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Remarks on Measurable Sets and Functions

Roy O. Davies [1]

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A. J. Goldman (On measurable sets and functions, J. Res. NBS **69B** (Math. and Math. Phys.) Nos. 1 and 2, 99–100 (1965)) conjectured that the Borel sets are characterized by their property of having measurable inverse images under all Lebesgue measurable functions: here it is pointed out that the existence of analytic non-Borel sets refutes this and a related conjecture. Also an error in Goldman's Theorem 2 is corrected.

Key Words: Measure, integration, real function.

We deal exclusively with subsets of the real line R, and with real-valued functions having R as domain. Let (BS) and (BF) denote the respective families of Borel sets and Borel-measurable functions, while (LS) and (LF) denote the respective families of Lebesgue-measurable sets and functions. Then $f\epsilon(LF)$ if and only if

$$f^{-1}(B)\epsilon(LS) \qquad \text{for all } B\epsilon(BS). \tag{1}$$

Recently Goldman [2] asked whether (1) characterized (BS), in the sense of the following

CONJECTURE: *If* S *is not in* (BS), *then there is an* fϵ(LF) *such that* $f^{-1}(S)$ *is not in* (LS).

We can *disprove* this conjecture as follows. Let

$$n=\{n(1), n(2), \ldots\}$$

be generic notation for an infinite sequence of positive integers. If $\mathfrak{F}$ is a family of sets, then any set

$$\bigcup_{n} \bigcap_{r=1}^{\infty} F(n(1), \ldots, n(r)),$$

where each $F(n(1), \ldots, n(r))\epsilon\mathfrak{F}$, is said to be "obtained from $\mathfrak{F}$ by operation $(\mathscr{A})$". If $\mathscr{A}(\mathfrak{F})$ consists of all sets obtainable from $\mathfrak{F}$ by operation $(\mathscr{A})$, then for any function f,

$$f^{-1}(\mathscr{A}(\mathfrak{F}))=\mathscr{A}(f^{-1}(\mathfrak{F})). \tag{2}$$

When $\mathfrak{F}=(BS)$, $\mathscr{A}(\mathfrak{F})$ is called the class of *analytic sets*, and it is known [3] that

$$(BS) \subset \mathscr{A}(BS) \text{ but } (BS) \neq \mathscr{A}(BS). \tag{3}$$

For any $f\epsilon(LF)$, it follows from (1) and (2) that

$$f^{-1}(\mathscr{A}(BS)) \subset \mathscr{A}(LS). \tag{4}$$

It is also known [4] that (LS) is closed under operation $(\mathscr{A})$, so that (4) implies

$$f^{-1}(\mathscr{A}(BS)) \subset (LS) \qquad \text{for all } f\epsilon(LF). \tag{5}$$

Considering $S\epsilon\mathscr{A}(BS)-(BS)$, as permitted by (3), we are led *via* (5) to a contradiction of the conjecture.

Denote functional composition by an asterisk $((f*g)(x)=f(g(x)))$, and let (LCF) be the class of functions f such that

$$g\epsilon(LF) \text{ implies } f*g\epsilon(LF).$$

Goldman (Theorem 4, *op cit*) also showed that we should have

$$(BF)=(LCF) \tag{6}$$

if the Conjecture were true. That (6) fails together with the Conjecture can be proved by choosing as f the characteristic function of some $S\epsilon\mathscr{A}(BS)-(BS)$; clearly f is not in (BF), but for any $B\epsilon(BS)$ we have $f^{-1}(B)$ a member of $\mathscr{A}(BS)$, namely R or ϕ or S or $R-S$,

[1] Department of Mathematics, The University, Leicester, United Kingdom.

[2] A. J. Goldman, On measurable sets and functions, J. Res. NBS **69B** (Math. and Math. Phys.) Nos. 1 and 2, 99–100 (1965).

[3] K. Kuratowski, Topologie **I**, 2d ed. (Warsaw, 1948), p. 391.

[4] K. Kuratowski, *op. cit.*, p. 64.

so that for any $g\epsilon(LF)$ it follows from (5) that

$$(f*g)^{-1}(B)=g^{-1}(f^{-1}(B))\epsilon(LS),$$

proving $f*g\epsilon(LF)$ and hence $f\epsilon(LCF)$.

Thus the problem of finding a satisfactory characterization of (LCF) remains open. If (QS) is the class of sets Q such that

$$g^{-1}(Q)\epsilon(LS) \qquad \text{for all } g\epsilon(LF),$$

then $f\epsilon(LCF)$ if and only if

$$f^{-1}(B)\epsilon(QS) \qquad \text{for all } B\epsilon(BS).$$

Hence characterizing (LCF) is closely related to characterizing (QS).

Finally, Goldman's Theorem 2 (*op cit*) should be amended to read as follows:

THEOREM: *For any* $\mathrm{B}\epsilon(\mathrm{BS})$ *and* $\mathrm{L}\epsilon(\mathrm{LS})$, *with sole exceptions* $(\mathrm{B}=\phi,\ \mathrm{L}\neq\phi)$ *and* $(\mathrm{B}=\mathrm{R},\ \mathrm{L}\neq\mathrm{R})$, *there is an* $\mathrm{f}\epsilon(\mathrm{LF})$ *such that* $\mathrm{L}=\mathrm{f}^{-1}(\mathrm{B})$.

PROOF: If $B=\phi$ and $L=\phi$, or $B=R$ and $L=R$, then any $f\epsilon(LF)$ will do. If $B=\phi$ and $L\neq\phi$, or $B=R$ and $L\neq R$, then no f will do. Finally, if $B\neq\phi$ and $B\neq R$, then we can define f on L so that $f(L)\subset B$, and on $R-L$ so that $f(R-L)\subset R-B$.

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