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Pairs of Nonsingular Matrices[1]

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Let R_1 and R_2 be m by n matrices of rank m. Let S_1 and S_2 be n-m by n matrices of rank n-m such that $R_1S_1^T=R_2S_2^T=0$. Then $R_1R_2^T$ is nonsingular if and only if $S_1S_2^T$ is nonsingular, in which case $R_2^T(R_1R_2^T)^{-1}R_1+\{S_2^T(S_1S_2^T)^{-1}S_1\}^T$ equals the identity matrix of order n.

Key Words: Matrices (pairs of nonsingular).

Let $\mathscr{R}_{mn}$ denote the set of m by n matrices of rank m, with entries from a field of characteristic 0. Given $R\epsilon\mathscr{R}_{mn}$ let $\mathscr{N}(R)$ denote the set of $S\epsilon\mathscr{R}_{n-m,\, n}$ such that $RS^T=0$. Note that $S\epsilon\mathscr{N}(R)$ if and only if $R\epsilon\mathscr{N}(S)$.

We assume as known the following results for $R\epsilon\mathscr{R}_{mn}$ and $S\epsilon\mathscr{N}(R)$:

LEMMA 1. $XR=0$ if and only if $X=0$.

LEMMA 2. $RY^T=0$ if and only if $Y=ZS$, for some matrix Z.

The obvious stipulations are that X, Y, and Z have m,n and n-m columns respectively, and Y and Z have the same number of rows. Using these lemmas we prove:

THEOREM. *If* R_1, $R_2\epsilon\mathscr{R}_{mn}$, $S_1\epsilon\mathscr{N}(R_1)$, $S_2\epsilon\mathscr{N}(R_2)$ *then* $R_1R_2^T$ *is nonsingular if and only if* $S_1S_2^T$ *is nonsingular, in which case*

$$R_2^T(R_1R_2^T)^{-1}R_1+\{S_2^T(S_1S_2^T)^{-1}S_1\}^T=I_n,$$

where I_n *is the identity matrix of order* n.

Suppose $R_1R_2^T$ is singular. Then there exists an $X\neq 0$ such that $XR_1R_2^T=0$.

Let $Y=XR_1$. By Lemma 1, $Y\neq 0$. We have $R_2Y^T=R_2R_1^TX^T=(XR_1R_2^T)^T=0$.

By Lemma 2, this implies that there exists a Z such that $Y=ZS_2$. Clearly $Z\neq \mathbf{0}$. We have $(S_1S_2^T)Z^T=S_1Y^T=S_1R_1^TX^T=0$ (because $S_1R_1^T=(R_1S_1^T)^T=0$), from which we conclude that $S_1S_2^T$ is singular.

The converse is proved in the same way, interchanging R_1 and S_1, R_2 and S_2. Thus $R_1R_2^T$ is singular if and only if $S_1S_2^T$ is singular.

Now suppose $R_1R_2^T$ and $S_1S_2^T$ are nonsingular. Let

$$\begin{aligned} M&=R_2^T(R_1R_2^T)^{-1}R_1+\{S_2^T(S_1S_2^T)^{-1}S_1\}^T-I_n\\ &=R_2^T(R_1R_2^T)^{-1}R_1+S_1^T(S_2S_1^T)^{-1}S_2-I_n. \end{aligned}$$

We wish to prove that $M=0$. We have

$$\begin{aligned} R_1M&=R_1R_2^T(R_1R_2^T)^{-1}R_1+R_1S_1^T(S_2S_1^T)^{-1}S_2-R_1\\ &=R_1+0-R_1\\ &=0. \end{aligned}$$

Similarly $S_2M=0$.

By Lemma 2, $R_1M=0$ implies that there exists an N such that $M^T=NS_1$. Then

$$0=(S_2M)^T=M^TS_2^T=NS_1S_2^T.$$

Since $S_1S_2^T$ is nonsingular, we conclude that $N=0$, whence $M=0$. This proves the theorem.

The following example shows that the characteristic roots (other than 0) of $R_1R_2^T$ and $S_1S_2^T$ need not have any relationship. Let

$$\left.\begin{aligned} R_i&=A_i(I_m\; 0_{m,\, n-m})\\ S_i&=B_i(0_{n-m,\, m}\; I_{n-m}) \end{aligned}\right\}\quad i=1,2$$

where A_1, A_2 and B_1, B_2 are *any* nonsingular matrices of orders m and $n-m$ respectively. Then R_1, R_2 $\epsilon\mathscr{R}_{mn}$, $S_1\epsilon\mathscr{N}(R_1)$, $S_2\epsilon\mathscr{N}(R_2)$, but $R_1R_2^T=A_1A_2^T$ and $S_1S_2^T=B_1B_2^T$. These latter two matrices are related only in the fact that both are nonsingular.

This is in direct contrast to the known result that a characteristic root of FG (F *and* G^T matrices of the same dimensions) is a characteristic root of GF, with the possible exception of a characteristic root equal to 0; i.e., FG may be singular and GF may be nonsingular, or vice-versa.

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[1] This work arose from some matrix-theoretic questions posed by Dr. U. Fano.

Yet this result can be translated into one similar to our theorem. Since 1 is either a characteristic root of both FG and GF or neither, it follows that $I-FG$ is nonsingular if and only if $I-GF$ is nonsingular. If both are nonsingular it is easy to show that

$$(I-FG)^{-1}-F(I-GF)^{-1}G=I_p,$$

where p is the number of rows of F (and columns of G).

Finally we note that if R_1, R_2, S_1, S_2 are as in the theorem and A is nonsingular matrix of order n, then

$$B=R_1AR_2^T,\ C=S_2A^{-1}S_1^T$$

are nonsingular together in which case

$$R_2^TB^{-1}R_1A+A^{-1}S_1^TC^{-1}S_2=I_n.$$

This follows from the fact that $R_1A\epsilon\mathscr{R}_{mn}$ and $S_1(A^T)^{-1}\epsilon\mathscr{N}(R_1A)$.

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