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On a Sequence of Points of Interest for Numerical Quadrature

Seymour Haber

Institute for Basic Standards, National Bureau of Standards, Washington, D.C.

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The distribution of 2 sequences of points, originally discussed by van der Corput and by K. F. Roth, is studied in detail. The results obtained disprove a conjecture of J. H. Halton and suggest a conjecture on the improvement of a theorem of Roth.

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1. If $x_1, x_2, \ldots, x_N$ are points in the unit S-dimensional cube G_S, we may approximate to an integral over G_S by the average of the values of the integrand at these points. Writing

$$\int_0^1 \ldots \int_0^1 f(x^{(1)}, x^{(2)}, \ldots, x^{(S)}) dx^{(1)} \ldots dx^{(S)} = \frac{1}{N} \sum_{i=1}^{N} f(x_i^{(1)}, \ldots, x_i^{(S)}) + R, \tag{1}$$

(where $x_i^{(1)}, x_i^{(2)} \ldots, x_i^{(S)}$ are the S coordinates of the point x_i), we call R the "error" of the quadrature formula (1). R depends on the integrand f and on the set of points $x_1, \ldots, x_N$; for R to be relatively small for some wide class of functions the set of points should be, in some sense, well distributed in the cube G_S. An appropriate sense of "well distributed" is that, for a region A in G_S, the number of points of the set which lie in A should be approximately N times the volume of A. Restricting attention to regions which are intervals with one vertex at the origin—i.e., which are of the form

$$\{x = (x^{(1)}, \ldots, x^{(S)}) | 0 \leq x^{(i)} < x_0^{(i)}, \; i = 1, 2, \ldots, S\}$$

for some given $x_0 = (x_0^{(1)}, \ldots, x_0^{(S)}) \epsilon G_S$—we shall let $S_N(x_0^{(1)}, \ldots, x_0^{(S)})$ denote the number of the points $x_1, \ldots x_N$ which lie in such an interval, and consider the quantity $|S_N(x_0^{(1)}, \ldots, x_0^{(S)}) - N x_0^{(1)} x_0^{(2)}, \ldots x_0^{(S)}|$.
Let

$$D_N^* = D_N^*(x_1, \ldots, x_N) = \sup_{x_0 \epsilon G_S} |S_N(x_0^{(1)}, \ldots, x_0^{(S)}) - N x_0^{(1)}, \ldots, x_0^{(S)}|. \tag{2}$$

D_N^* is very nearly the "discrepancy" (see, e.g., [1])[1] of the set of N points. E. Hlawka [2] has shown that if the integrand f in (1) is a function of bounded variation and $V(f)$ is its total variation, we have the bound

$$|R| \leq V(f) \frac{D_N^*}{N}. \tag{3}$$

[1] Figures in brackets indicate the literature references at the end of this paper.

Since it is very likely true that all continuous and bounded functions that one ever attempts to integrate numerically are of bounded variation, it is a matter of considerable interest to find sets of points for which D_N^* is as small as possible.

It is easy to see that $D_N^* \geqslant \frac{1}{2}$. For if $|S_N(x_1^{(1)}, 1, \ldots, 1) - Nx_1^{(1)}| = a < \frac{1}{2}$, then for any $\epsilon > 0$ $|S_N(x_1^{(1)}+\epsilon, 1, \ldots, 1) - N(x_1^{(1)}+\epsilon)| > 1 - a - N\epsilon$ and so is $> \frac{1}{2}$ if ϵ is sufficiently small. In one dimension, the sequence of points $x_i = \dfrac{2i-1}{2N}$, $i = 1, 2, \ldots, N$, has $D_N^* = \frac{1}{2}$ for every N. In two or more dimensions the situation is entirely different; K. F. Roth [3] has shown that for $S > 1$ dimensions there is a positive number $C(S)$ such that

$$D_N^* > C(S)(\log N)^{\frac{S-1}{2}} \tag{4}$$

for any set of N points. It is not known that this result is best possible; there is some reason to guess that the exponent $\dfrac{S-1}{2}$ may be replaceable by $S-1$. In his proof Roth introduced another measure of unevenness of distribution, which we shall denote "E_N". It is defined by

$$E_N = E_N(x_1, \ldots, x_N) = \left(\int_0^1 \cdots \int_0^1 |S_n(x_1^{(1)} \ldots, x^{(S)}) - Nx^{(1)}, \ldots, x^{(S)}|^2 dx^{(1)} \ldots dx^{(S)}\right)^{1/2}. \tag{5}$$

This L^2 norm is sometimes easier to handle than the maximum norm D_N^*, and Roth in fact proved (4) with E_N in place of D_N^*.

In 1935 van der Corput [4] raised the question of whether there exists an infinite sequence $x_1, x_2, \ldots$ of points in the inverval $[0, 1]$ such that $D_N^*(x_1, x_2, \ldots, x_N)$ is bounded in N. This was answered in the negative by Mrs. van Aardenne-Ehrenfest [5] in 1945. Roth, in [3], showed that this problem is equivalent to the problem of estimating a lower bound for D_N^* for fixed N in two dimensions. More generally, Roth showed the existence of a constant C such that: For any N and S, if there is a set of points $x_1, \ldots, x_N$ in G_S such that $D_N^*(x_1, \ldots, x_N) = A$, there is a sequence $x_1', \ldots, x_N'$ in G_{S-1} such that $\max_{n \leqslant N} D_n^*(x_1', \ldots, x_n') < CA$; and conversely. It follows that for any infinite sequence $x_1, x_2, \ldots$ in $[0, 1]$, $D_N^*(x_1, \ldots, x_N) \geqslant C'(\log N)^{1/2}$ for infinitely many values of N (where C' is some positive constant). Van der Corput found a sequence for which he showed that $D_N^* \leqslant \log_2 N + 1$ (here and below "$\log_2$" denotes the logarithm to the base two); it is constructed as follows: for each integer $n \geqslant 0$, write n as $2^{h_1} + 2^{h_2} + \ldots + 2^{h_k}$ with $h_1 < h_2 < \ldots < h_k$; then $x_n = \frac{1}{2}(2^{-h_1} + 2^{-h_2} + \ldots + 2^{-h_k})$. (We may picture this process as writing n as a numeral to the base 2 and reflecting this numeral in the "decimal point." Thus, in binary notation, $x_0 = 0$, $x_1 = 0.1$, $x_2 = 0.01$, $x_3 = 0.11$, $x_4 = 0.001$, etc.) Roth pointed out that then for each N, the sequence $\left(x_1, \frac{1}{N}\right)$, $\left(x_2, \frac{2}{N}\right)$, $\ldots$, $(x_N, 1)$ in G_2 has the property that $D_N^* \leqslant \log_2 N + 2$. We shall refer to the van der Corput sequence as "$\mathscr{S}$" and to the Roth sequence, for any N, as "$\mathscr{S}'$".

J. M. Hammersley [6] and J. H. Halton [7], interested in the application of evenly distributed sequences to high-dimensional numerical integration, suggested higher-dimensional analogs of these sequences. Defining $\varphi_k(n)$, for any integers $n \geqslant 0$ and $k > 1$, as the number produced by reflecting the representation of n to the base k in the "decimal point" (more formally, if $n = a_1k^{h_1} + a_2k^{h_2} + \ldots a_lk^{h_l}$ where $0 \leqslant a_i < k$ for $i = 1, \ldots, l$, then $\varphi_k(n) = \frac{1}{k}(a_1k^{-h_1} + \ldots + a_lk^{-h_l})$), Halton suggested the s-dimensional sequence defined by

$$x_n = (\varphi_2(n), \varphi_3(n), \varphi_5(n), \ldots, \varphi_{P_S}(n)),\ n = 0, 1, 2, \ldots \tag{6}$$

(where P_s is the Sth prime) and Hammersley suggested

$$x_n = \left(\frac{n}{N}, \varphi_2(n), \varphi_3(n), \ldots, \varphi_{P_{S-1}}(n)\right), \quad n = 1, 2, \ldots, N. \tag{7}$$

Halton proved that for the sequence (6), D_N^* and E_N are $\leq C(\log N)^K$ and for (7) they are $\leq C'$ (log $N)^{K-1}$, for some constants C and C'. He further conjectured that the exponents K and $K-1$ could be replaced by $K/2$ and $(K-1)/2$ respectively.

2. In this paper, we shall analyze the distribution of the sequences $\mathscr{S}$ and $\mathscr{S}'$ in detail. Our results are as follows:

THEOREM 1. *For the sequence* $\mathscr{S}$,

$$D_N^* \leq \frac{1}{3} \log_2 (N) + 0(1) \tag{8}$$

and the constant 1/3 is best possible—that is, with any smaller constant (8) *would not always be true.*

THEOREM 2. *For the sequence* $\mathscr{S}$,

$$E_N \leq \frac{1}{6} \log_2 N + 0(1) \tag{9}$$

and the constant 1/6 is best possible.

THEOREM 3. *For the sequence* $\mathscr{S}'$,

$$D_N^* = \frac{1}{3} \log_2 N + 0(1). \tag{10}$$

THEOREM 4. *For the sequence* $\mathscr{S}'$,

$$E_N = \frac{1}{8} \log_2 N + 0(1). \tag{11}$$

From these theorems it follows that Halton's conjecture is incorrect for the lowest-dimensional cases of his and Hammersley's sequences; it is very likely also to be incorrect in all dimensions.

3. Proofs of the Theorems:

We shall first make some notational conventions.

If a letter is used to represent a nonnegative integer, the same letter may be used to represent the finite string of zeros and ones which is the binary representation of that integer.

If α is a finite string of zeros and ones, we shall enumerate its digits from right to left, calling the rightmost one the "zeroth digit," and write "$\alpha = \alpha_k \ldots \alpha_1 \alpha_0$." The number of digits in α will be denoted "$||\alpha||$"; thus $||\alpha_k \alpha_{k-1} \ldots \alpha_0|| = k+1$. For $0 \leq i \leq k$, "$\alpha^{(i)}$" will denote the string $\alpha_i \alpha_{i-1} \ldots \alpha_1 \alpha_0$.

For a string $\alpha = \alpha_k \ldots \alpha_0$, "$.\alpha$" will denote the number $\frac{\alpha_k}{2} + \frac{\alpha_{k-1}}{4} + \ldots + \frac{\alpha_0}{2^{k+1}}$ (if α is null, $.\alpha$ will be taken to be zero). If α and β are strings, "$\alpha\beta$" will denote the string consisting of α and β in the order indicated; so, for example, "0.010α" will denote $\frac{1}{4} + \frac{1}{8}(.\alpha)$, and if $\alpha = 1101$, this will be the binary number 0.0101101, or 45/128.

If A is any real number, "$[A]$" will denote the greatest integer not greater than A, and "$\{A\}$" will denote $A - [A]$. The distance from A to the nearest integer, which equals the lesser of $\{A\}$ and $1 - \{A\}$, will be denoted "$< A >$."

Let us now fix an integer $N > 1$, and set $M = [\log_2 N]$. For any $x \epsilon [0,1]$, let $.a_0 a_1 a_2 \ldots$ be the nonterminating binary representation of x. Let $A = A(x)$ be the string $a_M a_{M-1} \ldots a_1 a_0$. We will first consider the sequence $\mathscr{S}=x_0, x_1, \ldots,$ with $x_n=\varphi_2(n)$. To estimate $S_n(x)$ we proceed as follows:

For x_n to be less than x, exactly one of the following $M+2$ conditions must hold:

$$(0'.)\quad a_0>n_0$$

$$(1'.)\quad a_0=n_0;\ a_1>n_1$$

$$\vdots$$

$$(M'.)\quad a_0=n_0,\ a_1=n_1,\ \ldots,\ a_{M-1}=n_{M-1};\ a_M>n_M$$

$$(M+1'.)\quad a_0=n_0,\ \ldots,\ a_{M-1}=n_{M-1},\ a_M=n_M.$$

These are equivalent, respectively, to the following conditions:

$$(0.)\quad n\equiv 0(\mathrm{mod}\ 2),\ a_0=1$$

$$(1.)\quad n\equiv a_0(\mathrm{mod}\ 2^2),\ a_1=1$$

$$(2.)\quad n\equiv a_0+2a_1(\mathrm{mod}\ 2^3),\ a_2=1$$

$$\vdots$$

$$(M.)\quad n\equiv a_0+2a_1+\ldots+2^{M-1}a_{M-1}(\mathrm{mod}\ 2^{M+1}),\ a_M=1$$

$$(M+1.)\quad n\equiv a_0+2a_1+\ldots+2^Ma_M(\mathrm{mod}\ 2^{m+1}).$$

The number of integers $0\leq n<N$ satisfying condition (0.) is

$$a_0\left(\left[\frac{N}{2}\right]+1\right);$$

the number satisfying condition (1.) is

$$a_1\left(\left[\frac{N-a_0}{2^2}\right]+1\right);$$

the number satisfying condition (2.) is

$$a_2\left(\left[\frac{N-a_0-2a_1}{2^3}\right]+1\right);$$

etc. At most one n satisfies condition $M+1$. Thus

$$\sum_{i=0}^{M} a_i\left(\left[\frac{N-(a_0+2a_1+\ldots+2^{i-1}a_{i-1})}{2^{i+1}}\right]+1\right)$$

differs from $S_N(x)$ by at most 1.

Now

$$Nx = N\left(\frac{a_0}{2} + \frac{a_1}{2^2} + \ldots\right) = \sum_{i=0}^{M} a_i \frac{N}{2^{i+1}} + \sum_{i=M+1}^{\infty} a_i \frac{N}{2^{i+1}}.$$

By the definition of M the second sum is between zero and 1. If we now set

$$d = d(x, N) = \sum_{i=0}^{M} a_i \left(\left[\frac{N - (a_0 + \ldots + 2^{i-1}a_{i-1})}{2^{i+1}}\right] + 1 - \frac{N}{2^{i+1}}\right), \tag{12}$$

then d differs from $S_N(x) - Nx$ by at most 2; and so in proving our Theorems, for which a difference which is 0(1) has no effect, we may deal with d instead of $S_N(x) - Nx$ throughout.

Now

$$\frac{a_0 + 2a_1 + \ldots + 2^{i-1}a_{i-1}}{2^{i+1}} = .0a_{i-1}a_{i-2} \ldots a_0$$

(in binary notation), and may be written "$.0A^{(i-1)}$"; it is equal to $\frac{1}{2}\varphi_2([2^i x])$. $\left\{\frac{N}{2^{i+1}}\right\}$ may also be written "$.N^{(i)}$." Since

$$\left[\frac{N - (a_0 + \ldots + 2^{i-1}a_{i-1})}{2^{i+1}}\right] + 1 - \frac{N}{2^{i+1}} = \left[\left[\frac{N}{2^{i+1}}\right] + (.N^{(i)} - .0A^{(i-1)})\right] - \left[\frac{N}{2^{i+1}}\right] + 1 - .N^{(i)},$$

we may write

$$d = \sum_{i=0}^{M} a_i b_i, \text{ where } b_i = \begin{cases} 1 - .N^{(i)} & \text{if } .N^{(i)} \geqslant .0A^{(i-1)} \\ -.N^{(i)} & \text{if } .N^{(i)} < .0A^{(i-1)}. \end{cases} \tag{13}$$

We will first obtain a lower bound for d: $b_i < 0$ only if $N_i = 0$; therefore

$$d \geqslant \sum_{\substack{N_i=0 \\ 0 \leqslant i \leqslant M}} a_i b_i \geqslant - \sum_{\substack{N_i=0 \\ 0 \leqslant i \leqslant M}} b_i \geqslant - \sum_{\substack{N_i=0 \\ 0 \leqslant i \leqslant M}} .N^{(i)}. \tag{14}$$

To estimate this last sum, we first note that if the string N (read from right to left) starts with a block of consecutive zeros, the $.N^{(i)}$ corresponding to those digits are zero; and so that block of zeros may be removed from N. Similarly a block of ones occurring at the leftmost end of N may be removed. What remains of N can then be broken into disjoint substrings of the forms 01, 001, 0001, . . . and 1, 11, 111, ... A substring 01 contributes one term to the sum $-.N^{(i)}$ for that i which is the index of the zero in the substring. Since that $.N^{(i)}$ is less than 1/2, the contribution of such a substring to the sum is less than 1/4 the length of the substring. Similarly each block of the form 001, 0001, . . . contributes to the sum a set of terms whose total is less than 1/4 the length of the block. Substrings of the form 1, 11, 111, . . . contribute nothing to the sum. It follows that

$$d > -\frac{M+1}{4}. \tag{15}$$

To obtain a sharp upper bound on d we shall require some lemmas:

LEMMA 1. *For a fixed integer* $\mathrm{M} \geqslant 1$ *and real number* $\mathrm{x} \epsilon [0,1]$, *and* $\mathrm{A} = \mathrm{A(x)}$ *as defined above, the maximum of* $\mathrm{d(x, N)}$ *over all* N *between* 0 *and* $2^{\mathrm{M}+1} - 1$, *is attained at* $\mathrm{N} = 2\mathrm{A} - 2^{\mathrm{M}+1}\mathrm{a_M}$. (*i.e.*, $\mathrm{N_0} = 0$, $\mathrm{N_i} = \mathrm{a_{i-1}}$ *for* $\mathrm{i} = 1, 2, \ldots, \mathrm{M}$.)

PROOF: $d(x, N)=\sum_{i=0}^{M} a_i(x)b_i(x, N)$. Changing any digit N_i of N changes b_j only for $j \geqslant i$. If a change in some N_i does not change the sign of some $b_j, j \geqslant i$, then it changes that b_j by just $\frac{1}{2^{j-i+1}}$. For any i between 0 and M, let j_0 be the 1'st $j \geqslant i$ such that $a_j=1$; it then follows that if changing N_i does not change the sign of b_j for any $j \geqslant 1$, the change in $a_{j_0}b_{j_0}$ dominates the sum of all the changes in d resulting from the change in N_i. On the other hand, no b_i with $j \geqslant i$ changes sign unless b_{j_0} does; so if the change in N_i does change the sign of some b_j, the change in $a_{j_0}b_{j_0}$ is still dominating. It follows that the N that maximizes d must also maximize $\sum_{i=0}^{K} a_i b_i$ for every K between zero and M.

Now let $a_{r_1}, a_{r_2} \ldots (r_i < r_2 < \ldots)$, be the 1's in A. To maximize $\sum_{i=0}^{r_1} a_i b_i$ we must clearly set $N_0=N_1=\ldots=N_{r_1}=0$. Proceeding by induction, we assume that $N_i=a_{i-1}$ for $i=1, 2, \ldots, r_k$, and try to define $N_{r_k+1}, \ldots, N_{r_{k+1}}$ so as to maximize $\sum_{i=0}^{r_{k+1}} a_i b_i$. If $r_{k+1}=r_k+1$, we must set $N_{r_k+1}=1$, for otherwise $b_{r_{k+1}}$ would be negative. If $r_{k+1} > r_k+1$, it is clear that $b_{r_{k+1}}$ is maximized by setting $N_{r_k+1}=1$, $N_{r_k+2}=N_{r_k+3}=\ldots=N_{r_{k+1}}=0$. In either case $N_i=a_{i-1}$ for $i=r_k+1, \ldots, r_{k+1}$, proving the lemma.

Let us now, for any finite string α of zeros and ones, let $\delta(\alpha)$ be the maximum of $d(x, N)$ for $0 \leqslant N < 2^{\|\alpha\|+1}$ and $A(x)=\alpha$. It remains to determine the maximum of $\delta(\alpha)$ over all strings α of a given length.

For any integer $L \geqslant 1$, let σ_L and σ_L' be the strings 10101 . . . 01 and 0101 . . . 01011, respectively, each of length $2L+1$ ($\sigma_1=101$, $\sigma_1'=011$).

LEMMA 2. *If* $0 \leqslant t \leqslant 1$, *the maximum of* $\delta(\alpha)-t(.\alpha)$ *over all strings* α *of length* $2L+1$ *occurs at* $\alpha=\sigma_L$; *if* $1 \leqslant t \leqslant 2$ *the maximum occurs at* $\alpha=\sigma_L'$.

PROOF: We first note that $\delta(\sigma_L')=\frac{2}{3}L+\frac{7}{9}+\frac{2}{9\cdot 4^L}$; and that $.\sigma_L=\frac{2}{3}-\frac{1}{6\cdot 4^L}$ and $.\sigma_L'=\frac{1}{3}+\frac{1}{6\cdot 4^L}$. The proof proceeds by induction on L. For $L=1$ the lemma is shown true by direct calculation. Assuming it true for L, if α is a string of length $2(L+1)+1$, then $\alpha=00\beta$ or 01β or 10β or 11β, where $\|\beta\|=2L+1$. In these 4 cases respectively we have:

$$\delta(\alpha)-t(.\alpha)=\delta(\beta)-\frac{t}{4}(.\beta)$$

$$\delta(\beta)-\left(1+\frac{t}{4}\right)(.\beta)+1-t/4$$

$$\delta(\beta)-\left(\frac{1}{2}+\frac{t}{4}\right)(.\beta)+1-t/2$$

$$\delta(\beta)-\left(\frac{3}{2}+\frac{t}{4}\right)(.\beta)+\frac{3}{2}-\frac{3t}{4};$$

we shall call these quantities δ_1, δ_2, δ_3, and δ_4. By the induction hypothesis, for $0 \leqslant t \leqslant 2$, δ_1 and δ_3 are maximized by taking $\beta=\sigma_L$, and δ_2 and δ_4 are maximized by $\beta=\sigma_L'$. Direct calculation then shows that δ_3 is the largest of the four when $0 \leqslant t \leqslant 1$, and δ_2 is largest when $1 \leqslant t \leqslant 2$, which proves the lemma.

Thus in particular the maximum of $\delta(\alpha)$ over all strings α of length $2L+1$ is $\delta(\sigma_L)$, which is equal to $\frac{1}{3}\|\alpha\|+0(1)$.

Furthermore, if α is a string of even length then $\alpha=0\beta$ or 1β, where $\|\beta\|=2L+1$ for some integer L. In the first case $\delta(\alpha)=\delta(\beta)$, and in the second $\delta(\alpha)=1-.\beta+\delta(\beta)$. Since $.\beta<1$, $\alpha=1\beta$

yields the larger δ, and then, by Lemma 2., $\delta(\alpha)$ is maximal when $\beta=\sigma_L$ or σ_L'; and once again the maximum of $\delta(\alpha)$ is $\frac{1}{3}\|\alpha\|+0(1)$.

Lemmas 1 and 2 together imply that for any positive integer M the maximum of $d(x, N)$ for $0 \leqslant x \leqslant 1$ and $0 \leqslant N < 2^{M+1}$ is $\frac{M+1}{3}+0(1)$, and Theorem 1. follows.

To prove Theorem 2., we shall first show:

LEMMA 3. $$\left(\int_0^1 (S_N(x)-Nx)^2\, dx\right)^{1/2}=\frac{1}{2}\sum_{i=1}^{\infty}\left\langle\frac{N}{2^i}\right\rangle+0(1).$$

PROOF: First of all, it is sufficient to prove the equation with $d(x, N)$ in place of $S_N(x)-Nx$. For fixed N, and $i=0,1,\ldots, M$ let $f_i(x)=a_i(x)b_i(x, N)$. Then $d(x, N)=f_0(x)+f_1(x)+\ldots+f_M(x)$, and

$$\int_0^1 d^2(x, N)dx=\left(\sum_{i=0}^{M}\int_0^1 f_i\right)^2+\sum_{i=0}^{M}\left(\int_0^1 f_i^2-\left(\int_0^1 f_i\right)^2\right)+2\sum_{0\leqslant i<j\leqslant M}\left(\int_0^1 f_if_j-\int_0^1 f_i\int_0^1 f_j\right). \tag{16}$$

To evaluate the first term on the right of (16), we first define C_i, for each i, by

$$C_i(x)=\begin{cases}1-.N^{(i)} \text{ if } .N^{(i)} \geqslant .0A^{(i+1)}+\frac{1}{2^{i+1}}\\ -.N^{(i)} \text{ otherwise}\end{cases} \tag{17}$$

and note that $C_i=b_i$ unless $.0A^{(i-1)} \leqslant .N^{(i)} < .0A^{(i-1)}+\frac{1}{2^{i+1}}$, when $C_i=b_i-1$. Now $.0A^{(i-1)}=\frac{1}{2}\varphi_2([2^ix])$ and so is constant on each interval $\left(\frac{r}{2^i}, \frac{r+1}{2^i}\right)$, $r=0, 1, \ldots, 2^i-1$; and its values on different intervals differ by at least 2^{-i}. Thus C_i and b_i are equal everywhere except perhaps on one interval of length 2^{-i}, so that if we set $F_i(x)=a_i(x)C_i(x)$, we have

$$\int_0^1 |F_i(x)-f_i(x)|dx \leqslant \frac{1}{2^i}, \text{ and } \left|\sum_{i=0}^{M}\int^1 F_i-\sum_{i=0}^{M}\int_0^1 f_i\right| \leqslant 2. \tag{18}$$

Since a_i is zero on one half of each interval $\left(\frac{r}{2^i}, \frac{r+1}{2^i}\right)$ and one on the other half, while C_i is constant on the interval, $\int_0^1 F_i(x)dx=\frac{1}{2}\int_0^1 C_1(x)dx$. The values $.0A^{(i-1)}+\frac{1}{2^{i+1}}$ takes on these 2^i intervals are just the numbers $\frac{1}{2^{i+1}}, \frac{2}{2^{i+1}}, \ldots, \frac{2^i}{2^{i+1}}$. So if $.N^{(i)} \geqslant \frac{1}{2}$, $C_i=1-.N^{(i)}$ throughout (0,1) and $\int_0^1 F_i=\frac{1}{2}(1-.N^{(i)})=\frac{1}{2}\left\langle\frac{N}{2^{i+1}}\right\rangle$. If $.N^{(i)} < \frac{1}{2}$, we may write $.N^{(i)}=K/2^{i+1}$, where K is an integer between 0 and 2^i-1. It's then easy to see that $.N^{(i)} < .0A^{(i-1)}+\frac{1}{2^{i+1}}$ on just 2^i-K of the above intervals. Then $C_i(x)=.N^{(i)}$ on a set of measure $1-K/2^i$ and $1-.N^{(i)}$ on a set of measure $K/2^i$; so that $\int_0^1 F_i=\frac{.N^{(i)}}{2}=\frac{1}{2}\left\langle\frac{N}{2^{i+1}}\right\rangle$ once again. Therefore, by (18),

$$\sum_{i=0}^{M}\int_0^1 f_i=\frac{1}{2}\sum_{i=0}^{M}\left\langle\frac{N}{2^{i+1}}\right\rangle+0(1)=\frac{1}{2}\sum_{i=1}^{\infty}\left\langle\frac{N}{2^i}\right\rangle+0(1). \tag{19}$$

Lemma 3. will now follow if we show that the second and third terms on the right in (16) are each $0(\log N)$. For the second term this is obvious, since $|f_i(x)| \leqslant 1$. To bound the 3d term we proceed as follows:

For $0 \leqslant i \leqslant M$, set $b_i(x) = -.N^{(i)} + e_i(x)$, and $y_i(x) = .0A^{(i-1)}$.

Then

$$e_i = \begin{cases} 1 \text{ if } .N^{(i)} \geqslant y_i \\ 0 \text{ if } .N^i < y_i \end{cases}, \tag{20}$$

and

$$\int_0^1 f_i f_i - \int_0^1 f_i \int_0^1 f_i = \frac{1}{4}\left(\int_0^1 e_i e_i - \int_0^1 e_i \int_0^1 e_i\right). \tag{21}$$

Now for any i and j, $i < j$, set $z_{i,j}(x) = y_j(x) - 2^{i-j} y_i(x)$. Recalling that $y_i = \frac{1}{2}\varphi_2([2^i x])$ and noting that $\varphi_2([2^i x]) = \{2^{j-i}\varphi_2([2^j x])\}$, we see that $z_{j,i}$ takes on, in each interval $\left(\frac{r}{2^i}, \frac{r+1}{2^i}\right)$, the 2^{j-i} values $\frac{l}{2^{j-i+1}}$, $l = 0, 1, \ldots, 2^{j-i} - 1$; and each value is taken on an interval of length 2^{-j}. Therefore $z_{j,i}$, and y_i are independent functions of x, and if we define $e_j^*(x) = e_{j,i}^*(x)$ by

$$e_j^* = \begin{cases} 1 \text{ if } .N^{(i)} \geqslant z_j \\ 0 \text{ if } .N^{(i)} < z_j \end{cases}, \tag{22}$$

we can conclude that

$$\int_0^1 e_i e_j^* - \int_0^1 e_i \int_0^1 e_j^* = 0. \tag{23}$$

By reasoning as we did above about b_i and C_i we can see that $e_i(x)$ and $e_j^*(x)$ are equal except on a set of measure at most 2^{i-j+1}, and there $|e_j^*(x) - e_j(x)| = 1$. Therefore, by (23)

$$\left|\int_0^1 e_i e_j - \int_0^1 e_i \int_0^1 e_j\right| = \left|\int_0^1 e_i(e_j - e_j^*) - \int_0^1 e_i \int_0^1 (e_j - e_j^*)\right| \leqslant 2\int_0^1 \left|e_j - e_j^*\right| \leqslant \frac{4}{2^{j-i}}.$$

From (21), therefore,

$$\sum_{0 \leqslant i < j \leqslant M} \left(\int_0^1 f_i f_j - \int_0^1 f_i \int_0^1 f_j\right) \leqslant \sum_{0 \leqslant i < j \leqslant M} \frac{1}{2^{i-j}} < M,$$

which completes the proof of the lemma.

To prove Theorem 2 we must now show that

$$\sum_{i=0}^{M} \left\langle \frac{N}{2^{i+1}} \right\rangle \leqslant \frac{1}{3} \log_2 N + 0(1), \tag{24}$$

and that the constant 1/3 in (24) is best possible. We have seen that $\sum_{i=0}^{M} \left\langle \frac{N}{2^{i+1}} \right\rangle = \sum_{i=0}^{M} \min\,(.N^{(i)}, 1 - .N^{(i)})$; and we shall show that the last sum is maximized (over all $0 \leqslant N < 2^{M+1}$) when N is of the form . . . 010101:

For $M = 0$, this is obvious. If $M = K$, then $N = 0N'$ or $1N'$, where N' is a string with $M = K - 1$. In the first case the sum is $\frac{1}{2}(.N') + \sum_{i=0}^{K-1} \min\,(.N^{(i)}, 1 - .N^{(i)})$, and in the second it is $\frac{1}{2}(1 - .N') + \sum_{i=0}^{K-1}$

min $(.N^{(i)}, 1-.N^{(i)})$. The first is larger if the leftmost digit of N' is 1, while the second is larger if the leftmost digit of N' is zero; which completes the induction. For N of the form specified, $\sum_{i=0}^{M}\left\langle\frac{N}{2^{i+1}}\right\rangle$ is equal to $1/3 \log_2 N+0(1)$, so (24) is established, together with the fact that the constant 1/3 cannot be improved upon.

To derive Theorems 3 and 4 from Theorems 1 and 2 we will use a device of K. F. Roth [3]. For any N, and $0\leq x\leq 1$ and $0\leq y\leq 1$, let $n=n(y)=[Ny]$. Then $S_N(x, y)$ for the sequence $\mathscr{S}'$ is equal to $S_n(x)$ for the sequence $\mathscr{S}$. Furthermore $|N xy-nx|\leq 1$, so that

$$\sup_{0\leq x,\, y\leq 1} |S_N(x, y)-Nxy| \text{ differs from } \sup_{\substack{0\leq x\leq 1\\ 0<n<N}} |S_n(x)-nx| \text{ by no more}$$

than 1. From Theorem 1,

$$|S_n(x)-nx|\leq\frac{1}{3}\log_2 n+0(1)\leq\frac{1}{3}\log_2 N+0(1);$$

and from the proof of Theorem 1, if n is the largest integer between 0 and N whose binary representation is of the form 10101 . . . 01,

$$\sup |S_n(x)-nx|=\frac{1}{3}\ \|n\|\ +0(1)=\frac{1}{3}\ \|N\|\ +0(1).$$

Theorem 3 follows.

To prove Theorem 4. we use the device of Roth to replace

$$\left(\int_0^1\int_0^1 (S_N(x,y)-Nxy)^2 dy\,dy\right)^{1/2} \text{ by } \left(\int_0^1\int_0^1 (S_{n(y)}(x)-xn(y))^2 dx\,dy\right)^{1/2};$$

the latter, in turn, differs by only 0(1) from

$$\left(\int_0^1\left(\frac{1}{2}\sum_{i=0}^{M}\left\langle\frac{n(y)}{2^{i+1}}\right\rangle\right)^2 dy\right)^{1/2}. \tag{25}$$

Setting $g_i(y)=\left\langle\frac{n(y)}{2^{i+1}}\right\rangle$, we can write

$$\int_0^1\left(\sum_{i=0}^{M}\left\langle\frac{n}{2^{i+1}}\right\rangle\right)^2 dy=\left(\sum_{i=0}^{M}\int_0^1 g_i\right)^2+\sum_{i=0}^{M}\left(\int_0^1 g_i^2-\left(\int_0^1 g_i\right)^2\right)+2\sum_{0\leq i<j\leq M}\left(\int_0^1 g_i g_j-\int_0^1 g_i\int_0^1 g_j\right). \tag{26}$$

As in the analysis of (16), we wish to show that the second and third terms on the right of (26) are each $0(M)$. For the second this is obvious; for the third we proceed so: For each i and $j>i$, set $g_{j,i}(y)=\left\langle\left\{\frac{n}{2^j}\right\}-2^{i-j}\left\{\frac{n}{2^i}\right\}\right\rangle$; $g_{j,i}$ is then a function of only the $M-j$th through $M-i+1$th digits of the binary representation of y, and so is independent of g_i, which is a function of the $M-i$th through Mth digits. Furthermore $\int_0^1|g_{j,i}(x)-g_j(x)|dx\leq\frac{2}{2^{j-i}}$; that the third term is $0(M)$ follows as in the proof of Lemma 3.

To evaluate the first term on the right of (26), we note that

$$\int_0^1 g_i(y)dy=\int_0^1\frac{[Ny]}{2^{i+1}}dy=\frac{1}{N}\sum_{r=0}^{N-1}\left\langle\frac{r}{2^{i+1}}\right\rangle. \tag{27}$$

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Since $\langle x\rangle = \langle \{x\} \rangle$, the sum in (27) can be written as

$$\left[\frac{N}{2^{i+1}}\right] \sum_{r=0}^{2^{i+1}-1} \left\langle\frac{r}{2^{i+1}}\right\rangle + E_i = \frac{2^{i+1}}{4}\left[\frac{N}{2^{i+1}}\right] + E_i, \tag{28}$$

where

$$0 \leqslant E_i = \sum_{r=2^{i+1}\left[\frac{N}{2^{i+1}}\right]}^{N-1} \left\langle\frac{r}{2^{i+1}}\right\rangle < \frac{2^{i+1}}{4}. \tag{29}$$

Therefore

$$N\int_0^1 g_i = \frac{2^{i+1}}{4}\left(\frac{N}{2^{i+1}} - \left\{\frac{N}{2^{i+1}}\right\} + \frac{4}{2^{i+1}}E_i\right)$$

$$= \frac{N}{4} + \frac{2^{i+1}}{4}\theta_i,. \tag{30}$$

where $|\theta_i| \nless 1$.

So

$$\sum_{i=0}^{M}\int_0^1 g_i = \frac{M+1}{4} + \frac{\theta}{N}\sum_{i=0}^{M} 2^{i-1} \tag{31}$$

where $|\theta| < 1$. Since $\sum_{i=0}^{M} 2^{i-1} < 2^M \leqslant N$, we can conclude that

$$\sum_{i=0}^{M}\int_0^1 g_i = \frac{M}{4} + 0(1) \tag{32}$$

and Theorem 4. follows.

4. An attempt to analyze the higher-dimensional sequences of Hammersley and Halton along the above lines was unsuccessful, due to the greater complication of those sequences. Analogs, for those sequences, of Theorems 1 and 3 would be very useful in application of the sequences to multiple numerical quadrature.

Intuitively, van der Corput's sequence $\mathscr{S}$ seems to be about as evenly distributed as possible. This feeling, together with Theorem 1, leads us to the conjecture in section 1 that K. F. Roth's lower bound on D_N^* can be improved.

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