JOURNAL OF RESEARCH of the National Bureau of Standards — B. Mathematics and Mathematical Physics
Vol. 70B, No. 1, January–March 1966

The Form Factor of the Fermi Model Spatial Distribution

L. C. Maximon and R. A. Schrack

(October 19, 1965)

A useful analytic expression for the form factor $F(q)=\int \rho(r)e^{i\mathbf{q}\cdot\mathbf{r}}d^3\mathbf{r}$ of the Fermi distribution $\rho(r)=\rho_0[1+e^{(r-c)/a}]^{-1}$ is derived. This expression consists of a simple term with elementary functions plus a rapidly convergent infinite series with terms of alternating sign. Tables of the form factor as a function of q for several values of the parameters c and a, as well as the numerical values of the normalization constant ρ_0 and the rms radius corresponding to these values of the parameters, are also given.

Key Words: Analytic, fermi distribution, form factor, momentum transfer, normalization constant, root-mean-square radius.

In the calculation of a cross section in the Born approximation the spatial distribution of the interaction may be represented by the inclusion of a form factor which is its Fourier transform. The reaction amplitude for a distributed interaction is then the form factor F times the reaction amplitude for a point interaction. The form factor squared thus enters expressions for the cross section. The concept of the form factor has been utilized in atomic physics for some time and now is used widely in nuclear and particle physics to characterize the effects of the spatial distribution of the interaction.

The form factor F is generally given as a function of the momentum q transferred by the incident particle (represented by a plane wave) to the interacting particle. If the spherically symmetric spatial distribution of the interacting particle is given by $\rho(r)$, then the form factor associated with it is given by the Fourier transform of $\rho(r)$:

$$F(q)=\int \rho(r)e^{i\mathbf{q}\cdot\mathbf{r}}d^3\mathbf{r}=\frac{4\pi}{q}\int_0^\infty \rho(r)r \sin qr\, dr. \tag{1}$$

The density is normalized so that

$$\int \rho(r)d^3\mathbf{r}=1., \tag{2}$$

from which

$$F(0)=1. \tag{3}$$

There are several collections of form factors for various models of the spatial distribution (for example, density distributions in nuclei) in the literature [1, 2, 3, 4].

One of the most widely used models is not included in most collections of form factors. It is defined by the spatial distribution:

$$\rho(r)=\rho_0/(1+\exp (r-c)/a) \tag{4}$$

where the normalization constant ρ_0 is chosen so that (2) is satisfied. The parameters a and c determine the shape of the spatial distribution.

This model has been named variously the Fermi smoothed uniform model, [1] the Fermi 2-parameter model, [4] the smoothed uniform model, [5] the Saxon-Woods distribution, [6] or simply the Fermi distribution [7].

An analytic expression for the form factor of the Fermi distribution has been presented but it is incomplete [8]. In this paper the complete expressions for this form factor, mean square radius of the density distribution and normalization constant will be presented together with tabulated evaluations. The effect of simplifying assumptions is examined.

The mean square radius of the distribution is given by

$$\langle r^2 \rangle = \int r^2 \rho(r) d^3\mathbf{r} \tag{5}$$

and may be obtained directly from our analytic expression for $F(q)$: Expanding (1) in a power series, we have, for small q,

$$F(q) = 1 - 1/6 \langle r^2 \rangle q^2 + \ldots \tag{6}$$

so that

$$\langle r^2 \rangle = -6 \left. \frac{\partial F}{\partial (q^2)} \right|_{q=0}. \tag{7}$$

Proceeding now to the derivation of $F(q)$, we have, substituting (4) in (1),

$$F(q) = \frac{4\pi\rho_0}{q} \int_0^\infty \frac{r \sin qr \, dr}{1+e^{(r-c)/a}} = -\frac{4\pi\rho_0}{q} \cdot \mathrm{Re} \frac{\partial}{\partial q} \int_0^\infty \frac{e^{iqr} \, dr}{1+e^{(r-c)/a}}. \tag{8}$$

Making the change of variables

$$x = e^{-r/a}, \quad b = e^{-c/a}, \quad \beta = qa \tag{9}$$

we then obtain

$$F(q) = -\frac{4\pi\rho_0 a^3}{b\beta} \mathrm{Re} \frac{\partial}{\partial \beta} \int_0^1 x^{-i\beta}(1+b^{-1}x)^{-1} dx. \tag{10}$$

The integral appearing here is one form of the hypergeometric function: [9]

$${}_2F_1(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a} dt. \tag{11}$$

Thus

$$\int_0^1 x^{-i\beta}(1+b^{-1}x)^{-1} dx = (1-i\beta)^{-1} \, {}_2F_1(1, 1-i\beta; 2-i\beta; -b^{-1}). \tag{12}$$

Transforming the hypergeometric function in (12) [9]

$${}_2F_1(a, b; c; z) = \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)} (-z)^{-a} {}_2F_1(a, 1-c+a; 1-b+a; z^{-1})$$

$$+ \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)} (-z)^{-b} \, {}_2F_1(b, 1-c+b; 1-a+b; z^{-1}) \tag{13}$$

we then have

$$F(q) = -\frac{4\pi\rho_0 a^3}{\beta} \operatorname{Re} \frac{\partial}{\partial\beta} \left\{ -\frac{1}{i\beta} \; {}_2F_1(1, i\beta; 1+i\beta; -b) + \frac{\pi b^{-i\beta}}{i \sinh \pi\beta} \right\} \tag{14}$$

where we have made use of the fact that if either of its first two parameters is zero, the hypergeometric function is unity, and also that [9]

$$\Gamma(1-i\beta)\Gamma(i\beta) = \frac{\pi}{i \sinh \pi\beta}. \tag{15}$$

Using the power series expansion for the hypergeometric function

$${}_2F_1(a, b; c; z) = 1 + \frac{ab}{c}\frac{z}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{z^2}{2!} + \ldots \tag{16}$$

we have, for the first term in (14)

$$-\frac{1}{i\beta} \; {}_2F_1(1, i\beta; 1+i\beta; -b) = -\frac{1}{i\beta} + \frac{b}{1+i\beta} - \frac{b^2}{2+i\beta} + \ldots \tag{17}$$

from which

$$\operatorname{Re} \frac{\partial}{\partial\beta} \left\{ -\frac{1}{i\beta} \; {}_2F_1(1, i\beta; 1+i\beta; -b) \right\} = -2\beta \left[\frac{b}{(1+\beta^2)^2} - \frac{2b^2}{(2^2+\beta^2)^2} + \frac{3b^3}{(3^2+\beta^2)^2} - \ldots \right]. \tag{18}$$

The evaluation of the second term in (14) is straightforward:

$$\operatorname{Re} \frac{\partial}{\partial\beta} \left\{ \frac{\pi b^{-i\beta}}{i \sinh \pi\beta} \right\} = \frac{-\pi}{\beta \sinh^2 \pi\beta} \left[\pi\beta \cosh(\pi\beta) \sin qc - qc \sinh(\pi\beta) \cos qc \right]. \tag{19}$$

Thus, finally, substituting (18) and (19) in (14) we have, in terms of the original parameters,

$$F(q) = \frac{4\pi^2 \rho_0 a^3}{(qa)^2 \sinh^2(\pi q a)} \left[\pi q a \cosh(\pi q a) \sin(qc) - qc \cos(qc) \sinh(\pi q a) \right] + 8\pi\rho_0 a^3 \sum_{n=1}^{\infty} (-1)^{n-1} \frac{n e^{-\frac{nc}{a}}}{[n^2 + (qa)^2]^2}. \tag{20}$$

This expression, without the infinite series, has been given previously by Blankenbecler [8]. To determine ρ_0, we take the limit of (20) as $q \to 0$. From (3), the left-hand side of (20) is unity, so that

$$\rho_0 = \left\{ \frac{4\pi c}{3} \left[(\pi a)^2 + c^2 \right] + 8\pi a^3 \sum_{n=1}^{\infty} (-1)^{n-1} \frac{e^{-\frac{nc}{a}}}{n^3} \right\}^{-1}. \tag{21}$$

For $\langle r^2 \rangle$, we have, substituting (20) in (7), and carrying out the differentiation,

$$\langle r^2 \rangle = \rho_0 \left\{ \frac{4\pi c}{3} \left[(\pi a)^2 + c^2 \right] \left[\frac{7}{5} (\pi a)^2 + \frac{3}{5} c^2 \right] + 96\pi a^5 \sum_{n=1}^{\infty} (-1)^{n-1} \frac{e^{-\frac{nc}{a}}}{n^5} \right\}. \tag{22}$$

The expressions (21) and (22) have been given previously by Elton [10].

TABLE 1

c	a	ρ_0	ρ_0'	$\langle r^2\rangle^{1/2}$	$\langle r^2\rangle'^{1/2}$
2.	0.	$2.984*10^{-2}$	$2.984*10^{-2}$	1.5492	1.5492
2.	0.2	$2.716*10^{-2}$	$2.716*10^{-2}$	1.7183	1.7183
2.	.4	$2.139*10^{-2}$	$2.140*10^{-2}$	2.1471	2.1473
2.	.6	$1.576*10^{-2}$	$1.580*10^{-2}$	2.7139	2.7156
2.	.8	$1.143*10^{-2}$	$1.157*10^{-2}$	3.3468	3.3531
2.	1.0	$8.365*10^{-3}$	$8.606*10^{-3}$	4.0129	4.0271
2.	1.2	$6.227*10^{-3}$	$6.554*10^{-3}$	4.6969	4.7220
2.	1.4	$4.725*10^{-3}$	$5.113*10^{-3}$	5.3914	5.4297
2.	1.6	$3.654*10^{-3}$	$4.079*10^{-3}$	6.0925	6.1459
2.	1.8	$2.875*10^{-3}$	$3.318*10^{-3}$	6.7979	6.8679
2.	2.	$2.298*10^{-3}$	$2.745*10^{-3}$	7.5063	7.5941

Calculations of $F(q)$, the normalization constant ρ_0, radius $\langle r^2\rangle$ are given respectively in expressions (20), (21), and (22). It is interesting to see under what conditions the summation terms appearing in these expressions can be dropped without serious effect on the accuracy of the calculation. In table 1 the primed values are obtained by dropping the sum terms, resulting in the following expressions:

$$\langle r^2\rangle' = \frac{1}{5}\left[7(\pi a)^2 + 3c^2\right] \tag{23}$$

$$\rho_0' = \left\{\frac{4\pi c}{3}\left[(\pi a)^2 + c^2\right]\right\}^{-1}. \tag{24}$$

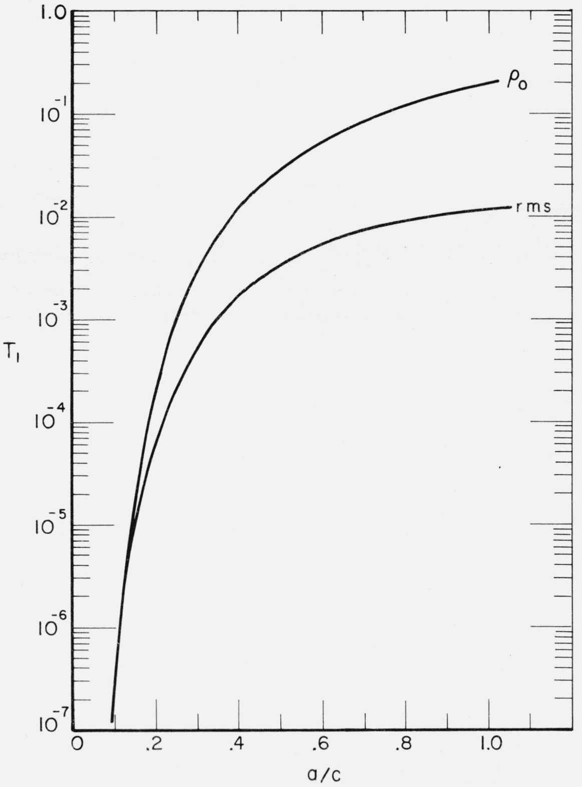

FIGURE 1. *The ordinate is the fraction* T_1 *that the sum terms contribute in the calculation of the normalization constant* ρ_0 *and the rms radius as a function of the ratio* a/c *of the Fermi model parameters.*

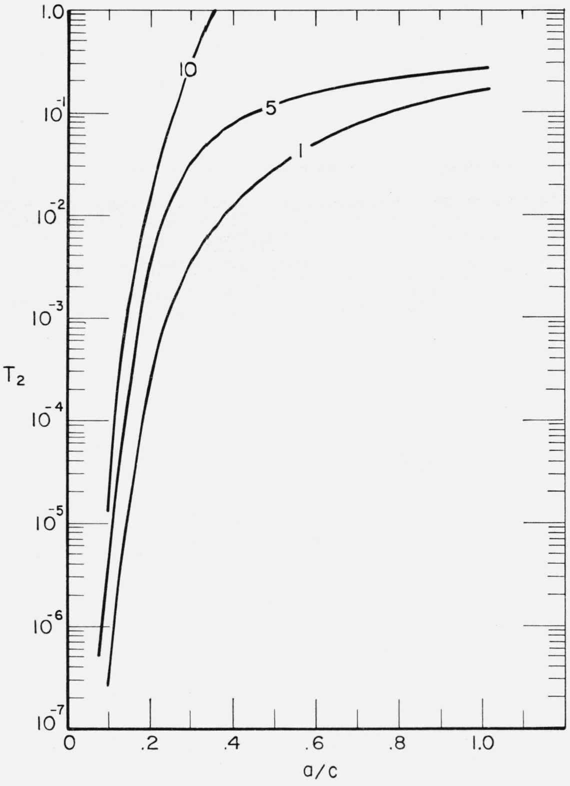

FIGURE 2. *The ordinate is the fraction* T_2 *that the sum terms contribute in the calculation of the form factor as a function of the ratio* a/c *of the Fermi model parameters.*
The curves are labeled with the value of $q\,\langle r^2\rangle^{1/2}$ to which they apply.

TABLE 2

$c=3.00$, $\rho_0=8.047667E-03$			$a=0.30$, rms $=2.577507E\ 00$	
	r/c	r	ρ/ρ_0	ρ
1	0.	0.	10.000E−01	8.047E−03
2	.05	.15	9.999E−01	8.047E−03
3	.10	.30	9.999E−01	8.047E−03
4	.15	.45	9.998E−01	8.046E−03
5	.20	.60	9.997E−01	8.045E−03
6	.25	.75	9.994E−01	8.043E−03
7	.30	.90	9.991E−01	8.040E−03
8	.35	1.05	9.985E−01	8.036E−03
9	.40	1.20	9.975E−01	8.028E−03
10	.45	1.35	9.959E−01	8.015E−03
11	.50	1.50	9.933E−01	7.994E−03
12	.55	1.65	9.890E−01	7.959E−03
13	.60	1.80	9.820E−01	7.903E−03
14	.65	1.95	9.707E−01	7.812E−03
15	.70	2.10	9.526E−01	7.666E−03
16	.75	2.25	9.241E−01	7.437E−03
17	.80	2.40	8.808E−01	7.088E−03
18	.85	2.55	8.176E−01	6.580E−03
19	.90	2.70	7.311E−01	5.883E−03
20	.95	2.85	6.225E−01	5.009E−03
21	1.00	3.00	5.000E−01	4.024E−03
22	1.05	3.15	3.775E−01	3.038E−03
23	1.10	3.30	2.689E−01	2.164E−03
24	1.15	3.45	1.824E−01	1.468E−03
25	1.20	3.60	1.192E−01	9.593E−04
26	1.25	3.75	7.586E−02	6.105E−04
27	1.30	3.90	4.743E−02	3.817E−04
28	1.35	4.05	2.931E−02	2.359E−04
29	1.40	4.20	1.799E−02	1.447E−04
30	1.45	4.35	1.099E−02	8.842E−05
31	1.50	4.50	6.693E−03	5.386E−05
32	1.55	4.65	4.070E−03	3.276E−05
33	1.60	4.80	2.473E−03	1.990E−05
34	1.65	4.95	1.501E−03	1.208E−05
35	1.70	5.10	9.111E−04	7.332E−06
36	1.75	5.25	5.528E−04	4.449E−06
37	1.80	5.40	3.354E−04	2.699E−06
38	1.85	5.55	2.034E−04	1.637E−06
39	1.90	5.70	1.234E−04	9.930E−07
40	1.95	5.85	7.485E−05	6.023E−07
41	2.00	6.00	4.540E−05	3.653E−07
42	2.05	6.15	2.754E−05	2.216E−07
43	2.10	6.30	1.670E−05	1.344E−07
44	2.15	6.45	1.013E−05	8.152E−08
45	2.20	6.60	6.144E−06	4.945E−08
46	2.25	6.75	3.727E−06	2.999E−08
47	2.30	6.90	2.260E−06	1.819E−08
48	2.35	7.05	1.371E−06	1.103E−08
49	2.40	7.20	8.315E−07	6.692E−09
50	2.45	7.35	5.043E−07	4.059E−09
51	2.50	7.50	3.059E−07	2.462E−09

TABLE 3

$c=3.00$, $\rho_0=6.337823E-03$			$a=0.60$, rms $=3.220687E\ 00$	
	r/c	r	ρ/ρ_0	ρ
1	0.	0.	9.933E−01	6.295E−03
2	.05	.15	9.914E−01	6.283E−03
3	.10	.30	9.890E−01	6.268E−03
4	.15	.45	9.859E−01	6.249E−03
5	.20	.60	9.820E−01	6.224E−03
6	.25	.75	9.770E−01	6.192E−03
7	.30	.90	9.707E−01	6.152E−03
8	.35	1.05	9.627E−01	6.101E−03
9	.40	1.20	9.526E−01	6.037E−03
10	.45	1.35	9.399E−01	5.957E−03
11	.50	1.50	9.241E−01	5.857E−03
12	.55	1.65	9.047E−01	5.734E−03
13	.60	1.80	8.808E−01	5.582E−03
14	.65	1.95	8.520E−01	5.400E−03
15	.70	2.10	8.176E−01	5.182E−03
16	.75	2.25	7.773E−01	4.926E−03
17	.80	2.40	7.311E−01	4.633E−03
18	.85	2.55	6.792E−01	4.305E−03
19	.90	2.70	6.225E−01	3.945E−03
20	.95	2.85	5.622E−01	3.563E−03
21	1.00	3.00	5.000E−01	3.169E−03
22	1.05	3.15	4.378E−01	2.775E−03
23	1.10	3.30	3.775E−01	2.393E−03
24	1.15	3.45	3.208E−01	2.033E−03
25	1.20	3.60	2.689E−01	1.705E−03
26	1.25	3.75	2.227E−01	1.411E−03
27	1.30	3.90	1.824E−01	1.156E−03
28	1.35	4.05	1.480E−01	9.383E−04
29	1.40	4.20	1.192E−01	7.555E−04
30	1.45	4.35	9.535E−02	6.043E−04
31	1.50	4.50	7.586E−02	4.808E−04
32	1.55	4.65	6.009E−02	3.808E−04
33	1.60	4.80	4.743E−02	3.006E−04
34	1.65	4.95	3.733E−02	2.366E−04
35	1.70	5.10	2.931E−02	1.858E−04
36	1.75	5.25	2.298E−02	1.456E−04
37	1.80	5.40	1.799E−02	1.140E−04
38	1.85	5.55	1.406E−02	8.913E−05
39	1.90	5.70	1.099E−02	6.963E−05
40	1.95	5.85	8.577E−03	5.436E−05
41	2.00	6.00	6.693E−03	4.242E−05
42	2.05	6.15	5.220E−03	3.308E−05
43	2.10	6.30	4.070E−03	2.580E−05
44	2.15	6.45	3.173E−03	2.011E−05
45	2.20	6.60	2.473E−03	1.567E−05
46	2.25	6.75	1.927E−03	1.221E−05
47	2.30	6.90	1.501E−03	9.514E−06
48	2.35	7.05	1.170E−03	7.412E−06
49	2.40	7.20	9.111E−04	5.774E−06
50	2.45	7.35	7.097E−04	4.498E−06
51	2.50	7.50	5.528E−04	3.503E−06

Figure 1 shows the fraction of the total that the sum terms contribute. In most cases (where $a/c < 0.3$) the short expressions, in which the sum terms are dropped, can be used with adequate accuracy. Figure 2 shows the effect of the sum terms on the form factor calculation. For values of $q\ \langle r^2 \rangle^{1/2} < 1$ the sum terms can be dropped when $a/c < 0.3$, but for large values of $q\ \langle r^2 \rangle^{1/2}$ the sum terms become important for all a/c. Tables 2 to 6 give values of the density distribution and tables 7 to 11 values of the form factor for various values of c and a. It should be noted that the model can be scaled. The important parameters in the description of a model are c and a/c. For example, if:

$$c_2 = mc_1 \tag{25}$$

and

$$\frac{a_2}{c_2}=\frac{a_1}{c_1} \tag{26}$$

then

$$\langle r_2^2\rangle = m^2 \langle r_1^2\rangle, \tag{27}$$

$$\rho_{0_2}=m^{-3}\rho_{0_1}, \tag{28}$$

TABLE 4

$c=3.00$	$\rho_0=4.668352E-03$		$a=0.90$	,rms = 4.070780E 00
	r/c	r	ρ/ρ_0	ρ
1	0.	0.	9.656E−01	4.508E−03
2	.05	.15	9.596E−01	4.480E−03
3	.10	.30	9.526E−01	4.447E−03
4	.15	.45	9.445E−01	4.409E−03
5	.20	.60	9.350E−01	4.365E−03
6	.25	.75	9.241E−01	4.314E−03
7	.30	.90	9.116E−01	4.256E−03
8	.35	1.05	8.972E−01	4.189E−03
9	.40	1.20	8.808E−01	4.112E−03
10	.45	1.35	8.622E−01	4.025E−03
11	.50	1.50	8.411E−01	3.927E−03
12	.55	1.65	8.176E−01	3.817E−03
13	.60	1.80	7.914E−01	3.694E−03
14	.65	1.95	7.625E−01	3.560E−03
15	.70	2.10	7.311E−01	3.413E−03
16	.75	2.25	6.971E−01	3.254E−03
17	.80	2.40	6.608E−01	3.085E−03
18	.85	2.55	6.225E−01	2.906E−03
19	.90	2.70	5.826E−01	2.720E−03
20	.95	2.85	5.416E−01	2.528E−03
21	1.00	3.00	5.000E−01	2.334E−03
22	1.05	3.15	4.584E−01	2.140E−03
23	1.10	3.30	4.174E−01	1.949E−03
24	1.15	3.45	3.775E−01	1.762E−03
25	1.20	3.60	3.392E−01	1.584E−03
26	1.25	3.75	3.029E−01	1.414E−03
27	1.30	3.90	2.689E−01	1.256E−03
28	1.35	4.05	2.375E−01	1.109E−03
29	1.40	4.20	2.086E−01	9.739E−04
30	1.45	4.35	1.824E−01	8.516E−04
31	1.50	4.50	1.589E−01	7.417E−04
32	1.55	4.65	1.378E−01	6.435E−04
33	1.60	4.80	1.192E−01	5.565E−04
34	1.65	4.95	1.028E−01	4.798E−04
35	1.70	5.10	8.840E−02	4.127E−04
36	1.75	5.25	7.586E−02	3.541E−04
37	1.80	5.40	6.497E−02	3.033E−04
38	1.85	5.55	5.555E−02	2.593E−04
39	1.90	5.70	4.743E−02	2.214E−04
40	1.95	5.85	4.044E−02	1.888E−04
41	2.00	6.00	3.445E−02	1.608E−04
42	2.05	6.15	2.931E−02	1.368E−04
43	2.10	6.30	2.492E−02	1.164E−04
44	2.15	6.45	2.118E−02	9.887E−05
45	2.20	6.60	1.799E−02	8.397E−05
46	2.25	6.75	1.527E−02	7.127E−05
47	2.30	6.90	1.295E−02	6.047E−05
48	2.35	7.05	1.099E−02	5.129E−05
49	2.40	7.20	9.316E−03	4.349E−05
50	2.45	7.35	7.897E−03	3.687E−05
51	2.50	7.50	6.693E−03	3.124E−05

TABLE 5

$c=3.00$	$\rho_0=3.387275E-3$		$a=1.20$	rms = 5.020226E 00
	r/c	r	ρ/ρ_0	ρ
1	0.	0.	9.241E−01	3.130E−03
2	.05	.15	9.149E−01	3.099E−03
3	.10	.30	9.047E−01	3.064E−03
4	.15	.45	8.933E−01	3.026E−03
5	.20	.60	8.808E−01	2.984E−03
6	.25	.75	8.670E−01	2.937E−03
7	.30	.90	8.520E−01	2.886E−03
8	.35	1.05	8.355E−01	2.830E−03
9	.40	1.20	8.176E−01	2.769E−03
10	.45	1.35	7.982E−01	2.704E−03
11	.50	1.50	7.773E−01	2.633E−03
12	.55	1.65	7.549E−01	2.557E−03
13	.60	1.80	7.311E−01	2.476E−03
14	.65	1.95	7.058E−01	2.391E−03
15	.70	2.10	6.792E−01	2.301E−03
16	.75	2.25	6.514E−01	2.206E−03
17	.80	2.40	6.225E−01	2.108E−03
18	.85	2.55	5.927E−01	2.008E−03
19	.90	2.70	5.622E−01	1.904E−03
20	.95	2.85	5.312E−01	1.799E−03
21	1.00	3.00	5.000E−01	1.694E−03
22	1.05	3.15	4.688E−01	1.588E−03
23	1.10	3.30	4.378E−01	1.483E−03
24	1.15	3.45	4.073E−01	1.380E−03
25	1.20	3.60	3.775E−01	1.279E−03
26	1.25	3.75	3.486E−01	1.181E−03
27	1.30	3.90	3.208E−01	1.087E−03
28	1.35	4.05	2.942E−01	9.966E−04
29	1.40	4.20	2.689E−01	9.110E−04
30	1.45	4.35	2.451E−01	8.302E−04
31	1.50	4.50	2.227E−01	7.543E−04
32	1.55	4.65	2.018E−01	6.836E−04
33	1.60	4.80	1.824E−01	6.179E−04
34	1.65	4.95	1.645E−01	5.573E−04
35	1.70	5.10	1.480E−01	5.015E−04
36	1.75	5.25	1.330E−01	4.504E−04
37	1.80	5.40	1.192E−01	4.038E−04
38	1.85	5.55	1.067E−01	3.614E−04
39	1.90	5.70	9.535E−02	3.230E−04
40	1.95	5.85	8.510E−02	2.883E−04
41	2.00	6.00	7.586E−02	2.570E−04
42	1.05	6.15	6.755E−02	2.288E−04
43	2.10	6.30	6.009E−02	2.035E−04
44	2.15	6.45	5.340E−02	1.809E−04
45	2.20	6.60	4.743E−02	1.606E−04
46	2.25	6.75	4.209E−02	1.426E−04
47	2.30	6.90	3.733E−02	1.264E−04
48	2.35	7.05	3.309E−02	1.121E−04
49	2.40	7.20	2.931E−02	9.929E−05
50	2.45	7.35	2.596E−02	8.792E−05
51	2.50	7.50	2.298E−02	7.783E−05

and

$$F_2(q)=F_1(mq). \tag{29}$$

The ability to scale the model allows one to use the given tables for any value of the parameter c. The data of table 1 have been reduced to a graph shown in figure 3. This graph shows the relationship of the root-mean-square radius to the model shape (a/c) for the case of $c=1$. To use the graph for other values of c multiply the abscissa scale by the value of c.

TABLE 6

	$c=3.00$, $\rho_0=2.478644\text{E}-03$		$a=1.50$, rms$=6.019361\text{E}\ 00$	
	r/c	r	ρ/ρ_0	ρ
1	0.	0.	8.808E−01	2.183E−03
2	.05	.15	8.699E−01	2.156E−03
3	.10	.30	8.581E−01	2.127E−03
4	.15	.45	8.455E−01	2.096E−03
5	.20	.60	8.320E−01	2.062E−03
6	.25	.75	8.176E−01	2.026E−03
7	.30	.90	8.022E−01	1.988E−03
8	.35	1.05	7.858E−01	1.948E−03
9	.40	1.20	7.685E−01	1.905E−03
10	.45	1.35	7.503E−01	1.860E−03
11	.50	1.50	7.311E−01	1.812E−03
12	.55	1.65	7.109E−01	1.762E−03
13	.60	1.80	6.900E−01	1.710E−03
14	.65	1.95	6.682E−01	1.656E−03
15	.70	2.10	6.457E−01	1.600E−03
16	.75	2.25	6.225E−01	1.543E−03
17	.80	2.40	5.987E−01	1.484E−03
18	.85	2.55	5.744E−01	1.424E−03
19	.90	2.70	5.498E−01	1.363E−03
20	.95	2.85	5.250E−01	1.301E−03
21	1.00	3.00	5.000E−01	1.239E−03
22	1.05	3.15	4.750E−01	1.177E−03
23	1.10	3.30	4.502E−01	1.116E−03
24	1.15	3.45	4.256E−01	1.055E−03
25	1.20	3.60	4.013E−01	9.947E−04
26	1.25	3.75	3.775E−01	9.358E−04
27	1.30	3.90	3.543E−01	8.783E−04
28	1.35	4.05	3.318E−01	8.224E−04
29	1.40	4.20	3.100E−01	7.684E−04
30	1.45	4.35	2.891E−01	7.165E−04
31	1.50	4.50	2.689E−01	6.666E−04
32	1.55	4.65	2.497E−01	6.190E−04
33	1.60	4.80	2.315E−01	5.737E−04
34	1.65	4.95	2.142E−01	5.308E−04
35	1.70	5.10	1.978E−01	4.903E−04
36	1.75	5.25	1.824E−01	4.522E−04
37	1.80	5.40	1.680E−01	4.164E−04
38	1.85	5.55	1.545E−01	3.829E−04
39	1.90	5.70	1.419E−01	3.516E−04
40	1.95	5.85	1.301E−01	3.225E−04
41	2.00	6.00	1.192E−01	2.955E−04
42	2.05	6.15	1.091E−01	2.704E−04
43	2.10	6.30	9.975E−02	2.472E−04
44	2.15	6.45	9.112E−02	2.259E−04
45	2.20	6.60	8.317E−02	2.062E−04
46	2.25	6.75	7.586E−02	1.880E−04
47	2.30	6.90	6.914E−02	1.714E−04
48	2.35	7.05	6.297E−02	1.561E−04
49	2.40	7.20	5.732E−02	1.421E−04
50	2.45	7.35	5.215E−02	1.293E−04
51	2.50	7.50	4.743E−02	1.176E−04

TABLE 7

	$c=3.00$ $a=0.30$, rms radius$=2.578$		
	$q\times$rms	q	Form factor
1	0.	0.	1.000E 00
2	.30	.116	9.851E−01
3	.60	.233	9.414E−01
4	.90	.349	8.721E−01
5	1.20	.466	7.818E−01
6	1.50	.582	6.765E−01
7	1.80	.698	5.630E−01
8	2.10	.815	4.478E−01
9	2.40	.931	3.371E−01
10	2.70	1.048	2.361E−01
11	3.00	1.164	1.485E−01
12	3.30	1.280	7.681E−02
13	3.60	1.397	2.181E−02
14	3.90	1.513	−1.688E−02
15	4.20	1.629	−4.077E−02
16	4.50	1.746	−5.213E−02
17	4.80	1.862	−5.369E−02
18	5.10	1.979	−4.828E−02
19	5.40	2.095	−3.861E−02
20	5.70	2.211	−2.704E−02
21	6.00	2.328	−1.545E−02
22	6.30	2.444	−5.196E−03
23	6.60	2.561	2.887E−03
24	6.90	2.677	8.440E−03
25	7.20	2.793	1.149E−02
26	7.50	2.910	1.234E−02
27	7.80	3.026	1.147E−02
28	8.10	3.143	9.429E−03
29	8.40	3.259	6.764E−03
30	8.70	3.375	3.952E−03
31	9.00	3.492	1.366E−03
32	9.30	3.608	−7.405E−04
33	9.60	3.725	−2.236E−03
34	9.90	3.841	−3.093E−03
35	10.20	3.957	−3.365E−03
36	10.50	4.074	−3.164E−03
37	10.80	4.190	−2.628E−03
38	11.10	4.306	−1.903E−03
39	11.40	4.423	−1.122E−03
40	11.70	4.539	−3.905E−04
41	12.00	4.656	2.143E−04
42	12.30	4.772	6.504E−04
43	12.60	4.888	9.053E−04
44	12.90	5.005	9.911E−04
45	13.20	5.121	9.373E−04
46	13.50	5.238	7.829E−04
47	13.80	5.354	5.699E−04
48	14.10	5.470	3.373E−04
49	14.40	5.587	1.177E−04
50	14.70	5.703	−6.559E−05
51	15.00	5.820	−1.990E−04

TABLE 8

$c=3.00 \quad a=0.60$
rms radius = 3.221

	$q\times$rms	q	Form factor
1	0.	0.	1.000E 00
2	.30	.093	9.851E−01
3	.60	.186	9.417E−01
4	.90	.279	8.734E−01
5	1.20	.373	7.856E−01
6	1.50	.466	6.850E−01
7	1.80	.559	5.783E−01
8	2.10	.652	4.718E−01
9	2.40	.745	3.711E−01
10	2.70	.838	2.800E−01
11	3.00	.931	2.011E−01
12	3.30	1.025	1.358E−01
13	3.60	1.118	8.383E−02
14	3.90	1.211	4.452E−02
15	4.20	1.304	1.637E−02
16	4.50	1.397	−2.394E−03
17	4.80	1.490	−1.366E−02
18	5.10	1.584	−1.923E−02
19	5.40	1.677	−2.074E−02
20	5.70	1.770	−1.959E−02
21	6.00	1.863	−1.687E−02
22	6.30	1.956	−1.345E−02
23	6.60	2.049	−9.902E−03
24	6.90	2.142	−6.635E−03
25	7.20	2.236	−3.865E−03
26	7.50	2.329	−1.685E−03
27	7.80	2.422	−1.013E−04
28	8.10	2.515	9.428E−04
29	8.40	2.608	1.534E−03
30	8.70	2.701	1.772E−03
31	9.00	2.794	1.757E−03
32	9.30	2.888	1.576E−03
33	9.60	2.981	1.305E−03
34	9.90	3.074	9.991E−04
35	10.20	3.167	7.006E−04
36	10.50	3.260	4.355E−04
37	10.80	3.353	2.178E−04
38	11.10	3.446	5.254E−05
39	11.40	3.540	−6.211E−05
40	11.70	3.633	−1.321E−04
41	12.00	3.726	−1.657E−04
42	12.30	3.819	−1.718E−04
43	12.60	3.912	−1.590E−04
44	12.90	4.005	−1.347E−04
45	13.20	4.099	−1.052E−04
46	13.50	4.192	−7.486E−05
47	13.80	4.285	−4.699E−05
48	14.10	4.378	−2.343E−05
49	14.40	4.471	−5.013E−06
50	14.70	4.564	8.158E−06
51	15.00	4.657	1.653E−05

TABLE 9

$c=3.00 \quad a=0.90$
rms radius = 4.071

	$q\times$rms	q	Form factor
1	0.	0.	1.000E 00
2	.30	.074	9.851E−01
3	.60	.147	9.419E−01
4	.90	.221	8.744E−01
5	1.20	.295	7.886E−01
6	1.50	.368	6.914E−01
7	1.80	.442	5.895E−01
8	2.10	.516	4.891E−01
9	2.40	.590	3.948E−01
10	2.70	.663	3.101E−01
11	3.00	.737	2.366E−01
12	3.30	.811	1.749E−01
13	3.60	.884	1.249E−01
14	3.90	.958	8.550E−02
15	4.20	1.032	5.544E−02
16	4.50	1.105	3.325E−02
17	4.80	1.179	1.749E−02
18	5.10	1.253	6.787E−03
19	5.40	1.327	−5.941E−05
20	5.70	1.400	−4.065E−03
21	6.00	1.474	−6.060E−03
22	6.30	1.548	−6.700E−03
23	6.60	1.621	−6.483E−03
24	6.90	1.695	−5.777E−03
25	7.20	1.769	−4.839E−03
26	7.50	1.842	−3.845E−03
27	7.80	1.916	−2.902E−03
28	8.10	1.990	−2.070E−03
29	8.40	2.063	−1.378E−03
30	8.70	2.137	−8.289E−04
31	9.00	2.211	−4.148E−04
32	9.30	2.285	−1.183E−04
33	9.60	2.358	8.052E−05
34	9.90	2.432	2.022E−04
35	10.20	2.506	2.658E−04
36	10.50	2.579	2.877E−04
37	10.80	2.653	2.816E−04
38	11.10	2.727	2.583E−04
39	11.40	2.800	2.259E−04
40	11.70	2.874	1.902E−04
41	12.00	2.948	1.551E−04
42	12.30	3.022	1.230E−04
43	12.60	3.095	9.525E−05
44	12.90	3.169	7.223E−05
45	13.20	3.243	5.388E−05
46	13.50	3.316	3.982E−05
47	13.80	3.390	2.945E−05
48	14.10	3.464	2.212E−05
49	14.40	3.537	1.717E−05
50	14.70	3.611	1.403E−05
51	15.00	3.685	1.219E−05

The Model Parameters. The model parameters c and a which appear in the analytic expression (4) for the density distribution are usually identified with geometric concepts. The parameter c is usually thought of as the radius for which the density is one-half the density at zero radius. The parameter a is usually related to the edge thickness t, defined as the measured distance in which the density falls from 0.9 to 0.1 of the density at zero radius. When $a \ll c$ it can be shown that:

$$t = 4a \ln 3 \simeq 4.4a. \tag{30}$$

TABLE 10

$c=3.00 \quad a=1.20$
rms radius $=5.020$

	$q\times$ rms	q	Form factor
1	0.	0.	1.000E 00
2	.30	.060	9.851E−01
3	.60	.120	9.421E−01
4	.90	.179	8.751E−01
5	1.20	.239	7.905E−01
6	1.50	.299	6.953E−01
7	1.80	.359	5.964E−01
8	2.10	.418	4.995E−01
9	2.40	.478	4.091E−01
10	2.70	.538	3.279E−01
11	3.00	.598	2.574E−01
12	3.30	.657	1.980E−01
13	3.60	.717	1.492E−01
14	3.90	.777	1.100E−01
15	4.20	.837	7.928E−02
16	4.50	.896	5.565E−02
17	4.80	.956	3.788E−02
18	5.10	1.016	2.482E−02
19	5.40	1.076	1.544E−02
20	5.70	1.135	8.893E−03
21	6.00	1.195	4.473E−03
22	6.30	1.255	1.613E−03
23	6.60	1.315	−1.295E−04
24	6.90	1.374	−1.096E−03
25	7.20	1.434	−1.541E−03
26	7.50	1.494	−1.653E−03
27	7.80	1.554	−1.564E−03
28	8.10	1.613	−1.367E−03
29	8.40	1.673	−1.125E−03
30	8.70	1.733	−8.767E−04
31	9.00	1.793	−6.455E−04
32	9.30	1.853	−4.436E−04
33	9.60	1.912	−2.759E−04
34	9.90	1.972	−1.424E−04
35	10.20	2.032	−4.054E−05
36	10.50	2.092	3.382E−05
37	10.80	2.151	8.526E−05
38	11.10	2.211	1.183E−04
39	11.40	2.271	1.372E−04
40	11.70	2.331	1.455E−04
41	12.00	2.390	1.462E−04
42	12.30	2.450	1.418E−04
43	12.60	2.510	1.342E−04
44	12.90	2.570	1.247E−04
45	13.20	2.629	1.145E−04
46	13.50	2.689	1.042E−04
47	13.80	2.749	9.425E−05
48	14.10	2.809	8.501E−05
49	14.40	2.868	7.660E−05
50	14.70	2.928	6.908E−05
51	15.00	2.988	6.241E−05

TABLE 11

$c=3.00 \quad a=1.50$
rms radius $=6.019$

	$q\times$ rms	q	Form factor
1	0.	0.	1.000E 00
2	.30	.050	9.851E−01
3	.60	.100	9.422E−01
4	.90	.150	8.755E−01
5	1.20	.199	7.917E−01
6	1.50	.249	6.978E−01
7	1.80	.299	6.007E−01
8	2.10	.349	5.059E−01
9	2.40	.399	4.178E−01
10	2.70	.449	3.388E−01
11	3.00	.498	2.702E−01
12	3.30	.548	2.121E−01
13	3.60	.598	1.641E−01
14	3.90	.648	1.252E−01
15	4.20	.698	9.414E−02
16	4.50	.748	6.983E−02
17	4.80	.797	5.107E−02
18	5.10	.847	3.679E−02
19	5.40	.897	2.608E−02
20	5.70	.947	1.817E−02
21	6.00	.997	1.241E−02
22	6.30	1.047	8.282E−03
23	6.60	1.096	5.383E−03
24	6.90	1.146	3.387E−03
25	7.20	1.196	2.046E−03
26	7.50	1.246	1.172E−03
27	7.80	1.296	6.259E−04
28	8.10	1.346	3.027E−04
29	8.40	1.395	1.278E−04
30	8.70	1.445	4.782E−05
31	9.00	1.495	2.560E−05
32	9.30	1.545	3.586E−05
33	9.60	1.595	6.194E−05
34	9.90	1.645	9.329E−05
35	10.20	1.695	1.237E−04
36	10.50	1.744	1.496E−04
37	10.80	1.794	1.698E−04
38	11.10	1.844	1.837E−04
39	11.40	1.894	1.919E−04
40	11.70	1.944	1.950E−04
41	12.00	1.994	1.940E−04
42	12.30	2.043	1.898E−04
43	12.60	2.093	1.832E−04
44	12.90	2.143	1.750E−04
45	13.20	2.193	1.657E−04
46	13.50	2.243	1.559E−04
47	13.80	2.293	1.459E−04
48	14.10	2.342	1.360E−04
49	14.40	2.392	1.265E−04
50	14.70	2.442	1.174E−04
51	15.00	2.492	1.088E−04

When $a \simeq c$ the geometrical interpretations of the analytic parameters no longer hold true because the density at zero radius is not ρ_0. For the sake of clarity let c^* and t^* be the actual geometrically measured distances and let

$$R_t = (t - t^*)/t \tag{31}$$

be the fractional change in t and

$$R_c = (c^* - c)/c \tag{32}$$

be the fractional change in c. Figure 4 shows the variation of R_t and R_c as a function of a/c. Note that for $a/c < 0.2$ the fractional change in the parameters is less than 1 percent and thus the usually accepted geometrical interpretations of a and c are valid.

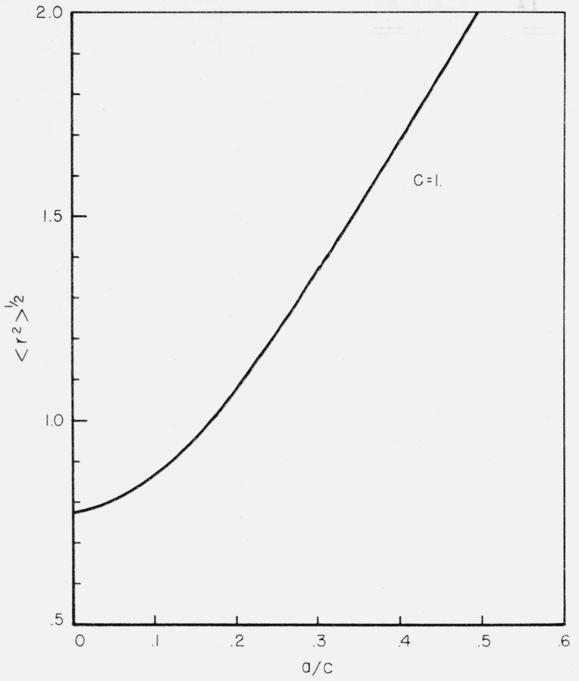

FIGURE 3. *Root-mean-square radius of the spatial distribution for the Fermi distribution with* c = 1. *as a function of* a/c.

To obtain the root-mean-square radius for other values of c simply multiply the result for $c = 1.$ by the value of c.

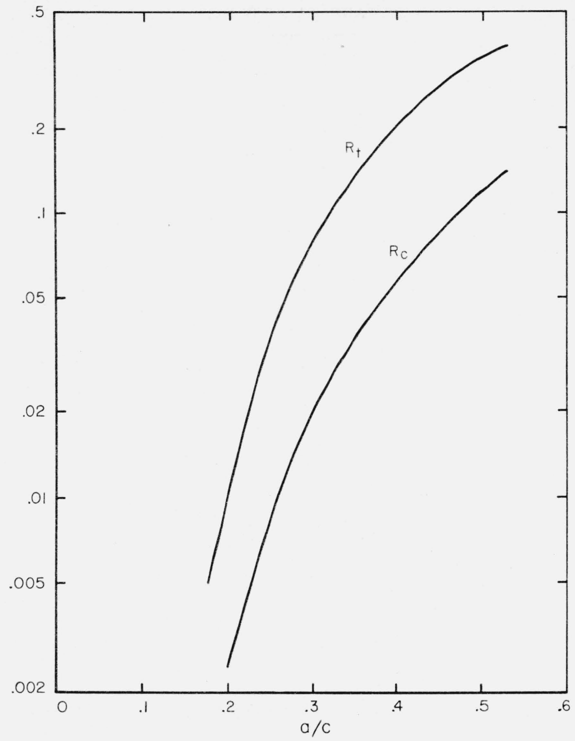

FIGURE 4. *The model parameters* t *and* c *in the analytic representation of the Fermi distribution are normally interpreted in geometrical terms respectively as a measure of the edge thickness and radius at one-half the density at zero radius.*

The curves R_t and R_c indicate respectively the errors in making this correlation between the analytic and geometrical interpretations of t and c.

References and Note

[1] Robert Hofstadter, Rev. Mod. Phys. **28,** 214 (1956).

[2] Robert Hofstadter, Ann. Rev. of Nuclear Science **7,** 231 (1957).

[3] Hofstadter and Herman, High Energy Electron Scattering Tables, (Stanford University Press, Stanford, California).

[4] G. R. Burleson and R. Hofstadter, Phys. Rev. **112,** 1282 (1958).

[5] D. R. Yennie, D. G. Ravenhall, and R. N. Wilson, Phys. Rev. **95,** 500 (1954).

[6] D. S. Saxon and R. D. Woods, Phys. Rev. **95,** 577 (1954).

[7] L. R. B. Elton, Nuclear Sizes (Oxford Univ. Press, London, 1961).

[8] R. Blankenbecler, Am. J. Phys. **25,** 279 (1957).

[9] Magnus and Oberhettinger, Functions of Mathematical Physics, p. 8 (Chelsea Pub. Co., New York, N.Y., 1954).

[10] It should be noted that if we neglect the infinite sums appearing in both (21) and (22) then we find the result given in ref. 7, on p. 28 eq (2.76) for ρ_0, and on p. 12 eq (2.30) and p. 28 eq (2.77) for $\langle r^2 \rangle$. Note, from p. 107 eq (C.4) that the term $10\pi^2a^2/c$ in eq (2.77) should read $10\pi^2a^2/3c^2$, and that after this correction has been made, the expression written there may be written more simply $R^2 = c^2\left(1 + \frac{7}{3}\frac{\pi^2a^2}{c^2}\right)$. The complete expressions for the moments of the distribution including the infinite sums, is given on p. 107 eq (C.2). Note, however, that the last terms in this equation should read e^{-mk} rather than e^{m-k}.

(Paper 70B1–170)