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Upper Bounds for the Determinant of a Row Stochastic Matrix

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If $A=(a_{ij})$ is a row stochastic matrix of order n then $|\det A| \leq (1-\Sigma_{j=1}^{n} \min_i a_{ij})^{n-1}$. If α is a 1 by n real vector such that every element of αA is nonnegative then $|\det A| \leq s(\alpha)/|g(\alpha)|$, where $s(\alpha)$ is the sum of the elements of α and $g(\alpha)$ is the element of α of greatest absolute value. The conditions for equality are determined in both cases.

Key Words: Determinants, matrices (row stochastic), upper bounds.

It is known that *if* A *is a row stochastic matrix then* $|\det \mathrm{A}| \leq 1$ *with equality if and only if* A *is a permutation matrix.* We will use this result to extend itself.

The result is equivalent to the following statement about nonnegative matrices.[1] Let A be a nonnegative matrix of order n, let $r_A^{(i)}$ denote the sum of the elements in the ith row, let

$$r_A = r_A^{(1)} r_A^{(2)} \ldots r_A^{(n)}$$

and

$$R_A = \mathrm{diag}\,(r_A^{(1)}, r_A^{(2)}, \ldots, r_A^{(n)}).$$

Note that $r_A=0$ if and only if some row of A is all 0's. Otherwise $R_A^{-1}A$ is row stochastic. *Thus* $|\det \mathrm{A}| \leq \mathrm{r_A}$ *with equality if and only if either* $\mathrm{r_A}=0$ *or* $\mathrm{A}=\mathrm{R_A P}$ *for some permutation matrix* P.

We use the following procedure. Given a row stochastic matrix A, let B be a nonsingular matrix such that $AB \geq 0$ (i.e., AB is nonnegative). Then the above result implies that

$$|\det A| \leq \frac{r_{AB}}{|\det B|}$$

with equality if and only if either $r_{AB}=0$ or $A=R_{AB}PB^{-1}$. The same procedure with $CA \geq 0$ yields a similar result. Since B (or C) need not be nonnegative we have room for improvement.

From this point on A will denote a row stochastic matrix of order n.

Lower case Greek letters will denote 1 by n real vectors including

$$\epsilon = (1\ 1\ \ldots\ 1) \text{ and } \delta_i = (0\ldots 0\ 1\ 0\ldots 0) = (\delta_{ij}).$$

[1] A matrix is nonnegative if each element is real and nonnegative. It is row stochastic if, in addition, each row sum equals 1.

For a vector α, $s(\alpha)$ will denote the sum of its elements, $g(\alpha)$ will denote its element of greatest absolute value, and $M_i(\alpha)$ will denote the matrix

$$M_i(\alpha) = \begin{bmatrix} \delta_1 \\ \cdot \\ \cdot \\ \cdot \\ \delta_{i-1} \\ \alpha \\ \delta_{i+1} \\ \cdot \\ \cdot \\ \cdot \\ \delta_n \end{bmatrix}$$

i.e., the identity matrix with the ith row replaced by α. For future clarity we note now that

$$s(\alpha) = \alpha\epsilon^T,$$

while $\epsilon^T\alpha$ is the matrix

$$\epsilon^T\alpha = \begin{bmatrix} \alpha \\ \alpha \\ \cdot \\ \cdot \\ \cdot \\ \alpha \end{bmatrix}$$

Finally $A\epsilon^T = \epsilon^T$. We shall prove:

THEOREM 1. *Let* A *be a row stochastic matrix of order* n, *and let*

$$\mathrm{c} = \sum_{j=1}^{n} \min_i \mathrm{a_{ij}}.$$

Then

$$|\det \mathrm{A}| \leq (1-\mathrm{c})^{n-1}$$

with equality if and only if

$$A=(1-s(\beta))P+\epsilon^T\beta$$

for some vector $\beta \geqslant 0$ *with* $1 \geqslant s(\beta)$, *and some permutation matrix* P.

THEOREM 2. *Let* A *be a row stochastic matrix and* α *a real, nonzero vector such that* $\alpha A \geqslant 0$. *Then*

$$|\det A| \leqslant \frac{s(\alpha)}{|g(\alpha)|}$$

with equality if and only if either $s(\alpha)=0$, *or* α *has exactly one positive element (in the i-th position) and*

$$A=M_i(\gamma)P$$

for some permutation matrix P, *with*

$$\gamma=\frac{1}{g(\alpha)}\{(g(\alpha)+s(\alpha))\delta_i-\alpha\}.$$

To prove Theorem 1 let $c_j=\min_i a_{ij}$ and let

$$B=I-\sum_{j=1}^{n} c_j\epsilon^T\delta_j.$$

The matrix $\epsilon^T\delta_j$ has 1's in the jth column and 0's elsewhere, so that $AB=(a_{ij}-c_j)$. By the definition of the c_j, AB is nonnegative.

The row sums of AB are the elements of the vector

$$AB\epsilon^T=A\epsilon^T-\sum_{j=1}^{n} c_j\epsilon^T\delta_j\epsilon^T=(1-c)\epsilon^T.$$

Thus $r_{AB}^{(i)}=1-c$ for all i, so that

$$r_{AB}=(1-c)^n.$$

It is a simple exercise (left to the reader) to prove that

$$\det B=1-c$$

and

$$B^{-1}=I+(1-c)^{-1}\sum_{j=1}^{n} c_j\epsilon^T\delta_j \qquad c\neq 1.$$

If $c\neq 1$ then, from our previous discussion, the necessity case of Theorem 1 follows with

$$\beta=(c_1\, c_2\, .\ .\ .\ c_n).$$

In the case of sufficiency, if $A=(1-s(\beta))P+\epsilon^T\beta$ then all the row sums of A equal $1(A\epsilon^T=\epsilon^T)$, A is nonnegative if $\beta\geqslant 0$, and $1\geqslant s(\beta)$, $\beta=(c_1c_2\, .\ .\ .\ c_n)$ with the c_j defined as above, and $|\det A|=(1-s(\beta))^{n-1}$.

A short discussion of the value of c will complete our proof. Since A is row stochastic the sum of all of its elements is n. Thus

$$n=\sum_{j=1}^{n}\sum_{i=1}^{n} a_{ij}\geqslant\sum_{j=1}^{n}\sum_{i=1}^{n} c_j=nc$$

with equality $(c=1)$ if and only if $a_{ij}=c_j$ for all i, j, in which case A is of rank 1. Then $\det A=1$ for $n=1$, and $\det A=0$ with $n>1$, and $A=\epsilon^T\beta$. This checks with the statement of Theorem 1 and completes our proof.

To prove Theorem 2 let $C=M_i(\alpha)$. The rows of CA are the rows of A except for the ith row which is αA. By hypothesis $\alpha A\geqslant 0$ so that CA is nonnegative.

The row sums of CA are 1 except for the ith row whose sum is

$$s(\alpha A)=(\alpha A)\epsilon^T=\alpha(A\epsilon^T)=\alpha\epsilon^T=s(\alpha).$$

Thus $r_{CA}=s(\alpha)$ and R_{CA} is the identity matrix transformed by replacing the 1 in the ith position on the diagonal by $s(\alpha)$. Note that, if $s(\alpha)\neq 0$, we have

$$R_{CA}^{-1}M_i(\alpha)=M_i(\alpha/s(\alpha)).$$

The determinant of C is the product of its diagonal elements. Let

$$\alpha=(a_1a_2\, .\ \ .\ a_n).$$

Then $\det C=a_i$. By our previous discussion $|\det A|\leqslant r_{CA}/|\det C|$. Thus

$$|\det A|\leqslant\frac{s(\alpha)}{|a_i|}.$$

This inequality holds for all i; the best of the inequalities is

$$|\det A|\leqslant\frac{s(\alpha)}{|g(\alpha)|}.$$

Suppose $g(\alpha)=a_i$. We have equality if and only if either $s(\alpha)=0$, (whence $\alpha A=0$, $\alpha\neq 0$, so that $\det A=0$) or, if $s(\alpha)\neq 0$:

$$A=C^{-1}R_{CA}P=(R_{CA}^{-1}M_i(\alpha))^{-1}P=(M_i(\alpha/s(\alpha))^{-1}P$$

for some permutation matrix P.

But $(M_i(\alpha/s(\alpha)))^{-1}=M_i(\gamma)$ with

$$\gamma=a_i^{-1}(-a_i\ .\ .\ .\ -a_{i-1}\ s(\alpha)\ -a_{i+1}\ .\ .\ .\ -a_n).$$

Since $A=M_i(\gamma)P$ is nonnegative we must have $a_j\leqslant 0$ for $j\neq i$. This completes the proof of Theorem 2.

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