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Error Bounds for Asymptotic Solutions of Differential Equations[1]

I. The Distinct Eigenvalue Case

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The method of Olver for bounding the error term in the asymptotic solutions of a second-order equation having an irregular singularity at infinity is extended to the general system of n first-order equations in the case when the eigenvalues of the lead coefficient matrix are distinct. Vector and norm bounds are given for the difference between an actual solution vector and a partial sum of a formal solution vector. Two cases are distinguished geometrically: In one it is possible to express the error vector by a single Volterra vector integral equation; in the other it is necessary to use a simultaneous pair of Volterra vector integral equations. Some new inequalities for integral equations are given in an appendix.

1. Introduction

Let us first establish our notation. To refer to an $m\times n$ matrix $\mathbf{A}$ we shall use $\mathbf{A}=[a_{ij}]$ $(i=1, 2, \ldots, m\ j=1,2, \ldots, n)$, while to refer to the (i, j)th element of $\mathbf{A}$ we shall use $\{\mathbf{A}\}_{ij}=a_{ij}$. We shall write $|\mathbf{A}|$ for the matrix of absolute values of the elements of $\mathbf{A}$, and for $\mathbf{A}$ and $\mathbf{B}$ real and of the same dimensions, $\mathbf{A}\leqq\mathbf{B}$ will imply that $\{\mathbf{A}\}_{ij}\leqq\{\mathbf{B}\}_{ij}$ for all (i, j). All the integration paths $\mathscr{P}$ we shall use consist of a finite number of Jordan arcs $t=t(s)$ $(a\leqq s\leqq b)$ on each of which $\frac{dt(s)}{ds}$ is continuous and non-vanishing. The notation $\sup_{t\in\mathscr{P}}|\mathbf{V}(t)|$ where $\mathbf{V}(t)$ is an $m\times n$ matrix will be used to denote the $m\times n$ matrix of non-negative numbers whose elements are the least upper bounds on $\mathscr{P}$ of the corresponding elements of $|\mathbf{V}(t)|$.

The system of differential equations we consider takes the form

$$\frac{dW}{dz}=z^r\left[\sum_{k=0}^{s-1}\mathbf{A}_k z^{-k}+\mathfrak{A}_s(z)z^{-s}\right]\mathbf{W} \tag{1.1}$$

where r and s are non-negative integers. Each $\mathbf{A}_k$ is a constant $n\times n$ matrix. The n^2 elements of $\mathfrak{A}_s(z)$ are holomorphic in some domain $\mathscr{D}$ (which may be part of a Riemann surface) extending to infinity. For any path $\mathscr{P}$ extending to infinity in $\mathscr{D}$, we suppose that $\sup_{z\in\mathscr{P}}|\{\mathfrak{A}_s(z)\}_{ij}|$ is finite for $i, j=1, 2, \ldots, n$. For notational convenience we shall write $\mathfrak{A}_0(z)=\mathfrak{A}(z)$, $\mathbf{A}_{-1}=\mathbf{O}$. The integer r denotes the rank of the system (1.1).

It is known ([2][3]) that the system (1.1) has a linearly independent set of n formal vector solutions which are asymptotic expansions of actual solutions of (1.1) as $z\to\infty$ in $\mathscr{D}$. The purpose of the present paper is to establish bounds for the difference between the actual solutions and the partial sums of formal solutions. This is achieved by an extension of the error analysis developed by Olver [1] for second-order differential equations.

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[1] An invited paper based on the author's Ph.D. thesis, University of Alberta (1965).
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[3] Figures in brackets indicate the literature references at the end of this paper.

2. Transformations on the Original System and Formal Solutions

For purposes of obtaining error bounds we require explicit rules for constructing formal solutions. In this section we therefore construct transformations which convert (1.1) into canonical form, and then we construct formal solutions for the transformed system.

THEOREM 2.1: *There exist constant* $n \times n$ *matrices* $\mathbf{T}_0, \mathbf{T}_1, \ldots, \mathbf{T}_{r+1}$ *such that the transformation*

$$\mathbf{X} = \mathfrak{T}(z)\mathbf{W} = \sum_{k=0}^{r+1} \mathbf{T}_k z^{-k} \mathbf{W} \tag{2.1}$$

reduces (1.1) *to the canonical form*

$$\frac{d\mathbf{W}}{dz} = z^r \mathfrak{C}(z)\mathbf{W}, \tag{2.2}$$

where $\mathfrak{C}(z)$ *can be expanded in the form*

$$\mathfrak{C}(z) = z^{r+1} \sum_{k=0}^{r+1} \mathbf{C}_k z^{-k} + z^{-2} \sum_{k=0}^{s-1} \mathbf{B}_k z^{-k} + \mathfrak{B}_s(z) z^{-s}. \tag{2.3}$$

Here the $\mathbf{C}_k$ *and* $\mathbf{B}_k$ *are constant* $n \times n$ *matrices,* $\mathbf{C}_k$ *being diagonal and* $\mathfrak{B}_s(z)$ *is an* $n \times n$ *matrix whose elements are holomorphic for all sufficiently large* z *in* $\mathscr{D}$*, and uniformly bounded on a path* $\mathscr{P}$ *extending to infinity in* $\mathscr{D}$*.*

We prove this important theorem here in order to furnish an explicit algorithm for computing the coefficient matrices $\mathbf{T}_k$ $(k=0, 1, \ldots, r+1)$. For an earlier proof see Birkoff [3].

Since $\mathbf{A}_0$ has distinct diagonal elements there exists a nonsingular matrix $\mathbf{T}_0$ such that $\mathbf{T}_0^{-1}\mathbf{A}_0\mathbf{T}_0 = \mathbf{C}_0$, where $\mathbf{C}_0$ is a diagonal matrix with distinct diagonal elements. Accordingly we assume that the transformation $\mathbf{X} = \mathbf{T}_0\mathbf{W}$ has been made and that $\mathbf{A}_0$ is already in diagonal form. Then we may take $\mathbf{T}_0 = \mathbf{I}$ in (2.1). On making the transformation (2.1) in (2.2), we obtain

$$\frac{d\mathbf{W}}{dz} = z^r \mathfrak{T}^{-1}(z) \left[\mathfrak{A}(z)\mathfrak{T}(z) - z^{-r} \frac{d\mathfrak{T}(z)}{dz} \right] \mathbf{W}. \tag{2.4}$$

We choose the matrices $\mathbf{T}_1, \mathbf{T}_2, \ldots, \mathbf{T}_{r+1}$ so that the matrix

$$\mathfrak{C}(z) = \mathfrak{T}^{-1}(z) \left[\mathfrak{A}(z)\mathfrak{T}(z) - z^{-r} \frac{d\mathfrak{T}(z)}{dz} \right] \tag{2.5}$$

has the form (2.3). For this purpose we need only consider the coefficients of $z^{-1}, z^{-2}, \ldots, z^{-r-1}$ in the equation

$$\mathfrak{T}(z)\mathfrak{C}(z) = \mathfrak{A}(z)\mathfrak{T}(z) - z^{-r} \frac{d\mathfrak{T}(z)}{dz}. \tag{2.6}$$

Expanding (2.6) and equating equal powers of z, we obtain

$$\sum_{k=0}^{l} (\mathbf{T}_{l-k}\mathbf{C}_k - \mathbf{A}_k\mathbf{T}_{l-k}) = \mathbf{O} \qquad (l=0, 1, \ldots, r+1). \tag{2.7}$$

With $l=0$, (2.7) gives $\mathbf{C}_0 = \mathbf{A}_0$. With $l=1$ we have

$$\mathbf{T}_1\mathbf{C}_0 - \mathbf{A}_0\mathbf{T}_1 + \mathbf{C}_1 - \mathbf{A}_1 = \mathbf{O}. \tag{2.8}$$

Now for arbitrary $\mathbf{T}_1$, the diagonal part of $\mathbf{T}_1\mathbf{C}_0-\mathbf{A}_0\mathbf{T}_1=\mathbf{T}_1\mathbf{A}_0-\mathbf{A}_0\mathbf{T}_1$ is zero. Thus $\mathbf{C}_1=\text{diag}(\mathbf{A}_1)$. The off-diagonal elements of $\mathbf{T}_1$ can be determined by the solution of

$$\{\mathbf{T}_1\}_{ij}(\{\mathbf{A}_0\}_{jj}-\{\mathbf{A}_0\}_{ii})=\{\mathbf{A}_1\}_{ij}$$

under the assumption that the diagonal elements of $\mathbf{A}_0$ are distinct. We set the undeterminable diagonal elements of $\mathbf{T}_1$ equal to zero.

Assume that for some positive integer $l \leqq r$ we have solved for $\mathbf{C}_1, \ldots, \mathbf{C}_l$ and the off-diagonal elements of $\mathbf{T}_1, \ldots, \mathbf{T}_l$ and that we have set the diagonal elements of $\mathbf{T}_1, \ldots, \mathbf{T}_l$ equal to zero. Then from (2.7) we have

$$\mathbf{T}_{l+1}\mathbf{A}_0-\mathbf{A}_0\mathbf{T}_{l+1}+\sum_{k=1}^{l}(\mathbf{T}_{l+1-k}\mathbf{C}_k-\mathbf{A}_k\mathbf{T}_{l+1-k})+\mathbf{C}_{l+1}-\mathbf{A}_{l+1}=\mathbf{O}. \tag{2.9}$$

Again, the diagonal part of $\mathbf{T}_{l+1}\mathbf{A}_0-\mathbf{A}_0\mathbf{T}_{l+1}$ is zero. Thus the diagonal part of

$$\sum_{k=1}^{l}(\mathbf{T}_{l+1-k}\mathbf{C}_k-\mathbf{A}_k\mathbf{T}_{l+1-k})+\mathbf{C}_{l+1}-\mathbf{A}_{l+1} \tag{2.10}$$

is also zero. This enables us to determine $\mathbf{C}_{l+1}$. The off-diagonal elements of $\mathbf{T}_{l+1}$ are determinable from the scalar equation

$$\{\mathbf{T}_{l+1}\}_{ij}(\{\mathbf{A}_0\}_{jj}-\{\mathbf{A}_0\}_{ii})+\left\{\sum_{k=1}^{l}(\mathbf{T}_{l+1-k}\mathbf{C}_k-\mathbf{A}_k\mathbf{T}_{l+1-k})\right\}_{ij}=0 \tag{2.11}$$

since the diagonal elements of $\mathbf{A}_0$ are distinct. The diagonal elements of $\mathbf{T}_{l+1}$ are again not determinable and we set these equal to zero. This proves by induction that each $\mathbf{T}_k$ and $\mathbf{C}_k$ $(k=0, 1, \ldots, r+1)$ can be determined as stated in the theorem.

To determine $\mathbf{B}_k$ and $\mathfrak{B}_s(z)$, we again expand (2.6) and equate coefficients of z^{-r-1-k}, $k \geqq 1$. We obtain

$$\mathbf{B}_k=\mathbf{A}_{k+r+2}-\sum_{l=1}^{\min(k,r+1)}(\mathbf{T}_l\mathbf{B}_{k-l}-\mathbf{A}_{k+r+2-l}\mathbf{T}_l)$$
$$-\sum_{l=k+1}^{r+1}(\mathbf{T}_l\mathbf{C}_{k+r+2-l}-\mathbf{A}_{k+r+2-l}\mathbf{T}_l)+k\mathbf{T}_k \tag{2.12}$$

for all $k \geqq 0$, where we define $\mathbf{T}_k=0$ for $k>r+1$. Since the elements of $\mathfrak{A}(z)$, $\mathfrak{T}(z)$, $\mathfrak{T}^{-1}(z)$ are holomorphic and bounded for all sufficiently large z in $\mathscr{D}$, the elements of $\mathfrak{B}_s(z)$ are similarly holomorphic and bounded.

This completes the proof of Theorem 2.1.

THEOREM 2.2: *The system* (2.2), *in which* $\mathfrak{C}(z)$ *has the canonical form* (2.3), *possesses a formal independent series solution matrix of the form*[4]

$$\widetilde{\mathbf{W}}(z)=\widetilde{\mathbf{U}}(z)\exp\mathfrak{Q}(z) \tag{2.13}$$

[4] Here and henceforth $\exp \mathbf{B}=\sum_{k=0}^{\infty}\frac{\mathbf{B}^k}{k!}$ for any square matrix $\mathbf{B}$.

where $\mathfrak{Q}(z)$ *is diagonal and has the form*

$$\mathfrak{Q}(z)=\sum_{k=0}^{r}\frac{\mathbf{Q}_k}{r+1-k}z^{r+1-k}+\mathbf{Q}_{r+1}\ln z=\sum_{k=0}^{r}\frac{\mathbf{C}_k z^{r+1-k}}{r+1-k}+\mathbf{C}_{r+1}\ln z \tag{2.14}$$

and [5]

$$\mathbf{U}(z)\hat{=}\mathbf{I}+\sum_{k=1}^{\infty}\mathbf{U}_k z^{-k}. \tag{2.15}$$

Proof: Substituting (2.13) into (2.2), canceling exp $\mathfrak{Q}(z)$ and equating coefficients of equal powers of z^{-1}, we find that

$$\sum_{j=0}^{\min(k,r+1)}(\mathbf{U}_{k-j}\mathbf{Q}_j-\mathbf{C}_j\mathbf{U}_{k-j})-(k-r-1)\mathbf{U}_{k-r-1}-\sum_{j=0}^{k-r-2}\mathbf{B}_j\mathbf{U}_{k-r-2-j}=\mathbf{O}(k=0,1,2,\ldots) \tag{2.16}$$

provided that we define $\mathbf{U}_j=\mathbf{O}$ for $j<0$. The proof of Theorem 2.2 will be complete after we prove the following lemma.

Lemma 2.1: *With* $\mathbf{U}_0=\mathbf{I}$ *and* $\mathfrak{Q}(z)$ *diagonal, equation* (2.16) *uniquely determines* $\mathbf{U}_j$ *and* $\mathbf{Q}_j$, $j=0, 1, 2, \ldots$.

Proof: With $k=0$ we have

$$\mathbf{Q}_0=\mathbf{C}_0. \tag{2.17}$$

With $k=1$, we have

$$\mathbf{U}_1\mathbf{C}_0-\mathbf{C}_0\mathbf{U}_1+\mathbf{Q}_1-\mathbf{C}_1=\mathbf{O}. \tag{2.18}$$

Now if $\mathbf{U}_1$ is an arbitrary $n\times n$ matrix, the diagonal part of $\mathbf{U}_1\mathbf{C}_0-\mathbf{C}_0\mathbf{U}_1$ is zero. By assumption, $\mathbf{Q}_1$ and $\mathbf{C}_1$ are diagonal; hence $\mathbf{Q}_1=\mathbf{C}_1$. Since the diagonal elements of $\mathbf{C}_0$ are distinct, the off-diagonal elements of $\mathbf{U}_1$ given by the solution of $\{\mathbf{U}_1\}_{ij}(\{\mathbf{C}_0\}_{jj}-\{\mathbf{C}_0\}_{ii})=0$ are zero. The diagonal elements of $\mathbf{U}_1$ are not determinable at this point and all we can say here is that $\mathbf{U}_1=\mathbf{D}_1$, where $\mathbf{D}_1$ is an arbitrary diagonal matrix.

Assume that for some integer k in the range $1\leqq k\leqq r$ we have found $\mathbf{Q}_j=\mathbf{C}_j$ and $\mathbf{U}_j=\mathbf{D}_j$ where $\mathbf{D}_j$ is an arbitrary diagonal matrix and $1\leqq j\leqq k$. Then solving for $\mathbf{U}_{k+1}$ and $\mathbf{Q}_{k+1}$ we have from (2.16)

$$\mathbf{U}_{k+1}\mathbf{C}_0-\mathbf{C}_0\mathbf{U}_{k+1}+\sum_{j=1}^{k}(\mathbf{D}_{k+1-j}\mathbf{C}_j-\mathbf{C}_j\mathbf{D}_{k+1-j})+\mathbf{Q}_{k+1}-\mathbf{C}_{k+1}=\mathbf{O} \tag{2.19}$$

or, since diagonal matrices commute,

$$\mathbf{U}_{k+1}\mathbf{C}_0-\mathbf{C}_0\mathbf{U}_{k+1}+\mathbf{Q}_{k+1}-\mathbf{A}_{k+1}=\mathbf{O}. \tag{2.20}$$

On comparing (2.20) with (2.18) it follows by exactly similar arguments to those used for (2.18) that $\mathbf{Q}_{k+1}=\mathbf{C}_{k+1}$ and that $\mathbf{U}_{k+1}=\mathbf{D}_{k+1}$, where $\mathbf{D}_{k+1}$ is an arbitrary diagonal matrix. We have thus proved by induction that for every integer k in the range $0\leqq k\leqq r+1$, $\mathbf{Q}_k=\mathbf{C}_k$ and $\mathbf{U}_k=\mathbf{D}_k$, where the $\mathbf{D}_k$ are undetermined diagonal matrices.

[5] The symbol "$\hat{=}$" here and henceforth denotes a formal equality.

With $k=r+2$, we have, from (2.16) and the above results,

$$\mathbf{U}_{r+2}\mathbf{C}_0-\mathbf{C}_0\mathbf{U}_{r+2}+\sum_{j=1}^{r+1}(\mathbf{D}_{r+2-j}\mathbf{C}_j-\mathbf{C}_j\mathbf{D}_{r+2-j})+\mathbf{B}_0\mathbf{D}_1=\mathbf{O}. \tag{2.21}$$

or, since the sum on the left is a sum of diagonal matrices,

$$\mathbf{U}_{r+2}\mathbf{C}_0-\mathbf{C}_0\mathbf{U}_{r+2}-\mathbf{B}_0-\mathbf{D}_1=\mathbf{O}. \tag{2.22}$$

The diagonal part of $\mathbf{U}_{r+2}\mathbf{C}_0-\mathbf{C}_0\mathbf{U}_{r+2}$ is zero; hence $\mathbf{D}_1=\mathbf{U}_1=-\operatorname{diag}\mathbf{B}_0$. The off-diagonal elements of $\mathbf{U}_{r+2}$ are given by the solution of $\{\mathbf{U}_{r+2}\}_{ij}(\{\mathbf{C}_0\}_{jj}-\{\mathbf{C}_0\}_{ii})=\{\mathbf{B}_0\}_{ij}$. The diagonal part $\mathbf{D}_{r+2}$ of $\mathbf{U}_{r+2}$ cannot be determined at this point.

Assume that for some positive integer l, the matrices $\mathbf{U}_1, \mathbf{U}_2, \ldots, \mathbf{U}_l$ as well as the off-diagonal elements of the matrices $\mathbf{U}_{l+1}, \mathbf{U}_{l+2}, \ldots, \mathbf{U}_{r+l+1}$ have been determined while the diagonal matrices consisting of the diagonals of $\mathbf{U}_k(l+1\leqq k\leqq r+l+1)$ are still undetermined. Then, from (2.16) we have

$$\mathbf{U}_{r+l+2}\mathbf{C}_0-\mathbf{C}_0\mathbf{U}_{r+l+2}+\sum_{j=1}^{r+1}(\mathbf{U}_{r+2+l-j}\mathbf{C}_j-\mathbf{C}_j\mathbf{U}_{r+l+2-j})+\sum_{j=1}^{l+1}\mathbf{B}_{j-1}\mathbf{U}_{l+1-j}-(l+1)\mathbf{U}_{l+1}=\mathbf{O}. \tag{2.23}$$

On writing $\mathbf{U}_j=\overline{\mathbf{U}}_j+\mathbf{D}_j$, $j>l$, where the diagonal elements of $\overline{\mathbf{U}}_j$ are zero, and using the fact that diagonal matrices commute, we find that

$$\overline{\mathbf{U}}_{r+l+2}\mathbf{C}_0-\mathbf{C}_0\overline{\mathbf{U}}_{r+l+2}+\sum_{j=1}^{r+1}(\overline{\mathbf{U}}_{r+l+2-j}\mathbf{C}_j-\mathbf{C}_j\overline{\mathbf{U}}_{r+l+2-j})-\sum_{j=1}^{l+1}\mathbf{B}_{j-1}\mathbf{U}_{l+1-j}-(l+1)(\overline{\mathbf{U}}_{l+1}+\mathbf{D}_{l+1})=\mathbf{O}. \tag{2.24}$$

Thus each $\mathbf{U}_j$, $j\geqq 1$, is uniquely determined. This completes the proof of Lemma 2.1 and of Theorem 2.2.

Clearly, the above procedure can be modified to compute a particular formal series solution vector independently of the others; the jth series solution vector is

$$\widetilde{\mathbf{W}}_j(z)=\widetilde{\mathbf{U}}_j(z)e^{q_j(z)} \tag{2.25}$$

where $\widetilde{\mathbf{U}}_j(z)$ is the jth vector in $\widetilde{\mathbf{U}}(z)$ and $q_j(z)=\{\Omega(z)\}_{jj}$. For later convenience we set

$$q_j(z)=\sum_{k=0}^{r}\frac{q_{jk}}{r+1-k}z^{r+1-k}+q_{j,\,r+1}\log z. \tag{2.26}$$

The representation (2.25) *illustrates that we have a one-to-one correspondence between the eigenvalues of* $\mathbf{A}_0$ *and the formal vector solutions.*

3. Error Bounds for the Formal Partial Sum Approximation

3.1. The Differential Equation for an Approximation

We denote the jth column of $\mathbf{U}_k$ $(k=0,1,\ .\ .\ .)$ as defined by Theorem 2.2 by $\mathbf{U}_{jk}$. Starting with the partial sum

$$\Phi_{jm}(z)=\left(\sum_{k=0}^{m-1}\mathbf{U}_{jk}z^{-k}\right)e^{q_j(z)} \tag{3.1}$$

we define a vector $\mathbf{R}_{jm}(z)$ by the differential equation

$$\frac{d\Phi_{jm}(z)}{dz}-z^r\mathfrak{C}(z)\Phi_{jm}(z)=\mathbf{R}_{jm}(z)e^{q_j(z)} \tag{3.2}$$

where $\mathfrak{C}(z)$ is defined by (2.3).

Expanding (3.2) and using equations (2.26) and (2.16), we obtain

$$\mathbf{R}_{jm}(z)=-\sum_{\mu=0}^{r+m+1}\left[\sum_{k=0}^{\min(\mu,\,r+1)}(\mathbf{Q}_k-\mathbf{I}q_{jk})\mathbf{U}^+_{\mu-k}+(\mu-r-1)\mathbf{U}^+_{j,\,\mu-r-1}\right.$$
$$\left.+\sum_{k=0}^{\mu-r-2}\mathbf{B}_k\mathbf{U}^+_{j,\,\mu-r-2-k}\right]z^{r-\mu}-\sum_{k=0}^{m-1}(\mathfrak{B}_{m-k}(z)\mathbf{U}^+_{jk})z^{-m-2} \tag{3.3}$$

where $\mathbf{U}^+_{jk}=\mathbf{U}_{jk}$ if $0\leqq k\leqq m-1$; $\mathbf{O}$ otherwise.

Consider the ith element $\mathbf{R}_{i\,jm}(z)$ of $\mathbf{R}_{jm}(z)$. If $i=j$ then using (2.16) again we obtain

$$\mathbf{R}_{j\,jm}(z)=m\mathbf{U}_{j\,jm}z^{-m-1}-\sum_{k=0}^{m-1}\{(\mathfrak{B}_{m-k}(z)\,\mathbf{U}_{jk})\}_j z^{-m-2} \tag{3.4}$$

where $\{\mathbf{V}_j\}_i$ denotes the ith element of the vector $\mathbf{V}_j$.

We also define

$$q_{ij}(z)=q_i(z)-q_j(z). \tag{3.5}$$

For $i\neq j$ we again make use of (2.16) in (3.3) to obtain

$$\mathbf{R}_{i\,jm}(z)=q'_{ij}(z)\mathbf{U}_{i\,jm}z^{-m}-z[q'_{ij}(z)-(q_{i0}-q_{j0})z^r]\mathbf{U}_{i\,jm}z^{-m-1}$$
$$-\sum_{\mu=m+1}^{m+r+1}\left[\sum_{k=\mu+1-m}^{\min(\mu,\,r+1)}(q_{ik}-q_{jk})\mathbf{U}_{ij,\,\mu-k}+(\mu-r-1)\mathbf{U}_{ij,\,\mu-r-1}\right.$$
$$\left.+\sum_{k=0}^{\mu-r-2}\{\mathbf{B}_k\mathbf{U}_{j,\,\mu-r-2-k}\}_i\right]z^{r-\mu}-\sum_{k=0}^{m-1}\{\mathfrak{B}_{m-k}(z)\mathbf{U}_{jk}\}_i\,z^{-m-2} \tag{3.6}$$

where $\mathbf{U}_{i\,jk}$ is the ith element of the vector $\mathbf{U}_{jk}$.

3.2. Two Possible Cases

In this section we shall set up integral equations for the error vector, that is, the difference between an actual solution vector of (2.2) and a partial sum of the formal solution vector (2.13).

Our aim is to construct actual solution vectors for which the formal solution vectors obtained in section 2 are asymptotic expansions, as $|z|\to\infty$. For this reason, and for purposes of obtaining good error bounds we assume $\mathscr{D}$ is such that we can choose our fixed end-points of integration, which we denote by ζ_k, at ∞. We shall show that under these conditions it is possible to express the error vector by at most a simultaneous pair of Volterra vector integral equations.

The error bounds we shall obtain are considerably sharper in the case when we can express the error by a single Volterra vector integral equation than in the case when we require a simultaneous pair of Volterra vector integral equations. It is thus desirable to use a single equation whenever possible.

We have already noted at the end of section 2 that with our construction of formal solutions there is a one-to-one correspondence between the eigenvalues of $\mathbf{C}_0$ and the vector solutions. Let λ_j be an eigenvalue of $\mathbf{C}_0$.

DEFINITION 3.1: *The eigenvalue* λ_j *will be called an extreme eigenvalue if there exists a path* $\mathscr{P}$ *lying in* $\mathscr{D}$, *joining some point* $z(|z|<\infty)$ *in* $\mathscr{D}$ *with* $\zeta_k(|\zeta_k|=\infty)$ *such that* Re $q_{ij}(t)$ *(see equation* (3.1)) *increases monotonically*[6] *as* t *traverses* $\mathscr{P}$ *from* z *to* ζ_k, *for* $i=1, 2, \ldots, n$. *Otherwise* λ_j *will be called an interior eigenvalue.*

Consider the eigenvalues of $\mathbf{C}_0$ as points in the complex plane enclosed by the smallest possible closed convex polygon Π. Except for rotation, the polygon Π and the differences between the various eigenvalues of $\mathbf{A}_0$ are left invariant under the transformations $\mathbf{W}=\mathbf{X}e^{\lambda z^{r+1}}$, $\mathbf{W}=\mathfrak{T}(z)\mathbf{X}$ ($\mathfrak{T}(z)$ as in Theorem 2.1), $z=\omega\zeta$, ω a constant such that $|\omega|=1$. Thus we may assume, without loss of generality, that the vector solution of interest corresponds to the eigenvalue $\lambda_j=0$. Moreover, in the case when zero is an extreme point of the polygon Π, we may assume that all other eigenvalues of $\mathbf{A}_0$ are within the closed sector $|\arg\lambda|\leqq\alpha\leqq\frac{\pi}{2}$ and that there is at least one eigenvalue on each of the rays $\arg\lambda=\alpha$ and $\arg\lambda=-\alpha$. In the case when zero is an interior point of Π, we assume that no eigenvalue (other than the zero eigenvalue) is located on the imaginary axis, that there are some eigenvalues of $\mathbf{A}_0$ in each of the sectors $|\arg\lambda|\leqq\alpha$ and $|\pi-\arg\lambda|\leqq\alpha$, $0\leqq\alpha<\frac{\pi}{2}$, and that there is at least one eigenvalue on each of the lines $\arg\lambda=\alpha$ and $\arg\lambda=-\alpha$.

Suppose for example that we have the case indicated in figure 1, that $\mathfrak{G}(z)$ is regular for all $|z|>\rho$, that $r=0$ and that $q_{i,r+1}=q_{i1}=0$ $(i=1, 2, 3, j)$. It is then clear that $|e^{\lambda_i z}|$ $(i=1, 2, 3, j)$ increases monotonically along any path for which $\left|\arg\frac{dz}{dx}\right|\leqq\frac{\pi}{2}-\alpha$. Thus assuming the points λ_i $(i=1, 2, 3, j)$ to be in the z-plane as indicated in figure 1, we can, for example, connect any point

[6] For any two points z_1, z_2 in the order z, z_1, z_2, ζ_k on $\mathscr{P}$ we have Re $q_{ij}(z_1)\leqq$ Re $q_{ij}(z_2)$.

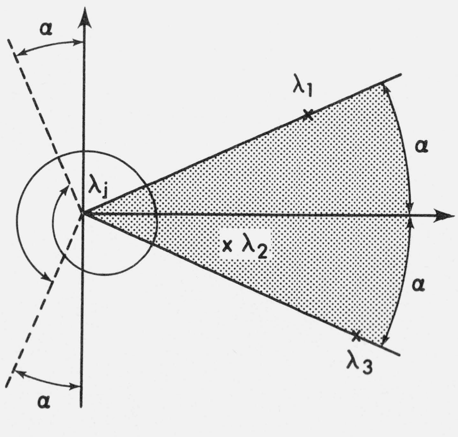

FIGURE 1.

in the interior of the overlapping sector $|\arg z| \leqq \frac{3}{4}\pi - \alpha - \delta$, $0 \leqq \delta < \frac{\pi}{4}$ of figure 1 with $+\infty$ by a path $\mathscr{P}$ consisting of at most three straight lines along each of which $\left|\arg \frac{dz}{dx}\right| \leqq \frac{\pi}{2} - \alpha$ and none of which passes through the origin.

For arbitrary rank $r+1$ it follows that, since $|\arg q_{ij}(t) - \arg [(q_{i0} - q_{j0})t^{r+1}/(r+1)]|$ can be made arbitrarily small by taking $|t|$ sufficiently large, we can connect any z ($|z|$ sufficiently large) in

$$\left|\arg z - \frac{2\pi k}{r+1}\right| \leqq \frac{1}{r+1}\left(\frac{3\pi}{2} - \alpha - \delta\right)\left(0 \leqq \delta < \frac{\pi}{4}\right), \ (k = 0, 1, \ldots, r)$$

to

$$\zeta_k = \infty \exp\left[\frac{2\pi k\sqrt{-1}}{r+1}\right]$$

by a path $\mathscr{P}$ along which Re $q_{ij}(t)$ increases monotonically for $i = 1, 2, \ldots, n$, provided that $\alpha < \frac{\pi}{2}$. In the case when $\alpha = \frac{\pi}{2}$ there may still be a region between two parallel straight lines (no longer a sector of positive angle) in which there is at least one path $\mathscr{P}$ joining z with ζ_k along which Re $q_{ij}(t)$ increases monotonically for all i.

Let us now consider a case in which λ_j is an interior point of the polygon. If the eigenvalues of $\mathbf{C}_0$ are as indicated in figure 2 then, given any number δ in $0 \leqq \delta < \pi - \alpha$ there is a path joining $-\infty$, any point z in the sector $\left|\arg z - \frac{\pi}{2}\right| \leqq \pi - \alpha - \delta$ and $+\infty$, consisting of at most four straight lines (none of which passes through the origin) such that Re $(\lambda_1 t)$, Re $(\lambda_2 t)$ increase monotonically while Re $(\lambda_3 t)$ decreases monotonically as t traverses $\mathscr{P}$ from $-\infty$ to ∞. If we require that a neighborhood of the origin is to be avoided by $\mathscr{P}$, $|z|$ may need to be taken large when δ is small. Similarly, it is easy to see that there is such a path joining $-\infty$, any point z in the sector $|\arg z + \frac{1}{2}\pi| \leqq \pi - \alpha - \delta$ and $+\infty$. Thus these two families of paths, one of which passes around the origin in a positive sense and one in the negative sense as t traverses $\mathscr{P}$ from $-\infty$ to $+\infty$, cover the complete neighborhood of infinity.

In the general case when λ_j is an interior eigenvalue of Π, we divide the integers $\mathbf{N} = 1, 2, \ldots, n$ into two disjoint classes $\mathbf{N}_1$ and $\mathbf{N}_2$ such that if λ_i is in $|\arg \lambda| \leqq \alpha$ then $i \in \mathbf{N}_1$ while if λ_i is in $|\pi - \arg \lambda| \leqq \alpha$ then $i \in \mathbf{N}_2$. Note that j may be taken either in $\mathbf{N}_1$ or in $\mathbf{N}_2$. Clearly there are two families of paths connecting $\zeta_k^{(1)} = \exp[(2k \pm 1)\pi\sqrt{-1}/(r+1)]$, and $\zeta_k^{(2)} = \exp[2k\pi\sqrt{-1}/(r+1)]$

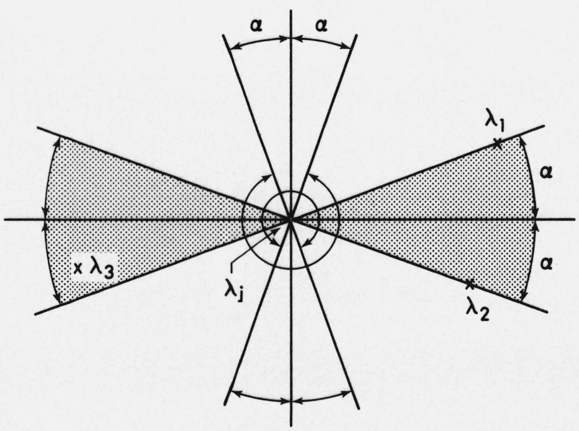

FIGURE 2.

($k=0, 1, \ldots, n$) such that as t traverses $\mathscr{P}$ from $\zeta_k^{(1)}$ to $\zeta_k^{(2)}$ Re $q_{ij}(t)$ decreases or increases monotonically according as $i \in \mathbf{N}_1$ or $i \in \mathbf{N}_2$. These two families of paths cover the sectors $\left| \arg z-(2k \pm\frac{1}{2})\frac{\pi}{r+1}\right| \leqq \frac{\pi-\alpha}{r+1}$.

Theorem 3.1: *Let λ_j be an eigenvalue of $\mathbf{C}_0$. If λ_j is a vertex of Π, then it is an extreme eigenvalue; if it is an interior point of Π then it is an interior eigenvalue.*

Let $\mathfrak{C}(z)$ be holomorphic in a domain $\mathscr{D}$ extending to infinity in the direction $\arg z=2\pi k/(r+1)$, for some integer k in $0 \leqq k \leqq r$. Let $\mathscr{E}$ be the smallest closed disc with center at the origin containing the zeros of $q'_{ij}(t)$ ($i=1, 2, \ldots, n$; $i \neq j$4, and let $\mathscr{D}^* = \mathscr{D}-\mathscr{E}-\{\infty\}$.[7] We define a region $\mathscr{D}(z, \zeta_k)$ to be the union of all points z such that there is a path $\mathscr{P}$ connecting z and $\zeta_k=\exp\,[2\pi k\sqrt{-1}/(r+1)]$ satisfying the following *extreme eigenvalue conditions*:[8]

(1) Except for ζ_k, $\mathscr{P}$ lies entirely in $\mathscr{D}^*$;

(2) For any two points t_1 and t_2 in the order ζ_k, t_1, t_2, z on $\mathscr{P}$, we have

$$|\exp\,[q_{ij}(t_2)-q_{ij}(t_1)]| \leqslant 1 \qquad (i=1, 2, \ldots, n);$$

$$\mathscr{V}_{\mathscr{P}}\,(t^{-1})=\int_{\mathscr{P}}|t^{-2}dt| \text{ is bounded.}$$

The variation symbol introduced in condition (3) above is more generally defined as follows: If $\mathbf{F}=\mathbf{F}(z)$ is a vector of holomorphic functions of z, we define

$$\mathscr{V}_{\mathscr{P}}(\mathbf{F})=\int_{\mathscr{P}}|d\mathbf{F}|=\int_{\mathscr{P}}\left|\frac{d\mathbf{F}}{dz}\,dz\right|. \tag{3.7}$$

Let us set

$$\mathbf{D}_j(z)=\mathfrak{Q}(z)-q_j(z)\mathbf{I}. \tag{3.8}$$

If a vector $\mathbf{W}_j(z)$ satisfies (2.2), then by (3.2) the error vector

$$\boldsymbol{\epsilon}_{jm}(z)=\{\mathbf{W}_j(z)-\Phi_{jm}(z)\}e^{-q_j(z)} \tag{3.9}$$

satisfies

$$\frac{d}{dz}\boldsymbol{\epsilon}_{jm}(z)-\mathbf{D}_j(z)\boldsymbol{\epsilon}_{jm}(z)=\frac{1}{z^2}\mathfrak{B}(z)\boldsymbol{\epsilon}_{jm}(z)-\mathbf{R}_{jm}(z). \tag{3.10}$$

where $\mathfrak{B}(z)=\mathfrak{B}_0(z)$. Now if $\boldsymbol{\epsilon}_{jm,\,k}(z)$ is a solution of

$$\boldsymbol{\epsilon}_{jm,\,k}(z)=\int_{\zeta_k}^{z} e^{\mathbf{D}_j(z)-\mathbf{D}_j(t)}[t^{-2}\mathfrak{B}(t)\boldsymbol{\epsilon}_{jm,\,k}(t)-\mathbf{R}_{jm,\,k}(t)]\,dt \tag{3.11}$$

where the path of integration is chosen as described above, then $\boldsymbol{\epsilon}_{jm,\,k}(z)$ also satisfies (3.10).

If λ_j is not an extreme eigenvalue, let $\mathfrak{A}(z)$ be holomorphic in a domain $\mathscr{D}$ which contains, or extends to infinity in, one of the sectors $\left|\arg z-(2k\pm\frac{1}{2})\frac{\pi}{r+1}\right| \leqq \frac{\pi-\alpha}{(r+1)}$ for some integer k in $0 \leqq k \leqq r$

[7] $\{\infty\}=\{e^{i\theta}\,|-\infty \leq \theta \leq \infty\}$.
[8] Compare [1], sec. 5.

and all z sufficiently large. We reorder, and if necessary relabel, the elements of the equation (3.10) so that we may write them in the form

$$\begin{aligned}\frac{d}{dz}\,\boldsymbol{\epsilon}_{jm}^{(1)}(z)-\mathbf{D}_j^{(1)}(z)\boldsymbol{\epsilon}_{jm}^{(1)}(z)&=z^{-2}\left[\mathfrak{a}_1(z)\boldsymbol{\epsilon}_{jm}^{(1)}(z)+\mathfrak{a}_2(z)\boldsymbol{\epsilon}_{jm}^{(2)}(z)\right]-\mathbf{R}_{jm}^{(1)}(z)\\ \frac{d}{dz}\,\boldsymbol{\epsilon}_{jm}^{(2)}(z)-\mathbf{D}_j^{(2)}(z)\boldsymbol{\epsilon}_{jm}^{(2)}(z)&=z^{-2}\left[\mathfrak{a}_3(z)\boldsymbol{\epsilon}_{jm}^{(1)}(z)+\mathfrak{a}_4(z)\boldsymbol{\epsilon}_{jm}^{(2)}(z)\right]-\mathbf{R}_{jm}^{(2)}(z)\end{aligned} \tag{3.12}$$

where the top line contains all rows of (3.10) such that $i\in\mathbf{N}_1$ and the bottom line contains all rows of (3.10) such that $i\in\mathbf{N}_2$ (see page 174). The diagonal matrix $\mathbf{D}_j(z)$ in (3.10) is accordingly subdivided into two diagonal matrices $\mathbf{D}_j^{(1)}(z)$ and $\mathbf{D}_j^{(2)}(z)$; the matrix $\mathfrak{B}(z)$ is accordingly partitioned into four blocks $\mathfrak{a}_k(z)$, $(k=1, 2, 3, 4)$ and $\mathbf{R}_{jm}(z)$ is split into two vectors in the same manner as $\boldsymbol{\epsilon}_{jm}(z)$ was split.

Let $\mathscr{E}$ be the smallest closed disk with center at the origin containing the zeros of $q_{ij}'(t)$ $(i=1, 2, \ldots, n;\ i\neq j)$ and let $\mathscr{D}^*=\mathscr{D}-\mathscr{E}-\{\infty\}$. We next define a region $\mathscr{D}(\zeta_k^{(1)}, \zeta_k^{(2)})$ to be the union of all points z $(z\neq\zeta_k^{(1)}, z\neq\zeta_k^{(2)})$ such that there is a path $\mathscr{P}$ connecting $\zeta_k^{(1)}$, z and $\zeta_k^{(2)}$ (in that order) where $\zeta_k^{(1)}$ is one of the points $\infty \exp\left[(2k\pm1)\pi\sqrt{-1}/(r+1)\right]$, and $\zeta_k^{(2)}=\infty\exp\left[2\pi k\sqrt{-1}/(r+1)\right]$, and satisfying the following *interior eigenvalue conditions:*

(1) Except for $\zeta_k^{(1)}$ and $\zeta_k^{(2)}$, $\mathscr{P}$ lies entirely in $\mathscr{D}^*$;

(2) For any two distinct points t_1 and t_2 in the order $\zeta_k^{(1)}$, t_1, t_2, $\zeta_k^{(2)}$, on $\mathscr{P}$ we have $|\exp\left[q_{ij}(t_2)-q_{ij}(t_1)\right]|\leqq 1$ if $i\in\mathbf{N}_1$, $|\exp\left[q_{ij}(t_1)-q_{ij}(t_2)\right]|\leqq 1$ if $i\in\mathbf{N}_2$;

(3) With a_1, a_2, a_3, and a_4 defined by equation (3.32) below, $\frac{1}{2}\{a_1+a_4+\sqrt{(a_1-a_4)^2+4a_2a_3}\}\,\mathscr{V}_{\mathscr{P}}(t^{-1})<1$.

It follows that the vector of analytic functions which satisfies

$$\begin{bmatrix}\boldsymbol{\epsilon}_{jm,\,k}^{(1)}(z)\\ \boldsymbol{\epsilon}_{jm,\,k}^{(2)}(z)\end{bmatrix}=\left\{\begin{matrix}\int_{\zeta_k^{(1)}}^{z}e^{\mathbf{D}_j^{(1)}(z)-\mathbf{D}_j^{(1)}(t)} & \mathbf{0}\\ \mathbf{0} & \int_{\zeta_k^{(2)}}^{z}e^{\mathbf{D}_j^{(2)}(z)-\mathbf{D}_j^{(2)}(t)}\end{matrix}\right\}\times\left\{t^{-2}\begin{bmatrix}\mathfrak{a}_1(t) & \mathfrak{a}_2(t)\\ \mathfrak{a}_3(t) & \mathfrak{a}_4(t)\end{bmatrix}\begin{bmatrix}\boldsymbol{\epsilon}_{jm,\,k}^{(1)}(t)\\ \boldsymbol{\epsilon}_{jm,\,k}^{(2)}(t)\end{bmatrix}-\begin{bmatrix}\mathbf{R}_{jm}^{(1)}(t)\\ \mathbf{R}_{jm}^{(2)}(t)\end{bmatrix}\right\}dt, \tag{3.13}$$

where the integrals are taken along a path $\mathscr{P}$ as described above, is a solution of (3.12).

3.3. Treatment of $\mathbf{R}_{i\,jm}(z)$

In this section we examine the integral

$$\mathbf{R}_{i\,jm}^*(z)=\int_{\zeta}^{z}e^{q_{ij}(z)-q_{ij}(t)}\,\mathbf{R}_{i\,jm}(t)dt \tag{3.14}$$

where the path $\mathscr{P}$ of integration satisfies either of the sets of conditions described in the previous section and ζ is a point appropriately chosen at infinity such that as t traverses $\mathscr{P}$ from ζ to z, Re $q_{ij}(t)$ decreases monotonically. The equations (3.4) and (3.6) show that when $i=j$, $\mathbf{R}_{i\,jm}(t)=\mathrm{O}(|t|^{-m-1})$, $|t|\to\infty$, but if $i\neq j$, then $\mathbf{R}_{ijm}(t)=\mathrm{O}(|t|^{r-m-1})$, $|t|\to\infty$. For this reason we cannot proceed directly as in [1] to bound the error.

When $i\neq j$ we first integrate (3.14) by parts to obtain a good bound for $\mathbf{R}_{i\,jm}^*(z)$. The equation (3.6) may be written

$$\mathbf{R}_{ijm}(z) = -e^{q_{ij}(z)}\frac{d}{dz}(e^{-q_{ij}(z)}\mathbf{U}_{ijm}z^{-m}) - z\left[q'_{ij}(z) - (q_{i0} - q_{j0})z^r + \frac{m}{z}\right]\mathbf{U}_{ijm}z^{-m-1}$$

$$-\sum_{\mu=m+1}^{m+r+1}\left[\sum_{k=\mu+1-m}^{\min(\mu,\, r+1)}(q_{ik} - q_{jk})\mathbf{U}_{ij,\,\mu-k} + (\mu - r - 1)\mathbf{U}_{ij,\,\mu-r-1}\right.$$

$$\left. + \sum_{k=0}^{\mu-r-2}\{\mathbf{B}_k\mathbf{U}_{j,\,\mu-r-2-k}\}_i\right]z^{r-\mu} - \sum_{k=0}^{m-1}\{(\mathfrak{B}_{m-k}(z)\mathbf{U}_{jk})\}_i\, z^{-m-2}. \tag{3.15}$$

The first term on the right of (3.15) is already in a form suitable for integration by parts. Consider now the polynomial $\mathbf{P}_{ijm}(z)$ consisting of all of the right-hand side, excluding the first and last terms. It is clear that the dominant term of $\mathbf{P}_{ijm}(z)$ is of order z^{r-m-1} as $|z| \to \infty$. On substituting $\mathbf{P}_{ijm}(t)$ for $\mathbf{R}_{ijm}(t)$ in (3.14), integrating by parts and denoting the resulting integral by $\mathbf{P}^*_{ijm}$, we obtain

$$\mathbf{P}^*_{ijm}(z) = -\frac{\mathbf{P}_{ijm}(z)}{q'_{ij}(z)} + \int_\zeta^z e^{q_{ij}(z) - q_{ij}(t)}\frac{d}{dt}\left(\frac{\mathbf{P}_{ijm}(t)}{q'_{ij}(t)}\right)dt. \tag{3.16}$$

Collecting terms, we obtain

$$\mathbf{R}^*_{ijm}(z) = \int_\zeta^z\left[m\mathbf{U}_{ijm}t^{-m-1} + (e^{q_{ij}(z) - q_{ij}(t)} - 1)\frac{d}{dt}\left(\frac{\mathbf{P}_{ijm}(t)}{q'_{ij}(t)}\right) - e^{q_{ij}(z) - q_{ij}(t)}\sum_{k=0}^{m-1}\{\mathfrak{B}_{m-k}(t)\mathbf{U}_{jk}\}_i t^{-m-2}\right]dt, \tag{3.17}$$

and since

$$|e^{q_{ij}(z) - q_{ij}(t)}| \leqq 1 \text{ and } \frac{d}{dt}\left(\frac{\mathbf{P}_{ijm}(t)}{q'_{ij}(t)}\right) = O(|t|^{-m-2}),$$

$|t| \to \infty$, the right of (3.17) is readily bounded. In the next section, we use this bound in obtaining vector and norm bounds for $\boldsymbol{\epsilon}_{jm}(z)$.

On the other hand for $i = j$, we see from (3.4) that

$$\mathbf{R}^*_{jjm}(z) = \int_\zeta^z\left[m\,\mathbf{U}_{jjm}t^{-m-1} - \sum_{k=0}^{m-1}\{\mathfrak{B}_{m-k}(z)\mathbf{U}_{jk}\}_j t^{-m-2}\right]dt \tag{3.18}$$

and obtaining a bound for this integrand is even easier than for (3.17).

3.4. Error Bounds for the Extreme Eigenvalue Case

We shall first obtain a vector bound. With t_1 and t_2 in the order ζ_k, t_1, t_2, z on $\mathscr{P}$, we define

$$(m+1)\boldsymbol{\gamma}_{ijm} = \sup_{t_1,\, t_2 \in \mathscr{P}}\left| e^{q_{ij}(t_2) - q_{ij}(t_1)}\left[t_1^{m+2}\frac{d}{dt}\left(\frac{\mathbf{P}_{ijm}(t)}{q'_{ij}(t)}\right)_{t=t_1}\right.\right.$$

$$\left.\left. + \sum_{k=1}^{m-1}\{\mathfrak{B}_{m-k}(t_1)\mathbf{U}_{jk}\}_i - \frac{d}{dt}\left(\frac{\mathbf{P}_{ijm}(t)}{q'_{ij}(t)}\right)_{t=t_1} t_1^{m+1}\right]\right| \qquad (i \neq j) \tag{3.19}$$

$$(m+1)\gamma_{jjm} = \sup_{t \in \mathscr{P}}\left|\sum_{k=0}^{m-1}\{\mathfrak{B}_{m-k}(t)\mathbf{U}_{jk}\}_j\right|.$$

We also denote by $\boldsymbol{\gamma}_{jm}$ the vector with ith element γ_{ijm}; i.e. $\{\boldsymbol{\gamma}_{jm}\}_i=\gamma_{ijm}$, and define

$$\mathbf{B}=\left[\sup_{t_1,\, t_2 \in \mathscr{P}}\left| e^{q_{ij}(t_2)-q_{ij}(t_1)}\{\mathfrak{B}(t_1)\}_{is}\right|\right],\ i,\ s=1,\ 2,\ \ldots,\ n. \tag{3.20}$$

In the part II of the present paper we shall show that there exists a unique vector $\boldsymbol{\epsilon}_{jm,k}(z)$ of holomorphic functions which tends to zero as $z\rightarrow\zeta_k$ and satisfies (3.11). The elements of $\boldsymbol{\epsilon}_{jm,k}(z)$ are holomorphic in a domain (actually a Riemann surface) which includes the region $\mathscr{D}(z,\ \zeta_k)$ defined by the extreme eigenvalue conditions in Section 3.2.

On combining equations (3.19) and (3.20) with (3.11) we obtain

$$|\boldsymbol{\epsilon}_{jm,k}(z)| \leqq \int_{\zeta_k}^{z} [|t^{-2}\mathbf{B}\boldsymbol{\epsilon}_{jm,k}(t)|+m|\mathbf{U}_{jm}t^{-m-1}|+(m+1)\boldsymbol{\gamma}_{jm}t^{-m-2}]|dt|. \tag{3.21}$$

Let σ be the value of s for which $t(s)=z$ $t(s)$ as defined in the Introduction. We apply Lemmas 1 and 2 of the appendix with $E(\sigma)=1$, $\mathbf{F}(s)ds=\mathbf{B}|t^{-2}dt|$, $\varphi(s)=|\boldsymbol{\epsilon}_{jm,k}(t)|$, $\mathbf{G}(s)ds=(m|\mathbf{U}_{jm}t^{-m-1}| +(m+1)\boldsymbol{\gamma}_{jm}|t^{-m-2}|)dt$ to obtain

THEOREM 3.2: *Corresponding to an extreme eigenvalue of* λ_j *of* $\mathbf{C}_0$, *the equation* (2.2) *possesses an actual solution vector* $\mathbf{W}_{jm,k}(z)$ *depending on* ζ_k *and an arbitrary positive integer* m *such that*

$$\mathbf{W}_{jm,k}(z)=\left[\sum_{s=0}^{m-1}\mathbf{U}_{js}z^{-s}+\boldsymbol{\epsilon}_{jm,k}(z)\right]e^{q_j(z)} \tag{3.22}$$

where

$$|\boldsymbol{\epsilon}_{jm,k}(z)| \leqq \exp\{\mathbf{B}\mathscr{V}_{\mathscr{P}}(t^{-1})\}\{|\mathbf{U}_{jm}|\mathscr{V}_{\mathscr{P}}(t^{-m})+\boldsymbol{\gamma}_{jm}\mathscr{V}_{\mathscr{P}}(t^{-m-1})\},\quad z\in\mathscr{D}(z,\ \zeta_k) \tag{3.23}$$

and $\mathscr{D}(z,\ \zeta_k)$ *is defined in section 3.2. In* (3.22) *each vector* $\mathbf{U}_{jk}$ $(k=0,\ 1,\ 2,\ \ldots)$ *is the* j*th column vector of the matrix* $\mathbf{U}_k$ *defined by Theorem 2.2;* $q_j(z)$ *is the* j*th diagonal element of the diagonal matrix* $\mathfrak{Q}(z)$ *(equation (2.13)). In* (3.23) $\mathbf{B}$ *is an* $n\times n$ *matrix of non-negative elements defined by equations* (2.3) *and* (3.20), *while the* i*th element* $(i=1,\ 2,\ \ldots,\ n)$ *of the vector* $\boldsymbol{\gamma}_{jm}$ *is defined by equation* (3.19).

Let us now obtain a norm bound for $\boldsymbol{\epsilon}_{jm,k}(z)$ in equation (3.22). For this purpose *we define a vector* $\mathbf{V}_j(t_2,\ t_1)$ *as follows: Let the* i*th element* $(m+1)\mathbf{V}_{ij}(t_2,\ t_1)$ *of* $(m+1)\mathbf{V}_j(t_2,\ t_1)$ *be the number inside the absolute value signs on the right of* (3.19). With t_1 and t_2 defined as for equation (3.19) we define [9]

$$B=\sup_{t\in\mathscr{P}}\|\mathfrak{B}(t)\|$$

$$\gamma_{jm}=\sup_{t_1,\, t_2\in\mathscr{P}}\|\mathbf{V}_j(t_2,\ t_1)\|. \tag{3.24}$$

THEOREM 3.3: *A norm bound for the vector* $\boldsymbol{\epsilon}_{jm,k}(z)$ *in Theorem 3.2 is given by*

$$\|\boldsymbol{\epsilon}_{jm,k}(z)\| \leqq \exp\{B\mathscr{V}_{\mathscr{P}}(t^{-1})\}\{\|\mathbf{U}_{jm}\|\mathscr{V}_{\mathscr{P}}(t^{-m})+\gamma_{jm}\mathscr{V}_{\mathscr{P}}(t^{-m-1})\}. \tag{3.25}$$

[9] Here and henceforth we assume that compatible matrix and vector norms are taken, i.e., $\|\mathbf{BV}\|\leqq\|\mathbf{B}\|\,\|\mathbf{V}\|$.

Put $t_2=z$, $t_1=t$, and let $\mathbf{V}_j(z,t)$ be defined as for (3.24). We split the vector $\mathbf{V}_j(z,t)$ into two vectors $\mathbf{V}_j^{(1)}(z,t)$ and $\mathbf{V}_j^{(2)}(z,t)$ in a manner corresponding to the way in which $\mathbf{R}_{jm}(t)$ was split in (3.13). We analogously split the vector $\mathbf{U}_{jm}$ into $\mathbf{U}_{jm}^{(1)}$ and $\mathbf{U}_{jm}^{(2)}$, such that the equation (3.13) may be written

$$\begin{bmatrix}\boldsymbol{\epsilon}_{jm,k}^{(1)}(z)\\ \boldsymbol{\epsilon}_{jm,k}^{(2)}(z)\end{bmatrix}=\begin{bmatrix}\int_{\zeta_k^{(1)}}^{z} & \mathbf{0}\\ \mathbf{0} & \int_{\zeta_k^{(2)}}^{z}\end{bmatrix}\left\{\begin{bmatrix}e^{\mathbf{D}_j^{(1)}(z)-\mathbf{D}_j^{(1)}(t)} & \mathbf{0}\\ \mathbf{0} & e^{\mathbf{D}_j^{(2)}(z)-\mathbf{D}_j^{(2)}(t)}\end{bmatrix}t^{-2}\right.$$

$$\left.\times\begin{bmatrix}\alpha_1(t) & \alpha_2(t)\\ \alpha_3(t) & \alpha_4(t)\end{bmatrix}\begin{bmatrix}\boldsymbol{\epsilon}_{jm,k}^{(1)}(t)\\ \boldsymbol{\epsilon}_{jm,k}^{(2)}(t)\end{bmatrix}+\begin{bmatrix}-m\mathbf{U}_{jm}^{(1)}t^{-m-1}+(m+1)\mathbf{V}_j^{(1)}(z,t)t^{-m-2}\\ -m\mathbf{U}_{jm}^{(2)}t^{-m-1}+(m+1)\mathbf{V}_j^{(2)}(z,t)t^{-m-2}\end{bmatrix}\right\}dt. \tag{3.26}$$

Again we shall first obtain a vector bound. To achieve this, we define

$$\mathbf{a}_k=\sup_{t\in\mathscr{P}}[\,|\{\alpha_k(t)\}_{is}|\,]\qquad (k=1,2,3,4)$$

$$\mathbf{C}_m^{(s)}(t)=m|\mathbf{U}_{jm}^{(s)}|\,|t^{-m-1}|+\sup_{t_1,\,t_2\in\mathscr{P}}|(m+1)\mathbf{V}_{jm}^{(s)}(t_2,t_1)|\,|t^{-m-2}| \tag{3.27}$$

where in the last of (3.27) t_1 and t_2 are on $\mathscr{P}$ in the order $\zeta_1, t_1, t_2, \zeta_2$ for $s=1$, and $\zeta_1, t_2, t_1, \zeta_2$ for $s=2$ respectively.

In addition, each element of the diagonal matrices $e^{\mathbf{D}_j^{(s)}(z)-\mathbf{D}_j^{(s)}(t)}$ $(s=1,2)$ on the right of (3.23) has an upper bound of 1.

In part II of the present paper we shall show that there exists a unique $n\times 1$ vector of functions satisfying (3.26) such that $\boldsymbol{\epsilon}_{jm,k}^{(1)}(z)\to\mathbf{0}$ as $z\to\zeta_k^{(1)}$, $\boldsymbol{\epsilon}_{jm,k}^{(2)}(z)\to\mathbf{0}$ as $z\to\zeta_k^{(2)}$. Furthermore, each element of this vector is holomorphic on a Riemann surface which includes the region $\mathscr{D}(\zeta_k^{(1)},\zeta_k^{(2)})$ defined by the interior eigenvalue conditions of section 3.2.

On substituting the above results in (3.26), we obtain

$$\begin{bmatrix}|\boldsymbol{\epsilon}_{jm,k}^{(1)}(z)|\\ |\boldsymbol{\epsilon}_{jm,k}^{(2)}(z)|\end{bmatrix}\leqq\begin{bmatrix}\int_{\zeta_k^{(1)}}^{z} & \mathbf{0}\\ \mathbf{0} & \int_{\zeta_k^{(2)}}^{z}\end{bmatrix}\left\{\begin{bmatrix}\mathbf{a}_1 & \mathbf{a}_2\\ \mathbf{a}_3 & \mathbf{a}_4\end{bmatrix}\begin{bmatrix}|\boldsymbol{\epsilon}_{jm,k}^{(1)}(t)|\\ |\boldsymbol{\epsilon}_{jm,k}^{(2)}(t)|\end{bmatrix}|t^{-2}|+\begin{bmatrix}\mathbf{C}_m^{(1)}(t)\\ \mathbf{C}_m^{(2)}(t)\end{bmatrix}\right\}|dt|. \tag{3.28}$$

On transforming to real variables as for (3.21), using Lemma 5 of the appendix and transforming back to complex variable notation, we obtain

THEOREM 3.3: *If, for an interior eigenvalue* λ_j *of* $\mathbf{C}_0$, *the interior eigenvalue conditions of section 3.2 are satisfied, then the equation* (2.2) *possesses an actual solution vector*

$$\begin{Bmatrix}\mathbf{W}_{jm,k}^{(1)}(t)\\ \mathbf{W}_{jm,k}^{(2)}(z)\end{Bmatrix}=\left\{\sum_{s=0}^{m-1}\begin{bmatrix}\mathbf{U}_{js}^{(1)}\\ \mathbf{U}_{js}^{(2)}\end{bmatrix}z^{-s}+\begin{bmatrix}\boldsymbol{\epsilon}_{jm,k}^{(1)}(z)\\ \boldsymbol{\epsilon}_{jm,k}^{(2)}(z)\end{bmatrix}\right\}e^{q_j(z)} \tag{3.29}$$

where

$$|\boldsymbol{\epsilon}_{jm,k}^{(1)}(z)|\leqq\exp\{\mathbf{a}_1\mathscr{V}_{\zeta_k^{(1)},z}(t^{-1})\}\psi_m^{(1)}(\zeta_k^{(1)},z)+\mathbf{C}_1(\zeta_k^{(1)},z)[\exp\{\mathbf{a}_4\mathscr{V}_{\mathscr{P}}(t^{-1})\}\psi_m^{(2)}(\zeta_k^{(1)},\zeta_k^{(2)})+\mathbf{B}],\ (z\in\mathscr{D}(\zeta_k^{(1)},\zeta_k^{(2)}))$$

and

$$\psi_m^{(s)}(u, v) = \int_u^v \mathbf{C}_m^{(s)}(t)|dt| \qquad (s=1, 2) \tag{3.30}$$

$$\mathbf{B} = [\mathbf{I}_2 - \mathbf{C}_2\mathbf{C}_1(\zeta_k^{(1)}, \zeta_k^{(2)})]^{-1}\mathbf{C}_2[\exp\{\mathbf{a}_1\mathscr{V}_{\mathscr{P}}(t^{-1})\}\psi_m^{(1)}(\zeta_k^{(1)}, \zeta_k^{(2)}) + \mathbf{C}_1(\zeta_k^{(1)}, \zeta_k^{(2)})\exp\{\mathbf{a}_4\mathscr{V}_{\mathscr{P}}(t^{-1})\}\psi_m^{(2)}(\zeta_k^{(1)}, \zeta_k^{(2)})]$$

$$\mathbf{C}_1(\zeta_k^{(1)}, z) = \mathbf{a}_1^{-1}[\exp\{\mathbf{a}_1\mathscr{V}_{\mathscr{P}}(t^{-1})\} - \mathbf{I}_1]\mathbf{a}_2;\ \mathbf{C}_2 = \mathbf{a}_4^{-1}[\exp\{\mathbf{a}_4\mathscr{V}_{\mathscr{P}}(t^{-1})\} - \mathbf{I}_2].$$

Here $\mathbf{I}_1$ *designates the* $\kappa\times\kappa$ *unit matrix, and* $\mathbf{I}_2$ *designates the* $(n-\kappa)\times(n-\kappa)$ *unit matrix,* κ *being the number of elements in* $\mathbf{N}_1$ (Section 3.2). *All integrals and variations on the right of* (3.30) *are taken along* $\mathscr{P}$. *The bound on the right of* (3.30) *is valid provided that every eigenvalue of the matrix* $\mathbf{C}_2\mathbf{C}_1(\zeta_k^{(1)}, \zeta_k^{(2)})$ *is less than* 1 *in magnitude. Moreover, an exactly similar result holds for* $\boldsymbol{\epsilon}_{jm,k}^{(2)}(z)$.

Let us now obtain a norm bound. We define

$$\psi_m^{(s)}(u, v) = \|\mathbf{U}_{jm}^{(s)}\|\mathscr{V}_{u,v}(t^{-m}) + \sup_{t_1,\, t_2\in\mathscr{P}}\|\mathbf{V}_j^{(s)}(t_2, t_1)\|\mathscr{V}_{u,v}(t^{-m-1}) \tag{3.31}$$

where t_1 and t_2 are any two points on $\mathscr{P}$ in the order $\zeta_k^{(1)}, t_1, t_2, \zeta_k^{(2)}$ for $s=1$ and in the order $\zeta_k^{(1)}$, $t_2, t_1, \zeta_k^{(2)}$ for $s=2$. Similarly, with the notation of (3.27) we put

$$a_k = \sup_{t\in\mathscr{P}}\|\alpha_k(t)\| \qquad (k=1, 2, 3, 4). \tag{3.32}$$

Thus we obtain an inequality similar to (3.29); using Lemma 6 and the inequality (32) of the appendix we then obtain

THEOREM 3.4: *Let* a_k $(k=1, 2, 3, 4)$ *be defined by* (3.32), *let* $\psi_m^{(s)}$ $(k=1, 2)$ *be defined by* (3.31), *and let*

$$\begin{aligned} f(z) &= \mathscr{V}_{\zeta_k^{(1)}, z}(t^{-1}) \\ F(z) &= \frac{a_2a_3}{(a_1+a_4)^2}\left[e^{(a_1+a_4)f(z)} - 1 - (a_1+a_4)f(z)\right] \end{aligned} \tag{3.33}$$

where the variations are taken along $\mathscr{P}$. *If* $F(\zeta_k^{(2)}) < 1$ *then the norm of the vector* $\boldsymbol{\epsilon}_{jm,k}^{(1)}(z)$ (equation (3.29)) *is bounded by the quantity*

$$(1-F(\zeta_k^{(1)}))^{-1}\{e^{a_1f(z)}[F(z)+1-F(\zeta_k^{(2)})]\psi_m(\zeta_k^{(1)}, z)$$

$$+\frac{a_2a_3}{(a_1+a_4)^2}[1-e^{-(a_1+a_4)f(z)}][e^{(a_1+a_4)f(\zeta^{(2)})} - e^{(a_1+a_4)f(z)}]\psi_m^{(1)}(z, \zeta_k^{(2)})$$

$$+\frac{a_2}{a_1+a_4}e^{a_1f(z)+a_4f(\zeta_k^{(2)})}[1-e^{-(a_1+a_4)f(z)}]\psi_m^{(2)}(\zeta_k^{(1)}, \zeta_k^{(2)}). \tag{3.34}$$

Moreover, an exactly similar result holds for $\boldsymbol{\epsilon}_{jm,k}^{(2)}(z)$.

It is noteworthy that the conditions of Theorems 3.3 and 3.4 can always be satisfied by taking contours $\mathscr{P}$ sufficiently far away from the origin.

4. Appendix. Some Inequalities for Volterra and Fredholm Integral Equations

4.1. Bellman's Lemma in Matrix Form

We begin with an extension of Bellman's Lemma [5] to matrices.

LEMMA 1: *Let* φ, $\mathbf{G}$ *and* $\mathbf{H}$ *be* $k \times 1$ *vectors and* $\mathbf{E}$ *and* $\mathbf{F}$ *be* $k \times k$ *matrices of continuous non-negative functions of a variable* σ *in an interval* $a \leqq \sigma \leqq b$ *such that* $\mathbf{F}(\sigma)\,\mathbf{E}(\sigma)$ *and* $\int^{\sigma} \mathbf{F}(s)\mathbf{E}(s)ds$ *commute. If*[10]

$$\varphi(\sigma) \leqq \mathbf{E}(\sigma) \int_a^{\sigma} [\mathbf{F}(s)\varphi(s) + \mathbf{G}(s)]ds + \mathbf{H}(\sigma) \qquad (a \leqq \sigma \leqq b) \tag{1}$$

then the following inequality also holds in $a \leqq \sigma \leqq b$:

$$\varphi(\sigma) \leqq \mathbf{H}(\sigma) + \mathbf{E}(\sigma) \int_a^{\sigma} \exp\left\{\int_s^{\sigma} \mathbf{F}(\tau)\mathbf{E}(\tau)d\tau\right\} \{\mathbf{G}(s) + \mathbf{F}(s)\mathbf{H}(s)\}ds. \tag{2}$$

The proof of Lemma 1 is a straightforward extension of the one-dimensional result in [5, page 135].

LEMMA 2: *With the conditions of Lemma 1, let* $\mathbf{E}^{-1}(\sigma)$ and $\mathscr{V}_{a,b}(\mathbf{E}^{-1}\mathbf{H})$ *exist for* σ *in* (a, b). *Then*

$$\varphi(\sigma) \leqq \mathbf{E}(\sigma) \exp\left\{\int_a^{\sigma} \mathbf{F}(\tau)\mathbf{E}(\tau)d\tau\right\} \left\{\mathbf{E}^{-1}(a)\mathbf{H}(a) + \mathscr{V}_{a,\sigma}(\mathbf{E}^{-1}\mathbf{H}) + \int_a^{\sigma} \mathbf{G}(s)ds\right\}. \tag{3}$$

To prove this result we integrate the right of (2) by parts.

If $\mathbf{E}^{-1}(s)\mathbf{H}(s)$ increases monotonically over (a,b), then $\mathscr{V}_{a,\sigma}(\mathbf{E}^{-1}\mathbf{H}) = \mathbf{E}^{-1}(\sigma)\mathbf{H}(\sigma) - \mathbf{E}^{-1}(a)\mathbf{H}(a)$; in this case the inequality (3) may be simplified. If, in addition, $\mathbf{E}(s)$ is a scalar quantity then (3) reduces to

$$\varphi(\sigma) \leqq \exp\left\{\int_a^{\sigma} \mathbf{F}(\tau)\mathbf{E}(\tau)d\tau\right\} \left\{\mathbf{H}(\sigma) + \mathbf{E}(\sigma)\int_a^{\sigma} \mathbf{G}(s)ds\right\}. \tag{4}$$

4.2. Bellman's Lemma for Fredholm Vector Integral Equations

The following lemma extends the above results to Fredholm integral equations.

LEMMA 3: *When the variables* σ, τ, *and* t *lie in the interval* (a, b), *let* $\mathbf{X}(\sigma, \tau)$ $(k \times k)$, $\varphi(\tau)$ $(k \times 1)$, *and* $\theta(\tau)$ $(k \times 1)$ *be matrices of non-negative continuous functions such that*

$$\mathbf{X}(\sigma, \tau)\,\mathbf{X}(\tau, t) \leqq \mathbf{X}(\sigma, t)\,\mathbf{X}(\tau, \tau) \tag{5}$$

and

$$\varphi(\sigma) \leqq \int_a^b \mathbf{X}(\sigma, \tau)\varphi(\tau)d\tau + \theta(\sigma). \tag{6}$$

Suppose that each eigenvalue of the matrix

$$\mathbf{F} = \int_a^b \mathbf{X}(\tau, \tau)d\tau \tag{7}$$

is less than 1 *in magnitude. Then*

$$\varphi(\sigma) \leqq \theta(\sigma) + \int_a^b \mathbf{X}(\sigma, \tau)(\mathbf{I} - \mathbf{F})^{-1}\theta(\tau)d\tau \quad (a \leqq \sigma \leqq b). \tag{8}$$

---

[10] If (a, b) is not compact we assume that the integral on the right of (1) exists and that each element of the resulting vector is uniformly bounded.

PROOF: From the conditions of the Lemma there exists a unique $k\times 1$ vector $\Psi(\sigma)$ of non-negative continuous functions such that we can replace the inequality (6) by the equality

$$\varphi(\sigma)=\int_a^b \mathbf{X}(\sigma, \tau)\varphi(\tau)d\tau+\theta(\sigma)-\Psi(\sigma). \tag{9}$$

The equation (9) is a Fredholm integral equation in φ which we can solve by successive approximations. Defining a sequence $\{\varphi_\nu\}(\nu=0, 1, \ldots)$ by $\varphi_0=\mathbf{O}$,

$$\varphi_{\nu+1}(\sigma)=\int_\alpha^b \mathbf{X}(\sigma, \tau)\varphi_\nu(\tau)d\tau+\theta(\sigma)-\Psi(\sigma), \qquad \nu=0, 1, \ldots, \tag{10}$$

and using (5), we easily establish by induction that

$$\mathbf{O}\leqq \varphi_{\nu+1}(\sigma)-\varphi_\nu(\sigma)\leqq \int_a^b \mathbf{X}(\sigma, \tau)\mathbf{F}^{\nu-1}(\theta(\tau)-\Psi(\tau))d\tau, \qquad \nu=1, 2, \ldots. \tag{11}$$

If the eigenvalues of $\mathbf{F}$ are all less than 1 in magnitude, we have

$$(\mathbf{I}-\mathbf{F})^{-1}=\sum_{\nu=0}^{\infty} \mathbf{F}^\nu \tag{12}$$

and clearly the sum of the power series on the right is a matrix of non-negative elements. On summing the inequalities (11), we obtain (8) with θ replaced by $\theta-\Psi$, since it is well known that this sum bounds the true solution of (9). Decreasing the magnitude of the elements of Ψ on the right of this sum increases the elements of φ. The inequality (8) now follows.

The proof of the following Lemma is similar to that of the above Lemma, and is omitted.

LEMMA 4: *With the conditions of Lemma 3 let* (5) *be replaced by*

$$\mathbf{X}(\sigma, \tau)\leqq \mathbf{G}(\sigma)\mathbf{H}(\tau) \tag{5'}$$

where $\mathbf{G}$ *and* $\mathbf{H}$ *are* $\mathrm{k}\times\mathrm{k}$ *matrices. Let each eigenvalue of*

$$\mathbf{F}=\int_\mathrm{a}^\mathrm{b} \mathbf{H}(\tau)\mathbf{G}(\tau)\mathrm{d}\tau \tag{7'}$$

be less than 1 *in magnitude. Then*

$$\varphi(\sigma)\leqq \theta(\sigma)+\int_\mathrm{a}^\mathrm{b} \mathbf{G}(\sigma)(\mathbf{I}-\mathbf{F})^{-1}\mathbf{H}(\tau)\theta(\tau)\mathrm{d}\tau \qquad (\mathrm{a}\leqq\sigma\leqq\mathrm{b}). \tag{8'}$$

4.3. Extensions to a Simultaneous Pair of Volterra Vector Integral Equations

Here we consider the pair of inequalities

$$\begin{aligned}\varphi_1(\sigma)&\leqq \int_a^\sigma \{[\boldsymbol{\alpha}\varphi_1(s)+\boldsymbol{\beta}\varphi_2(s)]\omega(s)+\theta_1(s)\}ds\\ \varphi_2(\sigma)&\leqq \int_\sigma^b \{[\boldsymbol{\gamma}\varphi_1(s)+\boldsymbol{\delta}\varphi_2(s)]\omega(s)+\theta_2(s)\}ds\end{aligned} \tag{13}$$

where φ_1 $(\kappa\times 1)$, φ_2 $((n-\kappa)\times 1)$, θ_1 $(\kappa\times 1)$, θ_2 $((n-\kappa)\times 1)$, ω (1×1) are matrices of nonnegative continuous functions on (a, b), and $n\geqq 2$, $0<\kappa<n$. Also, $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}$, and $\boldsymbol{\delta}$ are constant matrices of orders $\kappa\times\kappa$, $\kappa\times(n-\kappa)$, $(n-\kappa)\times\kappa$ and $(n-\kappa)\times(n-\kappa)$ respectively, whose elements are nonnegative.

On applying Lemma 1 to each of the equations (13), we obtain

$$\varphi_1(\sigma) \leqq \int_a^\sigma {}^{\alpha f(\sigma)-\alpha f(s)}\{\boldsymbol{\beta}\varphi_2(s)\omega(s)+\theta_1(s)\}\,ds$$

$$\varphi_2(\sigma) \leqq \int_\sigma^b {}^{\delta f(s)-\delta f(\sigma)}\{\boldsymbol{\gamma}\varphi_1(s)\omega(s)+\theta_2(s)\}\,ds \tag{14}$$

where

$$f(s)=\int_a^s \omega(\tau)d\tau. \tag{15}$$

Substituting the second of (14) for φ_2 in the first of (14), we get

$$\varphi_1(\sigma) \leqq \int_a^\sigma \int_s^b e^{\alpha f(\sigma)-\alpha f(s)}\;[\boldsymbol{\beta} e^{\delta f(\tau)-\delta f(s)}\;\{\boldsymbol{\gamma}\varphi_1(\tau)\omega(\tau)+\theta_2(\tau)\}d\tau\omega(s)+\theta_1(s)]ds. \tag{16}$$

The repeated integral (16) may be interpreted as a double integral over the shaded region indicated in figure 3. Splitting this region into a triangle and a rectangle and interchanging the order of integration, we obtain

$$\varphi_1(\sigma) \leqq \theta_3(\sigma)+\int_a^\sigma \int_a^\tau e^{\alpha f(\sigma)-\alpha f(s)}\;\boldsymbol{\beta} e^{\delta f(\tau)-\delta f(s)}\;\}\boldsymbol{\gamma}\varphi_1(\tau)\omega(\tau)+\theta_2(t)\}\omega(s)dsd\tau$$

$$+\int_\sigma^b \int_a^\sigma e^{\alpha f(\sigma)-\alpha f(s)}\;\boldsymbol{\beta} e^{\delta f(\tau)-\delta f(s)}\;\{\boldsymbol{\gamma}\varphi_1(\tau)\omega(\tau)+\theta_2(\tau)\}\omega(s)dsd\tau, \tag{17}$$

where

$$\theta_3(\sigma)=\int_a^\sigma e^{\alpha f(\sigma)-\alpha f(s)}\;\theta_1(s)ds. \tag{18}$$

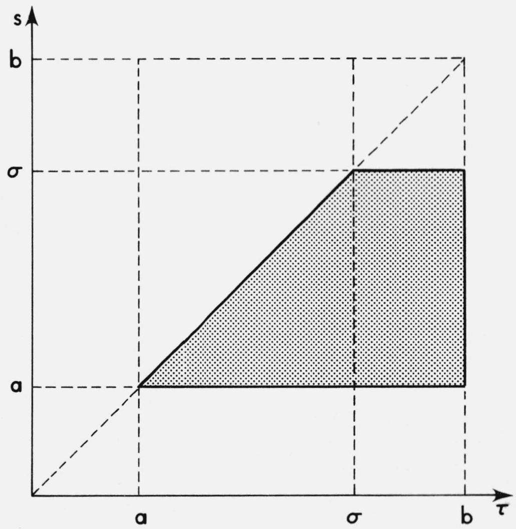

FIGURE 3.

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We evaluate the inner integrals in (17) to find that

$$\varphi_1(\sigma) \leqq \theta_3(\sigma) + \int_a^b \mathbf{X}(\sigma, \tau)\{\boldsymbol{\gamma}\varphi_1(\tau)\omega(\tau) + \theta_2(\tau)\} d\tau \tag{19}$$

where $\mathbf{X}(\sigma, \tau)$ satisfies

$$\boldsymbol{\alpha}\mathbf{X}(\sigma, \tau) + \mathbf{X}(\sigma, \tau)\boldsymbol{\delta} = \begin{cases} e^{\boldsymbol{\alpha} f(\sigma)}[\boldsymbol{\beta} e^{\boldsymbol{\delta} f(\tau)} - e^{-\boldsymbol{\alpha} f(\tau)}\boldsymbol{\beta}], \ \tau \leqq \sigma \\ [e^{\boldsymbol{\alpha} f(\sigma)}\boldsymbol{\beta} - \boldsymbol{\beta} e^{-\boldsymbol{\delta} f(\sigma)}] e^{\boldsymbol{\delta} f(\tau)}, \ \tau \geqq \sigma. \end{cases} \tag{20}$$

The equations (20) have unique solutions for $\mathbf{X}(\sigma, \tau)$ provided that $\mu_i + \nu_j \neq 0$ $(i = 1, 2, \ldots, \kappa; j = 1, 2, \ldots, n - \kappa)$, where μ_i and ν_j are the eigenvalues of $\boldsymbol{\alpha}$ and $\boldsymbol{\delta}$, respectively. Moreover, this condition can always be satisfied by an arbitrarily small change in the elements of $\boldsymbol{\alpha}$ and $\boldsymbol{\delta}$, if necessary. Nevertheless an exact explicit expression for the right hand side of (20) is generally not feasible, and in practice we make a slight sacrifice of sharpness of error bound in order to achieve a matrix bound which is simpler to evaluate. Several simplifications are possible; one of these is as follows:[11]

$$\int_a^\tau e^{\boldsymbol{\alpha} f(\sigma) - \boldsymbol{\alpha} f(s)} \boldsymbol{\beta} e^{\boldsymbol{\delta} f(\tau) - \boldsymbol{\delta} f(s)} \omega(s) ds \leqq \int_a^\tau e^{\boldsymbol{\alpha} f(\sigma) - \boldsymbol{\alpha} f(s)} \boldsymbol{\beta} e^{\boldsymbol{\delta} f(\tau)} \omega(s) ds$$
$$= \boldsymbol{\alpha}^{-1}[e^{\boldsymbol{\alpha} f(\sigma)} - e^{\boldsymbol{\alpha} f(\sigma) - \boldsymbol{\alpha} f(\tau)}]\boldsymbol{\beta} e^{\boldsymbol{\delta} f(\tau)} \leqq \boldsymbol{\alpha}^{-1}[e^{\boldsymbol{\alpha} f(\sigma)} - \mathbf{I}_\kappa]\boldsymbol{\beta} e^{\boldsymbol{\delta} f(\tau)}. \tag{21}$$

Similarly

$$\int_a^\sigma e^{\boldsymbol{\alpha} f(\sigma) - \boldsymbol{\alpha} f(s)} \boldsymbol{\beta} e^{\boldsymbol{\delta} f(\tau) - \boldsymbol{\delta} f(s)} \omega(s) ds \leqq \alpha^{-1}[e^{\boldsymbol{\alpha} f(\sigma)} - \mathbf{I}_\kappa]\boldsymbol{\beta} e^{\boldsymbol{\delta} f(\tau)} \equiv \mathbf{G}(\sigma)\mathbf{H}(\tau), \tag{22}$$

where

$$\mathbf{G}(\sigma) = \boldsymbol{\alpha}^{-1}[e^{\boldsymbol{\alpha} f(\sigma)} - \mathbf{I}_\kappa], \qquad \mathbf{H}(\tau) = \boldsymbol{\beta} e^{\boldsymbol{\delta} f(\tau)}. \tag{23}$$

Combining (22) and (23), we obtain

$$\varphi_1(\sigma) \leqq \theta_3(\sigma) + \int_a^b \mathbf{G}(\sigma)\mathbf{H}(\tau)\{\boldsymbol{\gamma}\varphi_1(\tau)\omega(\tau) + \theta_2(\tau)\} d\tau. \tag{24}$$

We apply Lemma 4 to this inequality to obtain:

LEMMA 5: *Let φ_1, φ_2, α, β, γ, δ, ω, θ, and θ_2 be defined as in* (13) *and the functions* $\mathbf{G}$ *and* $\mathbf{H}$ *as in* (23). *If the inequalities* (13) *hold and if each eigenvalue of the matrix*

$$\mathbf{F} = \int_a^b \mathbf{H}(\tau)\boldsymbol{\gamma}\mathbf{G}(\tau)\omega(\tau) d\tau \tag{25}$$

is less than 1 *in magnitude, then*

$$\varphi_1(\sigma) \leqq \theta(\sigma) + \int_a^b \mathbf{G}(\sigma)(\mathbf{I} - \mathbf{F})^{-1}\mathbf{H}(s)\boldsymbol{\gamma}\theta(s)\omega(s) ds, \tag{26}$$

[11] Since $\boldsymbol{\alpha}^{-1}[\exp(\boldsymbol{\alpha}) - \mathbf{I}] \equiv \sum_{k=0}^{\infty} \frac{\boldsymbol{\alpha}^k}{(k+1)!}$ the possibility of $\boldsymbol{\alpha}$ being singular is not excluded.

where

$$\theta(\sigma)=\theta_3(\sigma)+\int_a^b \mathbf{G}(\sigma)\mathbf{H}(\tau)\theta_2(\tau)d\tau \tag{27}$$

and θ_3 *is defined by* (18). *Moreover an exactly similar result holds for* $\varphi_2(\sigma)$.

In the case when $n=2$ and $\kappa=1$ the equations (20) can be solved explicitly. We then have (19) with

$$\mathbf{X}(\sigma,\tau)=\begin{cases}\dfrac{\boldsymbol{\beta}}{\boldsymbol{\alpha}+\boldsymbol{\delta}}\, e^{\boldsymbol{\alpha} f(\sigma)}[e^{\boldsymbol{\delta} f(\tau)}-e^{-\boldsymbol{\alpha} f(\tau)}], & \tau\leqq\sigma;\\[2mm] \dfrac{\boldsymbol{\beta}}{\boldsymbol{\alpha}+\boldsymbol{\delta}}\, e^{\boldsymbol{\delta} f(\tau)}[e^{\boldsymbol{\alpha} f(\sigma)}-e^{-\boldsymbol{\delta} f(\sigma)}], & \tau\geqq\sigma.\end{cases} \tag{28}$$

It is easily verified that for arbitrary points σ, τ and t on (a,b), $\mathbf{X}(\sigma,\tau)\mathbf{X}(\tau,t)\leqq\mathbf{X}(\sigma,t)\mathbf{X}(\tau,\tau)$; i.e., the conditions of Lemma 3 are satisfied. On applying Lemma 3 to the inequality (19) we obtain

LEMMA 6: *Let* φ_1, φ_2, $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$, $\boldsymbol{\delta}$, ω, θ_1 *and* θ_2 *be defined as in* (13), *with* $n=2$, $\kappa=1$, *and*

$$\mathbf{F}(\sigma)=\frac{\boldsymbol{\beta\gamma}}{(\boldsymbol{\alpha}+\boldsymbol{\delta})^2}\,[e^{(\boldsymbol{\alpha}+\boldsymbol{\delta})f(\sigma)}-1-(\boldsymbol{\alpha}+\boldsymbol{\delta})f(\sigma)], \tag{29}$$

where f(s) *is defined by* (15). *If the inequalities* (13) *hold and if* F(b) < 1, *then*

$$\varphi_1(\sigma)\leqq\theta_3(\sigma)+\frac{1}{1-\mathbf{F}(b)}\int_a^b \mathbf{X}(\sigma,\tau)[\boldsymbol{\gamma}\omega(\tau)\theta_3(\tau)+\theta_2(\tau)]d\tau \qquad (a\leqq\sigma\leqq b) \tag{30}$$

where $\mathbf{X}(\sigma,\tau)$ *is defined by* (28) *and* θ_3 *is defined by* (18).

On expanding the right hand side of (30) and using the inequality $\mathbf{X}(\sigma,\tau)\mathbf{X}(\tau,t)\leqq\mathbf{X}(\sigma,t)\mathbf{X}(\tau,\tau)$ once more, we get

$$\varphi_1(\sigma)\leqq\frac{\boldsymbol{\beta} e^{\boldsymbol{\alpha} f(\sigma)}}{1-\mathbf{F}(b)(\boldsymbol{\alpha}+\boldsymbol{\delta})}\Bigg[\int_a^\sigma\left\{\frac{\boldsymbol{\alpha}+\boldsymbol{\delta}}{\boldsymbol{\beta}}[\mathbf{F}(\sigma)-\mathbf{F}(s)+1-\mathbf{F}(b)]e^{-\boldsymbol{\alpha} f(s)}\theta_1(s)+[e^{\boldsymbol{\delta} f(s)}-e^{-\boldsymbol{\alpha} f(s)}]\theta_2(s)\right\}ds$$
$$+[1-e^{-(\boldsymbol{\alpha}+\boldsymbol{\delta})f(\sigma)}]\int_\sigma^b\left\{\frac{\boldsymbol{\gamma}}{\boldsymbol{\alpha}+\boldsymbol{\delta}}[e^{(\boldsymbol{\alpha}+\boldsymbol{\delta})f(b)}-e^{(\boldsymbol{\alpha}+\boldsymbol{\delta})f(s)}]e^{-\boldsymbol{\alpha} f(s)}\theta_1(s)+e^{\boldsymbol{\delta} f(s)}\theta_2(s)\right\}ds\Bigg]. \tag{31}$$

For ease of evaluation we make some over-estimates in (31) to obtain

$$\varphi_1(\sigma)\leqq\frac{1}{1-\mathbf{F}(b)}\Bigg\{e^{\boldsymbol{\alpha} f(\sigma)}[\mathbf{F}(\sigma)+1-\mathbf{F}(b)]\int_a^\sigma\theta_1(\tau)d\tau$$
$$+\frac{\boldsymbol{\beta\gamma}}{(\boldsymbol{\alpha}+\boldsymbol{\delta})^2}[1-e^{-(\boldsymbol{\alpha}+\boldsymbol{\delta})f(\sigma)}][e^{(\boldsymbol{\alpha}+\boldsymbol{\delta})f(b)}-e^{(\boldsymbol{\alpha}+\boldsymbol{\delta})f(\sigma)}]\int_\sigma^b\theta_1(\tau)d\tau$$
$$+\frac{\boldsymbol{\beta}}{\boldsymbol{\alpha}+\boldsymbol{\delta}}e^{\boldsymbol{\alpha} f(\sigma)+\boldsymbol{\delta} f(b)}[1-e^{-(\boldsymbol{\alpha}+\boldsymbol{\delta})f(\sigma)}]\int_a^b\theta_2(\tau)d\tau\Bigg\}. \tag{32}$$

In conclusion, we observe that if we replace all inequalities in Lemmas 1, 2, 3, and 4 by equalities, then the results of these lemmas are exact; hence the inequalities are sharp. However, since $\mathbf{X}(\sigma,\tau)<\mathbf{G}(\sigma)\mathbf{H}(\tau)$ almost everywhere in $a\leqq\sigma$, $\tau\leqq b$ the result of Lemma 5 is not sharp. The result of Lemma 6 also is not sharp because, although $\mathbf{X}(\sigma, \tau)\mathbf{X}(\tau, t) = \mathbf{X}(\sigma, t)\mathbf{X}(\tau, \tau)$ for σ, τ, t in the order σ, τ, t or t, τ, σ on (a,b), $\mathbf{X}(\sigma, \tau)\mathbf{X}(\tau, t) < \mathbf{X}(\sigma, t)\mathbf{X}(\tau, \tau)$ when this order of σ, τ and t is violated.

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5. References

[1] Olver, F. W. J., On the asymptotic solution of second order differential equations having an irregular singularity of rank one, with an application to Whittaker functions, J. Soc. Indust. Appl. Math. Numer. Anal. Ser. B **2**, 225–243 (1965).

[2] Trjitzinsky, W. J., Analytic theory of linear differential equations, Acta. Math. **62**, 167–226 (1933).

[3] Birkhoff, G. D., Singular points of ordinary differential equations, Trans. Amer. Math. Soc. **10**, 436–470 (1909).

[4] Olver, F. W. J., On error bounds for asymptotic expansions, with application to Bessel Functions of large argument; Asymptotic Solutions of Differential Equations and their Applications, C. H. Wilcox, ed., John Wiley, New York (1964).

[5] Beckenbach, E. F., and Bellman, R., Inequalities, Springer-Verlag (1961).

[6] Olver, F. W. J., and Stenger, F., Error bounds for asymptotic solutions of second-order differential equations having an irregular singularity of arbitrary rank, J. Soc. Indust. Appl. Math. Num. Anal. Ser. B **2**, 244–249 (1965).

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