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On *EPr* and Normal *EPr* Matrices[1]

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(1) *EPr* matrices A (that is, matrices A for which A and A^* have the same null space) are investigated. It is shown that if A is a complex EPr_1 matrix and B a complex EPr_2 matrix, and $AB=BA$, then AB is *EPr*. Other theorems about products of *EPr* matrices are established.

(2) Let A be a normal *EPr* matrix over an arbitrary field. A necessary and sufficient condition, involving the solvability (for X) of a matrix equation $XBX^*+AX+X^*A^*+C=0$, is found for the existence of a matrix N such that (i) $NN^*=I$ and (ii) $A^*=NA=AN$. Explicit solutions are given for two important classes of normal *EPr* matrices, namely (1) those satisfying the condition rank $A=$ rank AA^*, and (2) those of rank $n/2$, satisfying $AA^*=0$, over a field of characteristic $\neq 2$. An example is given to show that no such N need exist for characteristic $=2$.

(3) *EP* linear transformations on a finite-dimensional vector space are introduced, and the relation between them and *EPr* matrices is studied. It is shown that a linear transformation T of a complex vector space is *EP* if and only if rank $T=$ rank T^2.

Key Words: Matrix, normal, linear algebra.

1. Introduction

The concept of a normal matrix with entries from the complex field was introduced in 1918 by O. Toeplitz [19][3] who gave a necessary and sufficient condition that a complex matrix be normal. Since that time normal matrices and the generalization to linear transformations on a finite dimensional or infinite dimensional vector space [3, 7, 9, 18, 22, 23], to functional analysis, especially Hilbert space [5, 12, 13, 14], and to combinatorial analysis and the study of finite projective planes [1, 2, 4] have received a great deal of attention. Also, special types of normal matrices and linear transformations have been studied. But, until the appearance of [10] in 1959, no study had been made of normal matrices without restrictions on the underlying field.

First, results about *EPr* complex matrices, a concept introduced by H. Schwerdtfeger in [15] as a generalization of normality, were obtained and then in [10, 11] the notion of *EPr* was extended to matrices over arbitrary fields and applied to obtain results about normal matrices. One interesting feature of the study was the discovery that over an arbitrary field the concepts of normal and *EPr* are independent and that many of the well-known properties of complex normal matrices which do not carry over to an arbitrary field appear to generalize most naturally to matrices that are both *EPr* and normal. A matrix A of rank r is called *EPr* if A and A^* have the same null space.

The first section of this paper is concerned solely with developing the properties of *EPr* matrices. Real and complex *EPr* matrices are studied for their own inherent interest and a number of new results have been obtained. It is also shown how these results depend on the underlying field. Finally the structure of *EPr* matrices over an arbitrary field is developed, primarily for its use as a tool in section 3, which is devoted to normal matrices.

In section 3, a resumé of the known results concerning normal matrices over an arbitrary field is given. Then some questions raised in [10, 11] are considered and the concept of zero-type *EPr* matrix is introduced. For this new class of matrices a satisfactory solution is obtained to a problem dealt with in [10, 11]. Most of the results are partial results and interesting questions still remain to be answered.

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[3] Figures in brackets indicate the literature references at the end of this paper.

The material in section 4 is an outgrowth of an attempt to bring to bear on the study at hand the tools and methods of linear transformation theory. The concept of an *EP* linear transformation is introduced and related to the notion of *EPr* matrix. Various results are given, indicating how the study of these special linear transformations will yield results about *EPr* matrices. The concept of a θ-normal linear transformation, generalizing the concept of a normal linear transformation, is introduced and exploited to obtain anew the results of [10, 11].

We have tried to make this paper complete, readable and self-contained. It is hoped that because of our exposition here, some of the questions which we have left open (see sec. 5) will be resolved in the near future.

2. Structure Theory of *EPr* Matrices

1. In this section we develop the structure theory of *EPr* matrices over an arbitrary field F, essentially as given in [10]. However, in order to indicate clearly the difficulties encountered in extending results from the complex field to an arbitrary field because of the lack of a spectral theorem, to place in a proper setting those results which have not been extended, and to provide new results about *EPr* matrices with complex entries which give rise to additional questions over an arbitrary field, we begin with a comprehensive treatment of the complex case, basing the discussion on [10].

2. We begin by defining the notion of *EPr* matrix over the complex field.

Except for denoting the $k\times k$ identity matrix by I_k, subscripts on matrices will be used only to designate a row of a matrix; that is, A_i is the ith row of the matrix A. A^i will denote the ith column of A; when a superscript denotes a power of a matrix, the meaning will always be clear from the context.

DEFINITION: An $n\times n$ matrix A with entries from the complex field $\mathscr{C}$ is called *EPr* if it satisfies the following conditions.

(1) A has rank r.

(2) $\sum_{i=1}^{n} \alpha_i A_i=0$ if and only if $\sum_{i=1}^{n} \overline{\alpha}_i A^i=0$ ($\alpha_i \epsilon \mathscr{C}, i=1, \ldots, n$).

As indicated in the introduction, this paper is concerned with matrices that are both *EPr* and normal and whose elements are taken from an arbitrary field. Hence we note immediately that these concepts are not independent over $\mathscr{C}$. After proving theorem 1.1 which gives eight necessary and sufficient conditions that a matrix be *EPr*, we shall be able to give a simple proof that any normal complex matrix of rank r is *EPr*. The converse statement is false, though, since all nonsingular $n\times n$ matrices are necessarily *EPn*. For, assuming the $n\times n$ matrix A is nonsingular,

$$\sum_{i=1}^{n} \alpha_i A_i=0$$

if and only if

$$\alpha_1=\alpha_2=\ldots=\alpha_n=0$$

if and only if

$$\sum_{i=1}^{n} \overline{\alpha}_i A^i=0.$$

A portion of this next result is known [15], but is included for completeness. The proof presented is a slight modification and expansion of that found in [10].

THEOREM 1.1: *The following statements are equivalent.*

(1) A *is an* $n \times n$ EPr *matrix,*

(2) A *is unitarily similar to the direct sum of a nonsingular* $r \times r$ *matrix* D *and a zero matrix. That is,*

$$UAU^* = \left[\begin{array}{c|c} D & 0 \\ \hline 0 & 0 \end{array}\right]$$

where U *satisfies* $UU^* = I$ *and* D *is* $r \times r$ *and nonsingular.*

(3) A *is conjunctive to the direct sum of a nonsingular* $r \times r$ *matrix* D *and a zero matrix. That is*

$$QAQ^* = \left[\begin{array}{c|c} D & 0 \\ \hline 0 & 0 \end{array}\right]$$

where Q *is nonsingular,* D *is* $r \times r$ *and nonsingular.*

(4) A *is the matrix of a linear transformation* T *acting on* $\mathscr{C}_n$, *complex* n*-dimensional Euclidean space, and there are mutually orthogonal subspaces* V_1 *and* V_2 *of* $\mathscr{C}_n$ *such that* V_1 *has dimension* r,

$$T(V_1) = V_1, \ T(V_2) = 0$$

and

$$\mathscr{C}_n = V_1 \oplus V_2.$$

(5) A *has rank* r *and there is an* $n \times n$ *matrix* N *such that* $A^* = NA$.

(6) A *has rank* r *and there is a nonsingular* $n \times n$ *matrix* N *such that* $A^* = NA$.

(7) A *can be represented as*

$$A = P \left[\begin{array}{c|c} D & DX^* \\ \hline XD & XDX^* \end{array}\right] P^* = P \left[\begin{array}{c|c} I_r & 0 \\ \hline X & I \end{array}\right] \left[\begin{array}{c|c} D & 0 \\ \hline 0 & 0 \end{array}\right] \left[\begin{array}{c|c} I_r & X^* \\ \hline 0 & I \end{array}\right] P^*$$

where P *is a permutation matrix and* D *is an* $r \times r$ *nonsingular matrix.*

(8) $A\xi = 0$ *if and only if* $A^*\xi = 0$ *where* $\xi \epsilon \mathscr{C}_n$.

(9) $R(A) = R(A^*)$; *that is,* A *and* A^* *have the same range spaces.*

PROOF: The implications are proved in the following order:

$$(8) \Rightarrow (1) \Rightarrow (7) \Rightarrow (3) \Rightarrow (6) \Rightarrow (5) \Rightarrow (8);$$
$$(1) \Rightarrow (2) \Rightarrow (4) \Rightarrow (1);$$
$$(5) \Rightarrow (9) \Rightarrow (5).$$

(8) $\Rightarrow$ (1). Suppose $\sum_{i=1}^{n} \alpha_i A_i = 0$ $(\alpha_i \epsilon \mathscr{C}, i = 1, 2, \ldots, n)$.

Let

$$\xi = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \cdot \\ \cdot \\ \cdot \\ \alpha_n \end{bmatrix} \epsilon \mathscr{C}_n.$$

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Then

$$\xi^t A = [\alpha_1 \alpha_2 \ . \ . \ . \ \alpha_n] A = \sum_{i=1}^{n} \alpha_i A_i = 0.$$

Hence, taking the conjugate transpose of this equation, it follows that

$$A^* \bar{\xi} = 0.$$

Applying (8) we have

$$A \bar{\xi} = 0.$$

Thus

$$\sum_{i=1}^{n} \bar{\alpha}_i A^i = 0.$$

The implication in the other direction is proved similarly.

(1) $\Rightarrow$ (7). Let A be EPr and let the rows $A_{i_1}, A_{i_2}, \ . \ . \ ., A_{i_r}$ be linearly independent. If

$$\sum_{k=1}^{r} \beta_{i_k} A^{i_k} = 0,$$

then

$$\sum_{k=l}^{r} \bar{\beta}_{i_k} A_{i_k} = 0$$

and hence $\bar{\beta}_{i_1} = \bar{\beta}_{i_2} \ . \ . \ . = \bar{\beta}_{i_r} = 0$. Thus $\beta_{i_1} = \beta_{i_2} = . \ . \ . = \beta_{i_r} = 0$ and so the columns $A^{i_1}, A^{i_2}, \ . \ . \ .,$ A^{i_r} are linearly independent. Since the rank of A is r, the submatrix D formed by the elements in the intersection of rows $A_{i_1}, A_{i_2}, \ . \ . \ ., A_{i_r}$ and the columns $A^{i_1}, A^{i_2}, \ . \ . \ ., A^{i_r}$ is an $r \times r$ nonsingular matrix [15, p. 52]. This "crossing theorem" can be quickly proved as follows. There is a permutation matrix $\tilde{P}$ such that A is brought into the form

$$B = \tilde{P} A \tilde{P}^* = \left[\begin{array}{c|c} D & E \\ \hline F & G \end{array} \right]$$

by premultiplying A by $\tilde{P}$ and postmultiplying by $\tilde{P}^*$. Since the first block row of B is of the same rank r as is A and thus B, there is an $(n-r) \times r$ matrix H such that

$$[FG] = H[D, E],$$

and hence, by (1), such that

$$\begin{bmatrix} E \\ G \end{bmatrix} = \begin{bmatrix} D \\ F \end{bmatrix} H^*.$$

If D had rank $< r$, then $D\xi = 0$ would hold for some $\xi \epsilon \mathscr{C}_r$, and so

$$\begin{bmatrix} D \\ F \end{bmatrix} \xi = \begin{bmatrix} D\xi \\ HD\xi \end{bmatrix} = 0,$$

contradicting the assumption that the first block column of B has rank r.

With H as above, let

$$\tilde{Q} = \left[\begin{array}{c|c} I_r & 0 \\ \hline -H & I_{n-r} \end{array}\right].$$

It is readily verified that

$$\left[\begin{array}{c|c} D & 0 \\ \hline 0 & 0 \end{array}\right] = \tilde{Q}B\tilde{Q}^* = \tilde{Q}\tilde{P}A\tilde{P}^*\tilde{Q}^*$$

so [4]

$$A = \tilde{P}^{-1}\tilde{Q}^{-1} \left[\begin{array}{c|c} D & 0 \\ \hline 0 & 0 \end{array}\right] \tilde{Q}^{-*}\tilde{P}^{-*}$$

$$= \tilde{P}^{-1} \left[\begin{array}{c|c} I_r & 0 \\ \hline -\tilde{X} & I \end{array}\right] \left[\begin{array}{c|c} D & 0 \\ \hline 0 & 0 \end{array}\right] \left[\begin{array}{c|c} I_r & -\tilde{X}^* \\ \hline 0 & I \end{array}\right] \tilde{P}^{-*}.$$

Set $P = \tilde{P}^{-1}$, $X = -\tilde{X}$, $Q = \tilde{Q}^{-1}$.

(7) $\Rightarrow$ (3). Set

$$Q = \left[\begin{array}{c|c} I_r & 0 \\ \hline X & I \end{array}\right].$$

(3) $\Rightarrow$ (6). If

$$QAQ^* = \left[\begin{array}{c|c} D & 0 \\ \hline 0 & 0 \end{array}\right]$$

where Q is nonsingular, D is $r \times r$ nonsingular, then

$$A^* = Q^{-1}\left[\begin{array}{c|c} D^* & 0 \\ \hline 0 & 0 \end{array}\right]Q^{-*} = Q^{-1}\left[\begin{array}{c|c} D^*\ D^{-1} & 0 \\ \hline 0 & I \end{array}\right]QQ^{-1}\left[\begin{array}{c|c} D & 0 \\ \hline 0 & 0 \end{array}\right]Q^{-*} = NA$$

where

$$N = Q^{-1}\left[\begin{array}{c|c} D^*\ D^{-1} & 0 \\ \hline 0 & I \end{array}\right]Q$$

and N is nonsingular.

[4] We use the notation $\tilde{P}^{-*} = (\tilde{P}^{-1})^* = (\tilde{P}^*)^{-1}$.

(6) $\Rightarrow$ (5). Clear.

(5) $\Rightarrow$ (8). Let $A^*=NA$. Let $\eta(A)$ denote the null space of A and let $\xi\epsilon\eta(A)$, i.e., $A\xi=0$. Then $A^*\xi=0$. Thus $\eta(A)\subseteq\eta(A^*)$. But, since rank $A=$ rank A^*, it follows that dim $\eta(A)=n-$ rank A $=n$-rank $A^*=$ dim $\eta(A^*)$ and so $\eta(A)=\eta(A^*)$. Thus if $A^*\xi=0$, then $A\xi=0$.

(1) $\Rightarrow$ (2). By the well-known result of Schur [17], there is a unitary matrix V such that

$$B=VAV^*=\begin{bmatrix} \ddots & * \\ 0 & \ddots \end{bmatrix},$$

i.e., B is lower triangular. Moreover, the diagonal elements may be arranged in any order. We assume that the nonzero terms precede the zero terms on the diagonal of B. If $r=n$, there is nothing more to prove. Hence we assume $r<n$. Then 0 is a characteristic root of A and appears on the diagonal of B. Thus the last row of VAV^* is a row of zeros. Hence

$$0=0\cdot B_1+\ldots+0\cdot B_{n-1}+1\cdot B_n$$

Since B is EPr we have $0=0\cdot B^1+\ldots+0\cdot B^{n-1}+1\cdot B^n$. Repeating this argument shows that $B_i=0$ if and only if $B^i=0$ and so

$$VAV^*=\left[\begin{array}{c|c} D & 0 \\ \hline 0 & 0 \end{array}\right]$$

where D is $r\times r$ and nonsingular.

(2) $\Rightarrow$ (4) Let

$$UAU^*=\left[\begin{array}{c|c} D & 0 \\ \hline 0 & 0 \end{array}\right]$$

where $UU^*=I$ and D is $r\times r$ and nonsingular. Suppose

$$U=[u_{ij}]\ (1\leqq i, j\leqq n)$$

and

$$D=[d_{ij}]\ (1\leqq i, j\leqq r).$$

Let $e_1, e_2, \ldots, e_n$ denote the "natural basis" of $\mathscr{C}_n$, that is

$$e_i=\begin{bmatrix} \delta_{i1} \\ \delta_{i2} \\ \vdots \\ \delta_{in} \end{bmatrix}(i=1, \ldots, n)$$

where δ_{ij} is the Kronecker delta. Define the following linear transformations:

$$Re_i=\sum_{\lambda=1}^{r} d_{\lambda i}e_\lambda \qquad (i=1, \ldots, r)$$

$$Re_i=0 \qquad (i=r+1, \ldots, n)$$

$$Se_i = \sum_{\lambda=1}^{n} u_{\lambda i} e_\lambda \qquad (i=1,\ldots,n).$$

$$T = SRS^{-1}$$

Then S is unitary, that is, preserves the Euclidean inner product and the matrix of T is

$$U^{-1}\left[\begin{array}{c|c} D & 0 \\ \hline 0 & 0 \end{array}\right] U = A$$

relative to $e_1, \ldots, e_n$. Set

$$\overline{V}_1 = < e_1, \ldots, e_r >,$$

$$\overline{V}_2 = < e_{r+1}, \ldots, e_n >,$$

the spaces spanned by $e_1, \ldots, e_r$ and $e_{r+1}, \ldots, e_n$ respectively. Then set $V_1 = S\overline{V}_1$ and $V_2 = S\overline{V}_2$. Hence

$$TV_1 = T(S\overline{V}_1) = SR\overline{V}_1 = S\overline{V}_1 = V_1$$

and

$$TV_2 = T(S\overline{V}_2) = SR\overline{V}_2 = S(0) = 0$$

Finally, $0 = (\overline{v}_1, \overline{v}_2) = (S\overline{v}_1, S\overline{v}_2)$, $\overline{v}_i \epsilon \overline{V}_i (i=1, 2)$ so that V_1 and V_2 are orthogonal subspaces.

(4)$\Rightarrow$ (1). We prove this by showing that (4) implies (8). Suppose

$$A\xi = 0, \qquad \xi \epsilon \mathscr{C}_n, \qquad \xi = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}$$

Let $x_1, x_2, \ldots, x_n$ be the basis for $\mathscr{C}_n$ relative to which the matrix of T is A. Let V_1 have basis $v_1, \ldots, v_r$ and V_2 have basis $v_{r+1}, \ldots, v_n$. $A\xi = 0$ if and only if $\sum_{i=1}^{n} \alpha_i x_i \epsilon V_2$. Then, for $1 \leqq j \leqq r$,

$$\left(v_j, T^*\left(\sum_{i=1}^{n} \alpha_i x_i\right)\right) = \left(Tv_j, \sum_{i=1}^{n} \alpha_i x_i\right) = 0$$

since $Tv_j \epsilon V_1$ and $\sum_{i=1}^{n} \alpha_i x_i \epsilon V_2$, and for $r+1 \leqq j \leqq n$

$$\left(v_j, T^*\left(\sum_{i=1}^{n} \alpha_i x_i\right)\right) = \left(Tv_j, \sum_{i=1}^{n} \alpha_i x_i\right) = \left(0, \sum_{i=1}^{n} \alpha_i x_i\right) = 0.$$

Thus $T^*\left(\sum_{i=1}^{n} \alpha_i x_i\right)$ is in the orthogonal complement of $\mathscr{C}_n$, which is (0).

Hence $T^*\left(\sum_{i=1}^{n} \alpha_i x_i\right)=0$ so that $A^*\xi=0$.

(5) $\Rightarrow$ (9). $A^*x=NAx$ so $R(A) \subseteq R(A^*)$; since rank $A=$ rank A^*, the equality follows.
(9) $\Rightarrow$ (5). This follows from [16], p. 92.

COROLLARY 1: *If the complex matrix* A *is* EPr, *then* A^n *is* EPr ($n=1, 2, \ldots$).
PROOF: Apply (2). If

$$UAU^*=\left[\begin{array}{c|c} D & 0 \\ \hline 0 & 0 \end{array}\right],$$

then

$$UA^nU^*=\left[\begin{array}{c|c} D^n & 0 \\ \hline 0 & 0 \end{array}\right].$$

COROLLARY 2: *Every matrix is a product of* EP *matrices.*[5]
PROOF: It is well known that an $n \times n$ matrix of rank r may be written as

$$P\left[\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right]Q$$

where P and Q are nonsingular.

COROLLARY 3: *A complex normal matrix* A *is* EP.
PROOF: Since A is normal there is a unitary matrix U such that UAU^* is diagonal [15], and hence (2) of the theorem may be applied.

We shall return to such questions in section 3 and see that the situation is quite different over arbitrary fields.

An example is given in subsection 4 following, to show that corollary 3 need not be valid over an arbitrary field.

3. If A is an EPr_1 matrix and B is an EPr_2 matrix, then AB may not be an EP matrix. For example, over $\mathscr{C}$,

$$A=\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, B=\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

are normal, hence EP_1. But the product

$$C=AB=\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

is not EP_1 since $C\begin{pmatrix} 1 \\ 0 \end{pmatrix}=0$, but $C^*\begin{pmatrix} 1 \\ 0 \end{pmatrix}=\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

We shall prove that commutativity of A and B alters the above result.

For $V \subseteq \mathscr{C}_n$, let $V^\perp$ denote the orthogonal complement of V under the usual inner product. Also, let $\eta(A)$, $\eta(B)$, and $\eta(AB)$ be the null spaces of A, B, and AB respectively.

[5] A matrix will be called EP if it is EPr for some r.

LEMMA 1: *Let* A *and* B *be complex* n × n *matrices satisfying* AB = BA. *Then*

$$A^*B^*\mathscr{C}_n \subseteq [\eta(A) + \eta(B)]^\perp$$

PROOF: Let $v \epsilon \mathscr{C}_n$, $x \epsilon \eta(A)$, $y \epsilon \eta(B)$.

$$(A^*B^*v,\ x+y) = (v,\ BAx + ABy) = (v,\ 0) = 0$$

so that $A^*B^*v \epsilon [\eta(A) + \eta(B)]^\perp$

LEMMA 2: *With* A *and* B *as in lemma* 1, $A^*B^*\mathscr{C}_n \subseteq \eta(AB)^\perp$
PROOF: Let $v \epsilon \mathscr{C}_n$, $z \epsilon \eta(AB)$.

$$(A^*B^*v,\ z) = (v,\ BAz) = (v,\ 0) = 0$$

so $A^*B^*v \epsilon\ \eta(AB)^\perp$.

LEMMA 3: *Again, take* A *and* B *as in lemma* 1. *Then*

$$B^*(\eta(A)^\perp) \subseteq \eta(A)^\perp$$

PROOF: Let $x \epsilon \eta(A)^\perp$, $y \epsilon \eta(A)$.

$$ABy = B(Ay) = 0$$

so $By \epsilon \eta(A)$. Hence

$$(B^*x,\ y) = (x,\ By) = 0$$

and $B^*x \epsilon \eta(A)^\perp$

THEOREM 1.2: *Let* A *be a complex* EPr_1 *matrix and* B *a complex* EPr_2 *matrix satisfying* AB = BA. *Then*

$$\eta(A) + \eta(B) = \eta(AB).$$

PROOF: Clearly $\eta(A) + \eta(B) \subseteq \eta(AB)$. The reverse inclusion is obtained by showing that

$$[\eta(A) + \eta(B)]^\perp \subseteq \eta(AB)^\perp.$$

First note that

$$\eta(A) \subseteq \eta(A) + \eta(B)$$

so

$$[\eta(A) + \eta(B)]^\perp \subseteq \eta(A)^\perp.$$

Similarly, $[\eta(A) + \eta(B)]^\perp \subseteq \eta(B)^\perp$ and thus

$$[\eta(A) + \eta(B)]^\perp \subseteq \eta(A)^\perp \cap \eta(B)^\perp.$$

Denote $\eta(A)^\perp \cap \eta(B)^\perp$ by W.

We wish to show that B is one-one on $\eta(B)^{\perp}$, that is, if $x_1, x_2 \epsilon \eta(B)^{\perp}$ and $Bx_1 = Bx_2$ then $x_1 = x_2$. If $B(x_1 - x_2) = 0$ then $x_1 - x_2 \epsilon \eta(B) \cap \eta(B)^{\perp}$. However $\eta(B) \cap \eta(B)^{\perp} = (0)$ because, with the usual inner product, $\mathscr{C}_n$ has no isotropic vectors $\neq 0$. Similarly, it follows that A is one-one on $\eta(A)^{\perp}$.

Now we may prove that A^*B^* is one-one on W. If $x_1, x_2 \epsilon W$ and $A^*B^*(x_1 - x_2) = 0$, then $A^*(B^*(x_1 - x_2)) = 0$ and since A is EPr it follows that $A(B^*(x_1 - x_2)) = 0$. Thus $B^*(x_1 - x_2) \epsilon \eta(A)$. But by lemma 3, $B^*(x_1 - x_2) \epsilon \eta(A)^{\perp}$ since $x_1 - x_2 \epsilon \eta(A)^{\perp}$. Therefore $B^*(x_1 - x_2) = 0$ so that $B(x_1 - x_2) = 0$ and since $x_1 - x_2 \epsilon \eta(B)^{\perp}$, it follows that $x_1 - x_2 = 0$.

Also $A^*B^*[\eta(A) + \eta(B)]^{\perp} \subseteq A^*B^*\mathscr{C}_n \subseteq [\eta(A) + \eta(B)]^{\perp}$ by lemma 1, and because A^*B^* is one-one on W we have

$$A^*B^*[\eta(A) + \eta(B)]^{\perp} = [\eta(A) + \eta(B)]^{\perp}.$$

Finally,

$$[\eta(A) + \eta(B)]^{\perp} = A^*B^*[\eta(A) + \eta(B)]^{\perp} \subseteq A^*B^*\mathscr{C}_n \subseteq \eta(AB)^{\perp}$$

by lemma 2, completing the proof.

It is not needed in the above proof, but we can also show that $[\eta(A) + \eta(B)]^{\perp} = W$. For

$$\begin{aligned} A^*B^*W &\subseteq [\eta(A) + \eta(B)]^{\perp} \qquad \text{(by lemma 1)} \\ &\subseteq W \end{aligned}$$

and since A^*B^* is one-one on W it follows that $A^*B^*W = W$.

THEOREM 1.3: *If* A *is a complex* EPr_1 *matrix and* B *is a complex* EPr_2 *matrix such that* AB = BA, *then* AB *is an* EPr *matrix.*

PROOF: Suppose $ABz = 0$. Then $z \epsilon \eta(AB) = \eta(A) + \eta(B)$. Hence there exist $x \epsilon \eta(A)$, $y \epsilon \eta(B)$ such that $z = x + y$. Furthermore since $\eta(A) = \eta(A^*)$ and $\eta(B) = \eta(B^*)$ we have

$$B^*A^*z = B^*A^*x + A^*B^*y = 0 + 0 = 0$$

which completes the proof.

A theorem of N. Wiegmann [20, 9] asserts that the normality of the complex matrices A, B, and AB implies the normality of BA. The corresponding result about EPr matrices is false. For example,

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

are EP_2 and

$$AB = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is EP_1. However,

$$BA = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

has rank 1 and is not EP_1.

The following is an example of what can be said in the absence of commutativity.

THEOREM 1.4: *Let rank* $AB =$ *rank* $B = r_1$ *and rank* $BA =$ *rank* $A = r_2$. *If* AB *and* B *are* EPr_1 *and* A *is* EPr_2 *then* BA *is* EPr_2.

PROOF: If $A\xi = 0$, then $BA\xi = 0$ and therefore $\eta(A) \subseteq \eta(BA)$. However, since rank $A =$ rank BA, we have $\eta(A) = \eta(BA)$. Similarly, $\eta(B) = \eta(AB)$.

Then $BA\xi = 0$

$\Leftrightarrow A\xi = 0$

$\Leftrightarrow A^*\xi = 0$ since A is EPr_2

$\Leftrightarrow B^*A^*\xi = 0$

$\Leftrightarrow AB\xi = 0$ since AB is EPr_2

$\Leftrightarrow B\xi = 0$

$\Leftrightarrow B^*\xi = 0$

$\Leftrightarrow A^*B^*\xi = 0$.

Hence $\eta(BA) \subseteq \eta(A^*B^*) = \eta((BA)^*)$. But rank $BA =$ rank $(BA)^*$ and therefore $\eta(BA) = \eta(A^*B^*)$. Thus BA is EPr_2. We shall return to this theme again.

4. Now we are ready to consider EPr matrices over an arbitrary field F.

DEFINITION: Let F be a field and λ an involutory automorphism of F; that is λ is an automorphism of F such that λ^2 is the identity map. For $a \epsilon F$, let $\lambda(a) = \bar{a} \epsilon F$ and for $A = [a_{ij}]$ set $A^* = [b_{ij}]$ where $b_{ij} = \bar{a}_{ji}$. We say an $n \times n$ matrix A with entries from F is EPr if the following conditions are satisfied

(1) A has rank r,

(2) $\sum_{i=1}^{n} \alpha_i A_i = 0$ if and only if $\sum_{i=1}^{n} \bar{\alpha}_i A^i = 0$ $(\alpha_i \epsilon F,\ i = 1, \ldots, n)$.

We first prove an analogue of theorem 1.1.

THEOREM 1.1′: *Let* A *be an* $n \times n$ *matrix with entries in a field* F. *Then statements* (1), (3), (5), (6), (7), (8) *and* (9) *of theorem* 1.1 *are equivalent.*

PROOF: Examination of the proof of theorem 1.1 indicates that no properties peculiar to $\mathscr{C}$ were used and all properties of the complex conjugation that were essential carry over to λ; that is $\bar{a} = 0$ if and only if $a = 0$ and $\overline{ab} = \bar{a}\,\bar{b}$, $\overline{a+b} = \bar{a} + \bar{b}$ whether $\bar{a}$ denotes the complex conjugate of $a \epsilon \mathscr{C}$ or denotes $\lambda(a)$ for $a \epsilon F$.

It should be noted that statements (2) and (4) of theorem 1.1 may fail to be equivalent to statement (1). For example, consider

$$A = \begin{bmatrix} 1 & 3 & 1 \\ 0 & 3 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

over $GF(5)$. The only automorphism of $GF(5)$ is the identity (since the image of 1 is 1) and so $A^* = A^t$. A simple calculation shows that $\eta(A) = \eta(A^t)$ and a basis vector is $\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$, therefore, A is EP_2.

But there is no U satisfying $UU^*=I$ and such that

$$UAU^*=\left[\begin{array}{c|c} D & 0 \\ \hline 0 & 0 \end{array}\right]$$

where D is 2×2 and nonsingular. For the existence of such a matrix U would imply rank $A^2=$ rank $D^2=$ rank $D=$ rank A. However, from A^2 as calculated below, it is clear that rank $A^2<$ rank A.

We have seen that if a complex matrix A is EPr, then A^n is EPr. This situation does not hold for an arbitrary field. In fact, for the example above, we have

$$A^2=\begin{bmatrix} 1 & 3 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus rank $A^2=1$. But A^2 is not EP_1 since

$$A^2\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}=0,\ (A^2)^t\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}=\begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}.$$

Since $A^3=A^2$, *no* higher power of A is EP.

We point out that these conditions fail here because the proof of Schur's theorem, previously used, requires that the characteristic vectors of A not be isotropic vectors under the Euclidean inner product. Over $\mathscr{C}$, no nonzero vector is isotropic. Over $GF(5)$, the vector $\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$ is isotropic.

One should note that A has minimal polynomial $x^2(x-1)$. In fact, if x^2 does not divide the minimal polynomial of A, then rank $A^2=$ rank A. Hence $A^2\xi=0$ implies $A\xi=0$ so $A^*\xi=0$ and thus $(A^2)^*\xi=0$. Then A^2 is EP.

On the other hand the EP_2 matrix

$$A=\begin{bmatrix} 1 & 3 & 1 \\ 3 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

over $GF(5)$ has minimal polynomial $x^2(x-1)$, but A^2 is not EP.

We have

THEOREM 1.3′: *Let* A *be an* EPr_1 *matrix and* B *an* EPr_2 *matrix over a field* F. *If* AB = BA *and neither* η(A) *nor* η(B) *contains isotropic vectors, then* AB *is* EPr.

PROOF: The essential part of theorem 1.2 (from which theorem 1.3 follows immediately) which requires the complex field is the equality

$$\eta(A)\cap\eta(A)^{\perp}=(0)=\eta(B)\cap\eta(B)^{\perp}$$

But, under the hypothesis of theorem 1.3′, this holds.

It is of interest to have a theorem along the lines of theorem 1.3 which places no restrictions on F. We now give such a theorem, which will also show that any example of the type given at the beginning of this section must involve a matrix of size at least 3×3.

THEOREM 1.5: *Let* A *be an* EPr_1 *matrix and* B *an* EPr_2 *matrix over a field* F *such that* AB = BA. *If* r = *rank* AB = min(r_1, r_2), *then* AB *is* EPr.

PROOF: Suppose that $r=r_1$. If $A\xi=0$ where $\xi\epsilon F_n$, then $0=BA\xi=AB\xi$ so that $\eta(A)\subseteq\eta(AB)$.

Since $r=r_1$, it follows that $\eta(A)=\eta(AB)$. Thus, if $AB\xi=0$ then

$$\begin{aligned} A\xi&=0,\\ A^*\xi&=0,\\ B^*A^*\xi&=0 \end{aligned}$$

and so

$$\eta(AB)\subseteq\eta(B^*A^*).$$

But rank $AB=$ rank $(AB)^*$, so $\eta(AB)=\eta(B^*A^*)$ and therefore, AB is EPr_1.

The argument is similar if $r=r_2$.

Corollary: *If* A *is a* 2×2 EPr_1 *matrix over* F *and rank* $\mathrm{A}^2=\mathrm{r}_2$ *then* A^2 *is* EPr_2.

Proof: *Case 1.* $r_1=2$. Then A^2 is nonsingular, and hence is EP_2.

Case 2. $r_1=1$. If $r_2=1$, apply theorem 1.5. If $r_2=0$, then $A^2=0$ and the result is trivial.

Case 3. $r_1=0$. Trivial.

We note that theorem 1.4 remains valid for matrices with entries from an arbitrary field, and then show

Theorem 1.6: *Let* A, B, *and* AB *be* EPr *matrices over a field* F. *Then* BA *is an* EPr *matrix.*

Proof: It follows from theorem 1.4 that it suffices to prove $r=$ rank BA.

In view of the equality $(W^{\perp})^{\perp}=W$ for subspaces $W\subseteq F_n$ it follows that the formulas

$$\begin{aligned} &(\alpha) \qquad \text{rank } AB=r-\dim\,(\eta(A)\cap\eta(B^*)^{\perp}),\\ &(\beta) \qquad \text{rank } BA=r-\dim\,(\eta(B)\cap\eta(A^*)^{\perp}) \end{aligned}$$

(see [6], theorem 7.8) are valid. From (α) and the hypothesis that B is EPr we have

$$r=r-\dim\,(\eta(A)\cap\eta(B)^{\perp})$$

so

$$\eta(A)\cap\eta(B)^{\perp}=(0)$$

and thus $F_n=\eta(A)\oplus\eta(B)^{\perp}$ (by a dimension argument). Hence

$$(0)=(F_n)^{\perp}=(\eta(A)\oplus\eta(B)^{\perp})^{\perp}\supseteq\eta(A)^{\perp}\cap\eta(B)^{\perp\perp}=\eta(A)^{\perp}\cap\eta(B)$$

so that $\dim\,(\eta(B)\cap\eta(A)^{\perp})=0$. Then $r=$ rank BA.

Theorem 1.7: *Let* A *be an* EPr_1 *matrix and* B *an* EPr_2 *matrix over a field* F. *Then we may write* [6]

$$A^*=AN,\ B^*=BM,$$

for some nonsingular matrices N *and* M. *If* AB = BA, *then* AB *is* EPr *if and only if* $\mathrm{N}\eta(\mathrm{AB})\subseteq\eta(\mathrm{AB})$.

Proof: Since $B^*=M^{-*}B$ it follows that

$$B^*A^*=M^{-*}BAN=M^{-*}ABN.$$

Suppose AB is EPr and $AB\xi=0$. Then

$$0=B^*A^*\xi=M^{-*}ABN\xi=AB(N\xi)$$

which implies that $N\xi\epsilon\eta(AB)$.

---

[6] The following matrix N would be $(N^*)^{-1}$ in the notation of (6) of Theorem 1.1.

Conversely, if $N\eta(AB) \subseteq \eta(AB)$, then by repeating the above argument we can show that AB is EPr.

Theorem 1.8: *Let* A *be an* EPr_1 *matrix and* B *an* EPr_2 *matrix over* F. *Let* AB=BA *and let* $A^* = AN$, *where* N *is nonsingular. If* $\eta(A)+\eta(B)=\eta(AB)$, *then* $N\eta(AB) \subseteq \eta(AB)$.

Proof: Since A is EPr_1 then $\eta(A^*)=\eta(A)$ and therefore

$$N\eta(A) \subseteq \eta(A) \subseteq \eta(AB).$$

Hence it is sufficient to show that $N\eta(B) \subseteq \eta(AB)$.

First we note that $A^*\eta(B) \subseteq \eta(B)$. For if $x \epsilon \eta(B)=\eta(B^*)$, then

$$B^*A^*x = A^*B^*x = 0.$$

Moreover since B is EPr_2 it follows that $BA^*x=0$ and hence $A^*x \epsilon \eta(B)$.

Again let $x \epsilon \eta(B)$. Then

$$ABNx = BANx = BA^*x = 0$$

as required, completing the proof.

The converse of theorem 1.7 is false, as the following shows. Let p be a prime and let $F = GF(p)$. Let $A=B$ be a $p \times p$ matrix over F with all entries equal to 1. Then $A=B$ is EP_1 and $\eta(A)+\eta(B)=\eta(A)$. But $A^2=0$ so $\eta(AB)=\eta(A^2)=(GF(p))_n \neq \eta(A)+\eta(B)=\eta(A)$, since dim $\eta(A) = p-1$. However, $A=A'$ so the matrix N of theorem 1.7 may be chosen to be I.

We close this section by noting that theorems 1.6 and 1.7 have analogs for the matrix M.

3. A^* as Unitary Multiple of A

1. It is known that if A is a complex normal matrix then

$$A^* = NA = AN$$

where N satisfies $NN^*=I$, that is, N is unitary. By condition 6 of theorem 1.1, A is an EPr matrix. The matrix

$$A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$$

with entries from $GF(5)$ satisfies $AA^*=A^*A$ (where the automorphism used to define A^* is the identity). But

$$A\begin{pmatrix} 1 \\ 2 \end{pmatrix} = 0,\ A^*\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$$

so that A is not EPr.

Thus in order to be able to obtain a relationship of the form (1) it is necessary that A be both normal and EPr. The conjecture was made that these conditions are also sufficient. The main purposes of this section are to show that

(1) with some added hypotheses a relationship of the form (1) exists, but

(2) without any added hypotheses, the conjecture is not universally true.

The class of normal EPr matrices having such a relationship will be extended to zero-type[7] EPr matrices over a field of characteristic $\neq 2$ in subsection 2. Also, an example will be given of a 4×4 zero-type EP_2 matrix over an arbitrary field of characteristic 2 which does not have a relationship of type (1).

2. We begin by finding sufficient conditions for a relationship of type (1).

THEOREM 2.1 ([10]): *if* A *is normal and has the same rank as* AA*, *then* A *is* EPr.

Over the complex field an EPr matrix is not necessarily normal and over an arbitrary field the addition of the hypothesis rank $A=$ rank AA^* does not imply normality. For example

$$A=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

considered over $\mathscr{C}$ or over $GF(3)$ is EP_2 and satisfies rank $A=$ rank AA^*. But A is not normal.

THEOREM 2.2 ([10]): *Let* A *be an* $n\times n$ *matrix such that rank* $A=$ *rank* AA^*. *Then* A *is normal if and only if* $A^*=NA=AN$ *where* N *satisfies* $NN^*=I$.

THEOREM 2.3 ([11]): *Let* A *be an* $n\times n$ *matrix of rank* r. *Then* A *is a normal* EPr *matrix if and only if there is a nonsingular matrix* M *such that* $A^*=AM=MA$.

We do not repeat the proofs here. The essence of the argument will be repeated in the proof of theorem 2.4.

The main object of this section is to improve theorems 2.2 and 2.3, that is, to remove the restriction that A and AA^* have the same rank and, at the same time, obtain a matrix N such that $A^*=NA=AN$ and $NN^*=I$.

LEMMA: *Let* A *be an* EPr *matrix. We may express* A *as*[8]

$$A=T\left[\begin{array}{c|c} I & 0 \\ \hline X & I \end{array}\right]\left[\begin{array}{c|c} D & 0 \\ \hline 0 & 0 \end{array}\right]\left[\begin{array}{c|c} I & X^* \\ \hline 0 & I \end{array}\right]T^*. \tag{2}$$

Then A *is normal if and only if*

$$D(I+X^*X)D^*=D^*(I+X^*X)D. \tag{3}$$

PROOF: Suppose A is normal. Then

$$AA^*=T\left[\begin{array}{c|c} D(I+X^*X)D^* & D(I+X^*X)D^*X^* \\ \hline XD(I+X^*X)D^* & XD(I+X^*X)D^*X^* \end{array}\right]T^*$$

and

$$A^*A=T\left[\begin{array}{c|c} D^*(I+X^*X)D & D^*(I+X^*X)DX^* \\ \hline XD^*(I+X^*X)D & XD^*(I+X^*X)DX^* \end{array}\right]T^*.$$

Comparing the (1, 1) positions of $AA^*=A^*A$ shows that (3) holds.

Conversely, if

$$D(I+X^*X)D^*=D^*(I+X^*X)D,$$

[7] See above Theorem 2.5 for definition.

[8] Here T is a permutation matrix corresponding to the P in (7) of Theorem 1.1.

then

$$D(I+X^*X)D^*X^*=D^*(I+X^*X)DX^*,$$

$$XD(I+X^*X)D^*=XD^*(I+X^*X)D,$$

and

$$XD(I+X^*X)D^*X^*=XD^*(I+X^*X)DX^*$$

Hence, corresponding blocks of AA^* and A^*A are equal so that A is normal.

THEOREM 2.4: *Let* A *be a normal* EPr *matrix with* A *expressed as in* (2); *a necessary and sufficient condition that there exist a unitary* N *such that* $A^*=AN=NA$ *is that there exist an* $(n-r)\times(n-r)$ *matrix* G_0 *such that*

$$-I+XDD^{-*}D^{-1}D^*X^*-XDD^{-*}X^*G_0-G_0^*XD^{-1}D^*X^*+G_0^*(I+XX^*)G_0=0. \tag{4}$$

PROOF: Suppose that $A^*=NA$. Since T is nonsingular we may write N as follows:

$$N=T\left[\begin{array}{c|c} B & C \\ \hline F & G \end{array}\right]T^* \tag{5}$$

where B is $r\times r$, C is $r\times(n-r)$, F is $(n-r)\times r$ and G is $(n-r)\times(n-r)$. Substituting (5) into $A^*=NA$ and simplifying we obtain

$$T\left[\begin{array}{c|c} I_r & 0 \\ \hline X & I_{n-r} \end{array}\right]\left[\begin{array}{c|c} D^* & 0 \\ \hline 0 & 0 \end{array}\right]\left[\begin{array}{c|c} I_r & X^* \\ \hline 0 & I_{n-r} \end{array}\right]T^*=T\left[\begin{array}{c|c} B & C \\ \hline F & G \end{array}\right]T^*T\left[\begin{array}{c|c} I_r & 0 \\ \hline X & I_{n-r} \end{array}\right]\left[\begin{array}{c|c} D & 0 \\ \hline 0 & 0 \end{array}\right]\left[\begin{array}{c|c} I_r & X^* \\ \hline 0 & I_{n-r} \end{array}\right]T^*,$$

$$\left[\begin{array}{c|c} D^* & 0 \\ \hline XD^* & 0 \end{array}\right]=\left[\begin{array}{c|c} B & C \\ \hline F & G \end{array}\right]\left[\begin{array}{c|c} D & 0 \\ \hline XD & 0 \end{array}\right]=\left[\begin{array}{c|c} BD+CXD & 0 \\ \hline FD+GXD & 0 \end{array}\right].$$

Thus,

$$D^*=BD+CXD$$

so that

$$B=D^*D^{-1}-CX.$$

Also,

$$XD^*=FD+GXD.$$

Hence

$$F=XD^*D^{-1}-GX.$$

Thus the general solution of $A^*=NA$ is given by

$$N=T\left[\begin{array}{c|c} D^*D^{-1}-CX & C \\ \hline XD^*D^{-1}-GX & G \end{array}\right]T^*.$$

We further require that the matrix N obtained above satisfy $A^*=AN$. Using the representations (2) for A (and A^*) it follows that

$$T\left[\begin{array}{c|c} I_r & 0 \\ \hline X & I_{n-r}\end{array}\right]\left[\begin{array}{c|c} D^* & 0 \\ \hline 0 & 0\end{array}\right]\left[\begin{array}{c|c} I_r & X^* \\ \hline 0 & I_{n-r}\end{array}\right]T^*$$

$$=T\left[\begin{array}{c|c} I_r & 0 \\ \hline X & I_{n-r}\end{array}\right]\left[\begin{array}{c|c} D & 0 \\ \hline 0 & 0\end{array}\right]\left[\begin{array}{c|c} I_r & X^* \\ \hline 0 & I_{n-r}\end{array}\right]T^*T\left[\begin{array}{c|c} D^*D^{-1}-CX & C \\ \hline XD^*D^{-1}-GX & G\end{array}\right]T^*,$$

$$\left[\begin{array}{c|c} D^* & D^*X^* \\ \hline 0 & 0\end{array}\right]=\left[\begin{array}{c|c} D & DX^* \\ \hline 0 & 0\end{array}\right]\left[\begin{array}{c|c} D^*D^{-1}-CX & C \\ \hline XD^*D^{-1}-GX & G\end{array}\right]$$

$$=\left[\begin{array}{c|c} DD^*D^{-1}-DCX+DX^*XD^*D^{-1}-DX^*GX & DC+DX^*G \\ \hline 0 & 0\end{array}\right]$$

Hence,

$$D^*=DD^*D^{-1}-DCX+DX^*XD^*D^{-1}-DX^*GX=D(I+X^*X)D^*D^{-1}-D(C+X^*G)X \tag{6}$$

and

$$D^*X^*=DC+DX^*G$$

$$C=D^{-1}D^*X^*-X^*G. \tag{7}$$

Substituting (7) in (6) we have

$$D^*=D(I+X^*X)D^*D^{-1}-D(D^{-1}D^*X^*-X^*G+X^*G)X=D(I+X^*X)D^*D^{-1}-D^*X^*X$$

which is equivalent to (3). By the lemma it follows that

$$N=T\left[\begin{array}{c|c} D^*D^{-1}-D^{-1}D^*X^*X+X^*GX & D^{-1}D^*X^*-X^*G \\ \hline XD^*D^{-1}-GX & G\end{array}\right]T^* \tag{8}$$

satisfies $A^*=NA=AN$.

Finally, we require that G be chosen so that $N^*N=I$. By computing N^* from (8) and then multiplying out N^*N we obtain

$$N^*N=T\left[\begin{array}{c|c} P & Q \\ \hline R & S\end{array}\right]T^*$$

where

$$\begin{aligned}P=\;&D^{-*}DD^*D^{-1}-D^{-*}DD^{-1}D^*X^*X+D^{-*}DX^*GX-X^*XDD^{-*}D^*D^{-1}\\&+X^*XDD^{-*}D^{-1}D^*X^*X-X^*XDD^{-*}X^*GX+X^*G^*XD^*D^{-1}-X^*G^*XD^{-1}D^*X^*X\\&+X^*G^*XX^*GX+D^{-*}DX^*XD^*D^{-1}-D^{-*}DX^*GX-X^*G^*XD^*D^{-1}+X^*G^*GX\\&=I_r-X^*[I_{n-r}-XDD^{-*}D^{-1}D^*X^*+XDD^{-*}X^*G+G^*XD^{-1}D^*X^*-G^*(I+XX^*)G]X.\end{aligned} \tag{9}$$

$$\begin{aligned}Q=\;&D^{-*}DD^{-1}D^*X^*-D^{-*}DX^*G-X^*XDD^{-*}D^{-1}D^*X^*+X^*XDD^{-*}X^*G\\&+X^*G^*XD^{-1}D^*X^*-X^*G^*XX^*G+D^{-*}DX^*G-X^*G^*G\\&=X^*[I-XDD^{-*}D^{-1}D^*X^*+XDD^{-*}X^*G+G^*XD^{-1}D^*X^*-G^*(I+XX^*)G].\end{aligned} \tag{10}$$

$$R = Q^*. \tag{11}$$

$$S = XDD^{-*}D^{-1}D^*X^* - XDD^{-*}X^*G - G^*XD^{-1}D^*X^* + G^*XX^*G + G^*G$$

$$= XDD^{-*}D^{-1}D^*X^* - XDD^{-*}X^*G - G^*XD^{-1}D^*X^* + G^*(I + XX^*)G = S^*. \tag{12}$$

Hence

$$P = I_r - X^*(I_{n-r} - S)X,$$

$$Q = X^*(I_{n-r} - S),$$

$$R = (I_{n-r} - S)^*X = (I_{n-r} - S)X.$$

If for some G we have $N^*N = I$, then $S = I_{n-r}$ and hence (12) reduces to (4) as required.

Conversely, suppose there is a G_0 satisfying (4). If this value is substituted for G in (8) then (12) reduces to $I_{n-r} - S = 0$. Hence

$$P = I_r - 0 = I_r,$$

$$Q = 0,$$

$$R = 0$$

so that $N^*N = I$ as required completing the proof.

In ([10], p. 3) we have the added hypothesis that rank A = rank AA^*. From this it follows as **in [10] that $I + X^*X$ is nonsingular. Theorem 2.2 may be established by verifying that**

$$G_0 = X(I + X^*X)^{-1}(D^{-1}D^* - I)X^* + I$$

is a solution of (4).

DEFINITION: Let $n = 2r$ (r is a positive integer). An $n \times n$ ***EPr*** matrix A such that $AA^* = 0$, is called a zero-type matrix; that is the columns of A are pairwise orthogonal and individually isotropic with respect to the inner product induced on F_n by λ.)

THEOREM 2.5: *A zero-type matrix is normal.*

PROOF: Since A is ***EPr***

$$A = T\left[\begin{array}{c|c} I_r & 0 \\ \hline X & I_r \end{array}\right]\left[\begin{array}{c|c} D & 0 \\ \hline 0 & 0 \end{array}\right]\left[\begin{array}{c|c} I_r & X^* \\ \hline 0 & I_r \end{array}\right]T^*$$

where X is $r \times r$, so that

$$0 = AA^* = \left[\begin{array}{c|c} D(I + X^*X)D^* & 0 \\ \hline 0 & 0 \end{array}\right].$$

Since D is nonsingular, it follows that $I + X^*X = 0$. By the lemma before Theorem 2.4, A is normal.

The preceding proof did not use the fact that $n = 2r$. Nor does the following alternate proof: $AA^* = 0$ implies that each column of A^* lies in $\eta(A)$. Since A is EPr, it follows that $A^*A^* = 0$ and thus that $AA = 0$. Hence each column of A is in $\eta(A)$, and since A is EPr, $A^*A = 0$.

A similar proof shows that EPr matrix A is normal if $A^2 = 0$.[9]

[9] We are indebted to A. J. Goldman (National Bureau of Standards) for the observations of the preceeding two paragraphs.

COROLLARY: *If* A *is a zero-type matrix, then the square matrix* X *of theorem* 1.1, *part* (7) *satisfies*

$$I+X^*X=I+XX^*=0.$$

THEOREM 2.6: *If* A *is a zero-type matrix over a field of characteristic* $\neq 2$, *then there is a unitary matrix* N *such that* $A^*=NA=AN$.

PROOF: According to theorem 2.5 it suffices to find an $(n-r)\times(n-r)$ matrix G_0 satisfying (4). However, by the above corollary, for a zero-type matrix, $I+XX^*=0$ so that (4) reduces to

$$0=-I+XDD^{-*}D^{-1}D^*X^*-XDD^{-*}X^*G_0-G_0^*XD^{-1}D^*X^*$$

$$=-I+XDD^{-*}D^{-1}D^*X^*-XDD^{-*}X^*G_0-(XDD^{-*}X^*G_0)^*.$$

Set $Y=XDD^{-*}X^*G_0$. Then (4) becomes

$$0=-I+XDD^{-*}D^{-1}D^*X^*-Y-Y^*. \tag{13}$$

Set $H=-\frac{1}{2}(I-XDD^{-*}D^{-1}D^*X^*)$. Then $H=H^*$ and $Y=H$ is a solution of (13). Thus

$$G_0=(XDD^{-*}X^*)^{-1}H=-\frac{1}{2}(XDD^{-*}X^*)^{-1}(I-XDD^{-*}D^{-1}D^*X^*)$$

satisfies (4), completing the proof.

We now give an example to show that the requirement characteristic $\neq 2$ is necessary. Let

$$A=\begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

over a field of characteristic 2. Then

$$A=\left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{array}\right]\left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]\left[\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

so that A is EP_2. Moreover, $AA^*=0$ so that A is a zero-type matrix. Equation (13) of theorem 2.6 reduces to

$$Y+Y^*=\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

Thus, the impossibility in some cases of finding a unitary matrix N such that $A^*=AN=NA$ follows from the following theorem and theorem 2.5.

THEOREM 2.7: *Let* F *be a field of characteristic* 2 *and* λ *the identity. If* S *is a symmetric matrix over* F, *then a necessary and sufficient condition that there exist a matrix* R *such that* $R+R^*=S$ *is that* S *have only zeros on the diagonal.*

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PROOF: Suppose $R+R^*=S$. Because $\alpha+\alpha=0$ for $\alpha\epsilon F$, it is clear that the diagonal elements of S are zero.

Conversely, let the diagonal elements of S be zeros. Let R have zeros on and below the diagonal and be identical with S elsewhere. Then $R+R^*=S$, completing the proof.

A matrix M satisfying the conditions of theorem 2.3 is

$$M=\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

It would be desirable to relax the requirement $n=2r$ in the definition of zero-type matrix. Such a change, however, leads to difficulties in the ensuing matrix equation which we cannot presently handle. In particular, the condition $I+X^*X=0$ (which continues to hold) need not imply $I+XX^*=0$, and so (4) retains its formidable character. For example, consider

$$A=A^*=\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

over a field of characteristic 3. Then rank $A=1$ and $AA^*=0$. Suppose $X=(a,\ b)^t$ and $D=[d]$ in (7) of Theorem 1.1. For any permutation matrix T,

$$A=T^{-1}AT=\begin{bmatrix} d & d\bar{a} & d\bar{b} \\ da & da\bar{a} & da\bar{b} \\ db & db\bar{a} & db\bar{b} \end{bmatrix}$$

Hence $X=(1,\ 1)^t$, so that

$$I+XX^*=\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\neq 0$$

although $I+X^*X=0$. Note that $N=I$ is a unitary solution of $A^*=AN=NA$.

3. It was proved by Williamson [24] that any complex normal matrix A may be written as a polynomial in A^* with complex coefficients. An example is given in [10] to show that this is not necessarily the case for a normal matrix over an arbitrary field F. That example gives a normal but not EPr matrix. Instead of repeating this example we give a matrix A which is normal and EPr, yet A^* is not a polynomial in A. The matrix A of the example before theorem 2.7 suffices. $A^2=0$ so we try to find α and β in $GF(2)$ such that $A^*=\alpha A+\beta I$.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}=\alpha\begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}+\beta\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}=\begin{bmatrix} \alpha+\beta & 0 & \alpha & 0 \\ \alpha & \alpha+\beta & \alpha & \alpha \\ \alpha & 0 & \alpha+\beta & 0 \\ \alpha & \alpha & \alpha & \alpha+\beta \end{bmatrix}$$

so that comparing the (1, 3) and (2, 1) positions leads to

$$\alpha=1,\ \alpha=0.$$

Hence A^* is not a polynomial in A.

We close this section by stating the only known result concerning the expression of A^* as a polynomial in A and giving an example.

THEOREM 2.8 ([10]): *Let* A *be an* $n\times n$ *matrix over a field* F *and let* K *be a field containing* F *and the characteristic roots of* A. *If rank* $(A-\beta I)(A-\beta I)^*=$ *rank* $(A-\beta I)=r_\beta$ *and* $A-\beta I$ *is* EPr_β *for each* $\beta\epsilon K$, *then* A *is normal and* A^* *may be written as a polynomial in* A.

However, the hypotheses are not also necessary. For example

$$A=\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

over $GF(2)$ satisfies the conclusion of the theorem, but rank $A\neq$ rank AA^*. Moreover, this example is easily adapted to $GF(p)$, $p=$ prime.

4. *EP* Linear Transformations

1. In this section we introduce the notion of an *EP* linear transformation acting on a finite-dimensional vector space. We develop a theory for *EP* linear transformations and then show how some of the results of the previous sections may be derived by means of this new concept.

Finally, motivated by the well-known notion of normal linear transformation, we introduce the concept of θ-normality and apply it to the study of normal *EPr* matrices.

2. Notation: Throughout, V denotes an n-dimensional vector space over a field F, $V_n(F)$ [10] denotes the vector space of $m\times 1$ column vectors with components from F, $\hat{V}$ denotes the dual space of V (that is, the space of linear transformations from V to F), $L(V)$ denotes the space of linear transformations from V to V, and for $T\epsilon L(V)$, $\eta(T)$ and $R(T)$ denote the null space and range space of T respectively. $[T]_B$ denotes the matrix of T relative to the basis B of V. $\eta(A)$ and $R(A)$ denote the null space and range space of the $n\times n$ matrix A respectively.

We recall the essential facts concerning $T^*\epsilon L(\hat{V})$, the linear transformation dual to T. For $x\epsilon V$ and $\hat{y}\epsilon\hat{V}$ we set

$$T^*\hat{y}(x)=\overline{\hat{y}(Tx)},$$

where $\alpha\rightarrow\bar{\alpha}$ is an involution of F. Generally one sets $T^*\hat{y}(x)=\hat{y}(Tx)$, but our special purposes require the conjugation and no great change ensues. This gives a well-defined linear transformation on $\hat{V}$. Moreover, if $v_1, \ldots, v_n$ and $\hat{v}_1, \ldots, \hat{v}_n$ are dual bases, that is, $\hat{v}_i(v_j)=\delta_{ij}$, then the matrices of T relative to $v_1, \ldots, v_n$ and of T^* relative to $\hat{v}_1, \ldots, \hat{v}_n$ are conjugate transposes. These elementary facts will suffice for our purposes. For further details see the lectures by N. Jacobson [8, pp. 51–60].

DEFINITION: Let $T\epsilon L(V)$. T is an *EP* linear transformation if there is a basis $u_1, \ldots, u_n$ of V and a dual basis $\hat{u}_1, \ldots, \hat{u}_n$ of $\hat{V}$ such that

$$T\left(\sum_{i=1}^{n}\alpha_i u_i\right)=0 \text{ if and only if } T^*\left(\sum_{i=1}^{n}\alpha_i\hat{u}_i\right)=0.$$

Dual bases satisfying this requirement will be called special bases and the basis $u_1, \ldots, u_n$ will be called a special basis. (So if reference is made to a special basis, it is to be understood that there is a dual basis such that together these dual bases form special bases.)

[10] Previously denoted F_n.

Theorem 3.1: *Let* $T \epsilon L(V)$. T *is* EP *with special basis* B: $b_1, \ldots, b_n$ *if and only if* $[T]_B$ *is an* EP *matrix.*

Proof: Consider the isomorphism f_B: $V \to f_B(V) = V_n(F)$ defined by

$$f_B\left(\sum_{i=1}^{n} \alpha_i b_i\right) = (\alpha_1, \ldots, \alpha_n)^t. \tag{1}$$

Note that $E: f_B(b_1), \ldots, f_B(b_n)$ is the natural basis of $V_n(F)$. f_B induces an isomorphism $F_B: L(V) \to F_B(L(V)) = L(V_n(F))$ defined by

$$(F_B T)(f_B x) = f_B(Tx) \qquad (T \epsilon L(V),\ x \epsilon V).$$

Note that

$$[F_B T]_E = [T]_B. \tag{2}$$

If $\hat{B}: \hat{b}_1, \ldots, \hat{b}_n$ is the dual basis to B, then there is also an isomorphism $\hat{f}_B: \hat{V} \to \hat{f}_B(\hat{V}) = V_n(F)$ defined by

$$\hat{f}_B\left(\sum_{i=1}^{n} \beta_i \hat{b}_i\right) = (\beta_1, \ldots, \beta_n)^t$$

and $\hat{f}_B(\hat{b}_1), \ldots, \hat{f}_B(\hat{b}_n)$ is again the natural basis of $V_n(F)$. $\hat{f}_B$ induces an isomorphism $\hat{F}_B: L(\hat{V}) \to \hat{F}_B(L(\hat{V})) = L(\hat{V}_n(F))$ defined by

$$(\hat{F}_B \hat{T})(\hat{f}_B \hat{x}) = \hat{f}_B(\hat{T}\hat{x}) \qquad (\hat{T} \epsilon L(\hat{V}),\ \hat{x} \epsilon \hat{V}).$$

Note that

$$[\hat{F}_B \hat{T}]_E = [\hat{T}]_{\hat{B}}.$$

As remarked earlier, for any $T \epsilon L(V)$,

$$[T^*]_{\hat{B}} = [T]^*_B.$$

First assume T is EP with B as special basis. Then

$$0 = [T]_B f_B(x) = [F_B T]_E f_B(x) = (F_B T)(f_B(x)) = f_B(Tx)$$

implies that

$$0 = Tx = T\left(\sum_{i=1}^{n} \alpha_i b_i\right)$$

and thus that

$$0 = T^*\left(\sum_{i=1}^{n} \alpha_i \hat{b}_i\right) = T^* \hat{x}.$$

This in turn implies

$$0 = \hat{f}_B(T^* x) = (\hat{F}_B T^*)\hat{f}_B(\hat{x}) = [\hat{F}_B T^*]_E \hat{f}_B(\hat{x}) = [T^*]_{\hat{B}} \hat{f}_B(\hat{x}) = [T]^*_{\hat{B}}(\alpha_1, \ldots, \alpha_n)^t = [T]^*_B f_B(x).$$

Furthermore, the resoning is reversible. Thus for all $\xi = f_B(x) \epsilon V_n(F)$ we have

$$[T]_B \xi = 0 \qquad \text{iff} \qquad [T]_B^* \xi = 0,$$

and so $[T]_B$ is an EP matrix.

Next assume $[T]_B$ is an EP matrix. Then

$$Tx = T\left(\sum_{i=1}^{n} \alpha_i b_i\right) = 0$$

implies

$$0 = f_B(Tx) = (F_B T)(f_B(x)) = [F_B T]_E f_B(x) = [T]_B f_B(x),$$

which in turn implies

$$0 = [T]_B^* f_B(x) = [T^*]_{\hat{B}}(\alpha_1, \ldots, \alpha_n)^t = [\hat{F}_B T^*]_E \hat{f}_B\left(\sum_{i=1}^{n} \alpha_i \hat{b}_i\right)$$

$$= (\hat{F}_B T^*) \hat{f}_B\left(\sum_{i=1}^{n} \alpha_i \hat{b}_i\right) = \hat{f}_B\left(T^*\left(\sum_{i=1}^{n} \alpha_i \hat{b}_i\right)\right).$$

Thus $T^*\left(\sum_{i=1}^{n} \alpha_i \hat{b}_i\right) = 0$. Furthermore, the reasoning is reversible and so T is an EP transformation with B as special base.[11]

If there exist special bases for T it is not necessarily true that any pair of dual bases provides special bases. For example, consider the linear transformation T whose matrix relative to the natural basis e_1, e_2, e_3 of $V_3(F)$ is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Let $\hat{e_1}, \hat{e_2}, \hat{e_3}$ be the dual basis. Then $Te_2 = 0$. Also

$$T^* \hat{e_2}(e_1) = \overline{\hat{e_2}(Te_1)} = \overline{\hat{e_2}(e_1 + e_3)} = 0,$$

$$T^* \hat{e_2}(e_2) = \overline{\hat{e_2}(Te_2)} = \overline{\hat{e_2}(0)} = 0,$$

$$T^* \hat{e_2}(e_3) = \overline{\hat{e_2}(Te_3)} = \overline{\hat{e_2}(e_3)} = 0,$$

so that $T^* e_2 = 0$.

Suppose $T^*\left(\sum_{i=1}^{3} \alpha_i e_i\right) = 0$. Then

$$0 = T^*\left(\sum_{i=1}^{3} \alpha_i e_i\right)(e_1) = \overline{\sum_{i=1}^{3} \alpha_i e_i(e_1 + e_3)} = \overline{\alpha_1} + \overline{\alpha_3},$$

[11] We are indebted to A. J. Goldman (National Bureau of Standards) for simplifying an earlier version of this result and bringing the proof to its present simpler form.

$$0=T^*\left(\sum_{i=1}^{3}\alpha_i e_i\right)(e_2)=\overline{\sum_{i=1}^{3}\alpha_i e_i(0)}=0,$$

$$0=T^*\left(\sum_{i=1}^{3}\alpha_i e_i\right)(e_3)=\overline{\sum_{i=1}^{3}\alpha_i e_i(e_3)}=\overline{\alpha_3},$$

and hence $\alpha_1=\alpha_3=0$ so that $\sum_{i=1}^{3}\alpha_i e_i=\alpha_2 e_2$. But $T(\alpha_2 e_2)=0$. Thus T is EP with the natural basis and its dual as special bases.

However, consider the following dual bases. $v_1=e_1$, $v_2=e_1+e_2$, $v_3=e_3$ and $v_1=e_1-e_2$, $v_2=e_2$, $v_3=e_3$. $T(v_1-v_2)=0$. But

$$T^*(v_1-v_2)(v_1)=\overline{(v_1-v_2)(Tv_1)}=\overline{(e_1-2e_2)(e_1+e_3)}=1\neq 0$$

so that $T^*(v_1-v_2)\neq 0$. Hence these dual bases do not satisfy the requirements of the definition of special bases.

It is possible that every pair of dual bases be special bases. For example, this is the case whenever T is nonsingular.

Now we give an example of an *EP* linear transformation T with exactly one pair of special bases. Let T have the matrix

$$A=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$$

over $GF(2)$ relative to the natural basis e_1, e_2 of $V_2(GF(2))$. By the same type of calculation performed earlier one verifies that the dual bases e_1, e_2 and $\hat{e_1}$, $\hat{e_2}$ are special bases. By theorem 3.1, determining all special bases B is equivalent to finding all $2.\times 2$ nonsingular matrices P such that $[T]_B=\boldsymbol{P^{-1}AP}$ is EP_1. If

$$P=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$$

and $C=P^{-1}AP$ is EPr, then $(x,\ y)^t\epsilon\eta(C)$ implies

$$ax=by, \tag{3}$$

$$dx=cy \tag{4}$$

since the identity is the only automorphism of $GF(2)$, and

$$C\begin{pmatrix}x\\ y\end{pmatrix}=0=C^*\begin{pmatrix}x\\ y\end{pmatrix}.$$

By testing the three nonzero possibilities $(1,\ 0),(0,\ 1)$, and $(1,\ 1)$ for $(x,\ y)^t$, and eliminating the third since P is nonsingular, we find that

$$P=I,\qquad P=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$$

are the only solutions. Hence T has exactly one pair B, $\hat{B}$ of special bases.

The proof of the next theorem requires the following well-known result.
LEMMA 1: *Let* $T \epsilon L(V)$. *Then the following are equivalent.*

(i) rank $T = \text{rank } T^2$.
(ii) $\eta(T) \cap R(T) = (0)$.

THEOREM 3.2: *Let* $T \epsilon L(V)$ *and rank* $T =$ *rank* T^2. *Then* T *is* EP.
PROOF. Let rank $T = n - r$. Let V have a basis $u_1, \ldots, u_r, u_{r+1}, \ldots, u_n$ where the first r vectors form a basis of $\eta(T)$ and the remaining $(n-r)$ vectors form a basis of $R(T)$.

Suppose $Tu_i = \sum_{k=1}^{n} \alpha_{ki} u_k (i = 1, \ldots, n)$. Then

$$\sum_{k=1}^{r} \alpha_{ki} u_k = Tu_i - \sum_{k=r+1}^{n} \alpha_{ki} u_k. \tag{5}$$

The left-hand side of (5) is in $\eta(T)$ and the right-hand side is in $R(T)$. Thus

$$\sum_{k=1}^{r} \alpha_{ki} u_k = 0$$

and

$$Tu_i = \sum_{k=r+1}^{n} \alpha_{ki} u_k \epsilon R(T) \; (i = 1, \ldots, n).$$

If the dual basis is $\hat{u}_1, \ldots, \hat{u}_n$, then for $i = 1, \ldots, r$,

$$(T^* \hat{u}_i)(u_j) = \overline{\hat{u}_i(Tu_j)} = \begin{cases} \overline{\hat{u}_i(0)} = 0 \text{ for } j = 1, \ldots, r \\ \overline{\hat{u}_i\left(\sum_{k=r+1}^{n} \alpha_{kj} u_k\right)} = 0 \text{ for } j = r+1, \ldots, n, \end{cases}$$

so that $T^* \hat{u}_i = 0$. From rank $T = \text{rank } T^*$ [8, p. 59] it follows that $\hat{u}_1, \ldots, \hat{u}_r$ form a basis for $\eta(T^*)$. Hence T is EP, completing the proof.
COROLLARY 1: *Let* $T \epsilon L(V)$. *There is a positive integer* k *such that* T^k *is* EP.
PROOF. The sequence of subspaces

$$\eta(T) \subseteq \eta(T^2) \subseteq \ldots$$

terminates. That is, there is a positive integer k such that rank $T^k = \text{rank } T^{k+1}$. Thus rank $T^k = \text{rank } T^{2k}$.
COROLLARY 2: *If* A *is an* $n \times n$ *matrix such that rank* $A =$ *rank* A^2, *then there is a nonsingular matrix* P *such that* $P^{-1}AP$ *is* EPr.
PROOF. If T is the linear transformation whose matrix relative to the natural basis is A, then T is EP and so has a special basis B. By theorem 3.1,

$$[T]_B = P^{-1}AP$$

is EPr where P is the matrix changing coordinates from the natural basis to the special basis.
COROLLARY 3: *Let* A *be as in corollary* 2. *Then any linear transformation* T *whose matrix relative to some basis is* A *is* EP.

REMARK. The condition of theorem 3.2 is not necessary that T be EP. For example, consider the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \text{ over } GF(2).$$

A is EPr, since it is symmetric, so the linear transformation T of theorem 3.1 is EP. However, rank $T=1$ and rank $T^2=0$.

Furthermore, this type of example can be constructed for any nonzero characteristic.

The situation is better over the complex field.

THEOREM 3.3: *The linear transformation* T *on a vector space* V *over the complex field is* EP *if and only if rank* $\mathrm{T} =$ *rank* T^2.

PROOF: We may assume that T is singular. Suppose that T is EP and has a special basis B. $[T]_B$ is EPr so applying theorem 1.1,

$$P[\ T\ ]_B P^* = \left[\begin{array}{c|c} D & 0 \\ \hline 0 & 0 \end{array}\right]$$

where $PP^*=I$ and D is nonsingular. Since D^2 is nonsingular, rank $T^2 =$ rank $P[T^2]_B P^* =$ rank T.

The converse is included in theorem 3.2.

3. DEFINITION. Let $T \epsilon L(V)$ and V and $\hat{V}$ have dual bases $\hat{B} : \hat{u}_1, \ldots, \hat{u}_n$ and $\hat{B} : \hat{u}_1, \ldots, \hat{u}_n$. Let θ_B denote the linear transformation of V onto $\hat{V}$ given by

$$\theta_B \left(\sum_{i=1}^{n} \alpha_i u_i \right) = \sum_{i=1}^{n} \alpha_i \hat{u}_i.$$

Then $\theta_B^{-1} T^* \theta_B \epsilon L(V)$. We say that T is $\theta_B -$ *normal* if T and $\theta_B^{-1} T^* \theta_B$ commute.

Suppose $Tu_i = \sum_{k=1}^{n} t_{ki} u_k (i=1, \ldots, n)$. Then

$$(\theta_B^{-1} T^* \theta_B) u_i = \theta_B^{-1} T^* \hat{u}_i = \theta_B^{-1} \sum_{k=1}^{n} \bar{t}_{ik} \hat{u}_k = \sum_{k=1}^{n} \bar{t}_{ik} u_k (i=1, \ldots, n)$$

and so *the matrices of* T *and* $\theta_B^{-1} \mathrm{T}^* \theta_B$ *relative to the basis* $\mathrm{u}_1, \ldots, \mathrm{u}_n$ *are conjugate transposes.*

Let T have matrix A relative to the basis $e_1, \ldots, e_n$ and let $u_i = \sum_{k=1}^{n} p_{ki} e_k (i=1, \ldots, n)$, $P=[p_{ij}]$. Then

$$[T]_B = P^{-1} A P,\ [\theta_B^{-1} T^* \theta_B]_B = [T]_B^* = P^* A^* P^{-*}.$$

Hence, in matrix terms, T *is* $\theta_B -$ *normal if and only if* PAP^{-1} *and* $\mathrm{P}^{-*}\mathrm{A}^*\mathrm{P}^*$ *commute.*

LEMMA 2: *Let* $\mathrm{T} \epsilon \mathrm{L(V)}$ *and let* θ_B *be the mapping of the preceding definition. Suppose the underlying automorphism is the identity. Then* $(\theta_B^{-1} \mathrm{T}^* \theta_B)^* = \theta_B \mathrm{T} \theta_B^{-1}$.

PROOF. We use the bases $\hat{B} : u_1, \ldots, u_n$ and $\hat{B} : \hat{u}_1, \ldots, \hat{u}_n$ employed in defining θ_B. The proof is arranged to indicate why assuming λ to be the identity is required.

Suppose $Tu_i = \sum_{k=1}^{n} \beta_{ki} u_k (i=1, \ldots, n)$ so that $T^* \hat{u}_i = \sum_{k=1}^{n} \bar{\beta}_{ik} \hat{u}_k\ (i=1, \ldots, n)$. If $x = \sum_{i=1}^{n} \alpha_i u_i \epsilon V$,

then

$$
\begin{aligned}
[(\theta_B^{-1}T^*\theta_B)^*u_i](x) &= \overline{\hat{u}_i(\theta_B^{-1}T^*\theta_B x)} \\
&= \overline{\hat{u}_i\left(\theta_B^{-1}T^* \sum_{j=1}^{n} \alpha_j \hat{u}_j\right)} \\
&= \overline{\hat{u}_i\left(\theta_B^{-1} \sum_{j,\,k=1}^{n} \alpha_j \bar{\beta}_{jk} \hat{u}_k\right)} \\
&= \overline{\hat{u}_i\left(\sum_{j,\,k=1}^{n} \alpha_j \bar{\beta}_{jk} u_k\right)} \\
&= \sum_{j=1}^{n} \bar{\alpha}_j \beta_{ji}
\end{aligned}
$$

and

$$
\begin{aligned}
(\theta_B T \theta_B^{-1} \hat{u}_i)\,(x) &= (\theta_B T u_i)\,(x) \\
&= \left(\theta_B \sum_{k=1}^{n} \beta_{ki} u_k\right)(x) \\
&= \left(\sum_{k=1}^{n} \beta_{ki} \hat{u}_k\right)\left(\sum_{j=1}^{n} \alpha_j u_j\right) \\
&= \sum_{j=1}^{n} \beta_{ji} \alpha_j .
\end{aligned}
$$

Since λ is the identity, $\alpha_j = \bar{\alpha}_j$; thus $(\theta_B^{-1}T^*\theta_B)^*$ and $\theta_B T \theta_B^{-1}$ agree on a basis and so they are equal.

LEMMA 3: *Let* A *be an* EPr *matrix over a field* F. *Then rank* $A^2 =$ *rank* AA^*.

PROOF: We have shown that $A^* = NA$ where N is nonsingular, so $AA^* = A^2 N^{-*}$.

REMARK: $A = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix}\right]$ shows the converse is false.

Before stating the next theorem we indicate the notation to be used. Let $T \epsilon L(V)$ and $n - r = \text{rank } T = \text{rank } T^2$. Then $\eta(T) \cap R(T) = (0)$ and hence we may choose a basis B: $u_1, \ldots, u_n$ of V such that $u_1, \ldots, u_r$ form a basis of $\eta(T)$ and $u_{r+1}, \ldots, u_n$ form a basis of $R(T)$. Moreover, T is nonsingular on $R(T)$ and thus $Tu_{r+1}, \ldots, Tu_n$ also form a basis of $R(T)$. As in theorem 3.3,

$\hat{B}$: $\hat{u}_1, \ldots, u_r, Tu_{r+1}, \ldots, Tu_n$ and $\hat{B}$: $\widehat{u_1}, \ldots, \widehat{u_r}, \widehat{(Tu_{r+1})} \ldots, \widehat{(Tu_n)}$

form a pair of special bases for T. Let

$$
\begin{aligned}
&v_i = u_i,\ \hat{v}_i = \hat{u}_i \qquad (i = 1, \ldots, r), \\
&v_i = Tu_i,\ \hat{v}_i = \widehat{(Tu_i)} \qquad (i = r+1, \ldots, n),
\end{aligned}
$$

and

$$
\theta_B\left(\sum_{i=1}^{n} \alpha_i v_i\right) = \sum_{i=1}^{n} \alpha_i \hat{v}_i .
$$

Finally, suppose the underlying automorphism on F is the *identity*. Then we have the following analog of theorem 2.2.

THEOREM 3.4: *With the notation of the preceding pargraph suppose that* T *is* θ_B*-normal. Then there is a linear transformation* $R = R_\theta \epsilon L(V)$ *such that*

(i) $\theta_B^{-1}T^*\theta_B = RT = TR$,

(ii) R *is unitary relative to the inner product*

$$(x, y) = \theta_B y(x), \qquad x, y \epsilon V,$$

(*that is*, $(Rx, Ry) = (x, y)$ *for all* x, y),

(iii) N, *the matrix of* R *relative to the basis* B, *satisfies* $NN^* = I$.

PROOF: (i) Define R as follows.

$$Ru_i = u_i \qquad (i = 1, \ldots, r)$$

$$R(Tu_i) = \theta_B^{-1}T^*\theta_B u_i \qquad (i = r+1, \ldots, n)$$

so that R is defined on all of V since $V = \eta(T) \oplus R(T)$. Since T is EP with special bases $u_1, \ldots, u_n$ and $\hat{u}_1, \ldots, \hat{u}_n$ it follows that

$$RTu_i = R0 = 0 \qquad (i = 1, \ldots, r),$$

$$\theta_B^{-1}T^*\theta_B\hat{u}_i = \theta_B^{-1}T^*\hat{u}_i = \theta_B^{-1}0 = 0 \qquad (i = 1, \ldots, r).$$

Hence $\theta_B^{-1}T^*\theta_B = RT$. Moreover,

$$TRu_i = Tu_i = 0 \qquad (i = 1, \ldots, r),$$

$$\begin{aligned} TR(Tu_i) &= T(RTu_i) \\ &= T(\theta_B^{-1}T^*\theta_B u_i) \\ &= \theta_B^{-1}T^*\theta_B Tu_i \qquad \text{(by } \theta\text{-normality)} \\ &= RT(Tu_i) \end{aligned}$$

and so $TR = RT$.

(Note that $\eta(T) \cap R(T) = (0)$ is used only to insure that R can be defined on all of V.)

(ii) For $i, j = 1, \ldots, r$, $(Rv_i, Rv_j) = (Ru_i, Ru_j) = (u_i, u_j) = (v_i, v_j)$.
For $i, j = r+1, \ldots, n$,

$$\begin{aligned} (Rv_i, Rv_j) = (RTu_i, RTu_j) &= (\theta_B^{-1}T^*\theta_B u_i, \theta_B^{-1}T^*\theta_B u_j) \\ &= \theta_B\theta_B^{-1}T^*\theta_B u_j(\theta_B^{-1}T^*\theta_B u_i) \\ &= \theta_B u_j(T\theta_B^{-1}T^*\theta_B u_i) \\ &= \theta_B u_j(\theta_B^{-1}T^*\theta_B Tu_i) \\ &= (\theta_B^{-1}T^*\theta_B)^*\theta_B u_j(Tu_i) \\ &= \theta_B T\theta_B^{-1}\theta_B u_j(Tu_i) \qquad \text{(by lemma 2)} \end{aligned}$$

$$=\theta_B Tu_j(Tu_i)$$

$$=(Tu_i,\ Tu_j)$$

$$=(v_i,\ v_j).$$

For $i=1, \ldots, r$ and $j=r+1, \ldots, n$,

$$(v_i,\ v_j)=(u_i,\ Tu_j)=(v_i,\ v_j)=\theta_B v_j(v_i)=\hat{v}_j(v_i)=0$$

and

$$(\mathrm{R}v_i,\ Rv_j)=(Ru_i,\ RTu_j)=(u_i,\ \theta_B^{-1}T^*\theta_B u_j)=\theta_B\theta_B^{-1}T^*\theta_B u_j(u_i)=\theta_B u_j(Tu_i)=\theta_B u_j(0)=0.$$

Finally,

$$\left(\sum_{i=1}^{n}\alpha_i v_i, \sum_{j=1}^{n}\beta_j v_j\right)=\sum_{j=1}^{n}\beta_j\hat{v}_j\left(\sum_{i=1}^{n}\alpha_i v_i\right)$$

$$=\sum_{i=1}^{n}\beta_i\alpha_i$$

$$=\sum_{i=1}^{n}\alpha_i\beta_i$$

$$=\sum_{i=1}^{n}\alpha_i\hat{v}_i\left(\sum_{j=1}^{n}\beta_j v_j\right)$$

$$=\left(\sum_{j=1}^{n}\beta_j v_j, \sum_{i=1}^{n}\alpha_i v_i\right)$$

so for $i=r+1, \ldots, n$ and $j=1, \ldots, r$ it follows that

$$(v_i,\ v_j)=(Tu_i,\ u_j)=(u_j,\ Tu_i)$$

$$=(Ru_j,\ RTu_i) \qquad \text{(from the preceding case)}$$

$$=(RTu_i,\ Ru_j)$$

$$=(Rv_i,\ Rv_j).$$

Hence $(x,\ y)=(Rx,\ Ry)$ for all $x,\ y \epsilon V$.

(iii) Set $Rv_i=\sum_{j=1}^{n} r_{ji}v_j$ $(i=1, \ldots, n)$. Then

$$\delta_{ij}=\hat{v}_j(v_i)=(v_i,\ v_j)$$

$$=(Rv_i,\ Rv_j)$$

$$=\left(\sum_{\lambda=1}^{n} r_{\lambda i}v_{\lambda}, \sum_{\mu=1}^{n} r_{\mu j}v_{\mu}\right)$$

$$=\sum_{\lambda=1}^{n} r_{\lambda i}r_{\lambda j}$$

so that $N^*N=I$, completing the proof.

COROLLARY 1: *Let* A *be a normal* n×n *matrix (relative to the identity automorphism) satisfying rank* A=*rank* AA*. *Then there is a matrix* N *such that* N*N=I *and* A*= NA=AN.

PROOF: By [10] A is EPr and so it follows from lemma 3 and Theorem 3.2 that T, the linear transformation whose matrix relative to the natural basis of $V_n(F)$, E: $e_1, \ldots, e_n$ is A, is EP with $e_1, \ldots, e_n$ and $\hat{l}_1, \ldots, \hat{l}_n$ as special bases.

Set $\theta_E\left(\sum_{i=1}^{n} \alpha_i e_i\right)=\sum_{i=1}^{n} \alpha_i \hat{e}_i$. Since A is normal, and

$$[T]_E=A, \; [\theta_E^{-1}T^*\theta_E]_E=A^*,$$

it follows that T is θ_E—normal. By (i) of Theorem 3.4,

$$A^*=[\theta_E^{-1}T^*\theta_E]_E=[RT]_E=[TR]_E=NA=AN.$$

5. Some Open Problems

(1) If A is EP and normal is A^2 necessarily EP? (See corollary 1 of theorem 1.1.)

Remarks: (i) If the conclusion of theorem 2.6 is valid for all normal, EP matrices, then the answer is yes.

(ii) One should note that A^2 is EP_0 for the example given in connection with theorem 2.6.

(2) When can a matrix of rank r be expressed as a product of EPr matrices? (See corollary 2 of theorem 1.1).

(3) If A, B, and AB are EP, it does not follow that BA is EP. What additional condition will guarantee that BA is EP? In particular, A of the example preceding theorem 1.4 is not normal. Will normality suffice? (Wiegmann's theorem 20, 9 answers this in the affirmative over the complex field).

(4) If A is an EP matrix, can x^3 divide the minimal polynomial of A?

Remarks: (i) If A is a complex EPr matrix, then

$$UAU^*=\left[\begin{array}{c|c} D & 0 \\ \hline 0 & 0 \end{array}\right]$$

where D is an $r\times r$ nonsingular matrix and $UU^*=I$. This implies that x^2 does not divide the minimal polynomial of A.

(ii) The matrix $$A=\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

over $GF(2)$ shows that x^2 can divide the minimal polynomial if A has entries from an arbitrary field.

(5) When is the sum of EP matrices EP?

Remark: A trivial sufficient condition is the following. $A^*=NA$, $B^*=MB$, N and M nonsingular and $N=M$.

(6) Characterize the matrices A with the property that A^* is a polynomial in A.

(7) Find what additional condition a normal matrix A must satisfy so that $A=SU=US$, $S=S^*$ and $UU^*=I$.

(8) Find necessary and sufficient conditions that a linear transformation T on a finite-dimensional vector space be EP.

(9) Let T be a linear transformation on a finite-dimensional vector space V, B_1 a basis of V and $[T]B_1=C$. If T is EP, then there is a special basis ϵ such that $[T]B_2=A$ is an EP matrix and so C is similar to an EP matrix. Thus determining the EP linear transformations amounts to finding all matrices similar to an EP matrix. Find them.

(10) Let A be an EP matrix, B_1 and B_2 bases of a vector space, and $[T_1]_{B1},=A$, $[T_2]_{B2}=A$. Then T_1 and T_2 are EP and $T_1=R^{-1}T_2R$ for some R. Does R have any special properties?

(11) Prove theorems about EP linear transformations which are analogues of those about EP matrices.

Remark: Here are two easily established results about products.

(i) If S and T are linear transformations on a finite-dimensional vector space whose minimal polynomials have 0 as a simple root, and $ST=TS$, then ST is EP.

(ii) If S and T are linear transformations on a finite-dimensional vector space that are EP and commute, S and T have a common special basis, and the null spaces of S and T contain no isotropic vectors, then ST is EP.

(12) Let T_1 and T_2 be EP linear transformations on a vector space V with special bases B_1, B_2 respectively. If T_1T_2 is EP, determine a relation between a special basis of T_1T_2 and B_1, B_2.

Remark: In this connection, consider the following. Let A be an EP matrix and rank A^2 $=$rank A so that A^2 is also EP. If T is a linear transformation such that $[T]_B=A$, then T is EP with special basis B and $[T^2]_B=A^2$ so T^2 is EP with special basis B.

(13) Can some basis of the null space of an EP linear transformation T always be extended to a special basis of T?

Remark: This has played a distinguished role in section 4.

(14) The examples of special bases given are extreme. Find examples which are not.

(15) Characterize those EP linear transformations T(over an arbitrary field) that satisfy rank $T=$rank T^2.

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