$. Then (with the notation (10)), (6) holds throughout E_p. For example, if for $n=1,2, \ldots, K_n(x, t)$ is as in example III of section 10, then throughout E_p we have

$$\left|f(x_1, \ldots, x_p)-P_{n_1, \ldots, n_p}(f, x_1, \ldots, x_p)\right| \leq 3 \sum_{r=1}^{p} \omega_r(n_r^{-1/2}).$$

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