

Inclusion of the Third Zonal Harmonic in an Accurate Reference Orbit of an Artificial Satellite*

John P. Vinti¹

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The previous paper gave the general theory and physical principles involved in so modifying the author's spheroidal potential for an oblate planet as to permit exact inclusion of the effects of the third zonal harmonic of the planet's gravitational field. The present paper carries out the computational details necessary to derive the resulting orbit, which now corresponds to a potential fitted exactly through the third zonal harmonic and to about two-thirds of the fourth.

The accuracy of the orbit itself, as a solution for the given potential, depends on the accuracy of solution of a certain cubic equation. The paper works out this solution through terms of the third order in J_2 , the coefficient of the second zonal harmonic, but its accuracy, and thus that of the secular terms, may be increased at will. Periodic terms are carried through the second order, but their accuracy may also be increased.

An obvious advantage of accounting for J_3 in this way is the absence of small denominators in e or $\sin I$ that occur in a perturbation theory. Another is the resulting increase in accuracy, through terms in J_2^2 , of the long-periodic third harmonic terms.

Key Words: Third zonal harmonic, increased accuracy, spheroidal reference orbit, artificial satellite.

1. Introduction

The previous paper² [V1966] developed the theory and physical principles for inclusion of the third zonal harmonic of a planet's gravitational potential in an accurate reference orbit of an artificial satellite. The aim of the present paper is to work out the resulting generalized reference orbit in such detail as to permit practical calculation.

To begin, let ρ , η , φ be oblate spheroidal coordinates, satisfying the equations

$$x + iy = r \cos \theta \exp i\varphi = [(\rho^2 + c^2)(1 - \eta^2)]^{1/2} \exp i\varphi \quad (1.1)$$

$$z = r \sin \theta = -\delta + \rho\eta. \quad (1.2)$$

Here x , y , z are rectangular coordinates of a satellite in a Cartesian frame, with origin O at the center of mass of an oblate planet, Oz pointing along the polar axis, and Ox pointing toward its vernal equinox. Also r , θ , φ are the planetocentric distance, latitude, and right ascension.

If r_e is the equatorial radius and if

$$c_2 = r_e^2 J_2 \left(1 - \frac{1}{4} J_3^2 J_2^{-3}\right) \quad (2)$$

$$\delta = -\frac{1}{2} r_e J_2^{-1} J_3, \quad (3)$$

then the gravitational potential

$$V = -\mu(\rho^2 + c^2\eta^2)^{-1}(\rho + \eta\delta) \quad (4)$$

leads to separability of the problem of satellite motion [V1966]. Here $\delta \approx 7$ km for the earth and

*Research supported by the National Aeronautical and Space Administration, Washington, D.C.

¹ Present Address: Department of Mathematics, North Carolina State University, Raleigh, North Carolina.

² Since there will be so many references to my own papers, I abbreviate my name to the initial V in the body of the paper.

the potential (4) leads to a fit of

$$V = -\frac{\mu}{r} \left[1 - \sum_2^{\infty} \left(\frac{r_e}{r} \right)^n J_n P_n(\sin \theta) \right] \quad (5)$$

exactly through the third zonal harmonic. It also gives about two-thirds of the fourth harmonic and negligible values, of order J_2^3 or smaller, for the higher harmonics.

By V1959, if α_1 is the energy, α_3 the z-component of angular momentum, and α_2 the separation constant, the generalized momenta p_ρ , p_η , and p_φ are given by

$$p_\varphi = \alpha_3 \quad (6.1), \quad p_\rho = \pm (\rho^2 + c^2)^{-1/2} F(\rho)^{1/2} \quad (6.2), \quad p_\eta = \pm (1 - \eta^2)^{-1} G(\eta)^{1/2}. \quad (6.3)$$

Here $F(\rho)$ and $G(\eta)$ are the quartics

$$F(\rho) = c^2 \alpha_3^2 + (\rho^2 + c^2)(-\alpha_2^2 + 2\mu\rho + 2\alpha_1\rho^2) \quad (7.1)$$

$$G(\eta) = -\alpha_3^2 + (1 - \eta^2)(\alpha_2^2 + 2\mu\eta\delta + 2\alpha_1 c^2 \eta^2). \quad (7.2)$$

The Hamilton-Jacobi function $W(\rho, \eta, \varphi)$ is then

$$W = \int p_\varphi d\varphi + \int p_\rho d\rho + \int p_\eta d\eta = \alpha_3 \varphi + \int_{\rho_1}^{\rho} \pm (\rho^2 + c^2)^{-1/2} F(\rho)^{1/2} d\rho + \int_{\eta_1}^{\eta} \pm (1 - \eta^2)^{-1} G(\eta)^{1/2} d\eta. \quad (8)$$

The quartic $F(\rho)$ has two physical zeros, ρ_1 and ρ_2 , and the quartic $G(\eta)$ has two physical zeros, η_1 and η_0 . The possible motions are thus confined to the toroidal region

$$0 < \rho_1 \leq \rho \leq \rho_2 \quad (9.1)$$

$$-1 \leq \eta_1 \leq \eta \leq \eta_0 \leq 1. \quad (9.2)$$

If β_1 , β_2 , and β_3 are constants, the orbit is then given by

$$t + \beta_1 = \frac{\partial W}{\partial \alpha_1}, \quad \beta_2' = \frac{\partial W}{\partial \alpha_2}, \quad \beta_3' = \frac{\partial W}{\partial \alpha_3}. \quad (10)$$

With ρ_1 and η_1 as lower limits of the integrals in (8), these become

$$t + \beta_1 = R_1 + c^2 N_1 \quad (11.1)$$

$$\beta_2' = -\alpha_2 R_2 + \alpha_2 N_2 \quad (11.2)$$

$$\varphi = \beta_3' - c^2 \alpha_3 R_3 + \alpha_3 N_3. \quad (11.3)$$

Here the ρ -integrals are given by

$$R_1 = \int_{\rho_1}^{\rho} \pm \rho^2 F^{-1/2} d\rho, \quad R_2 = \int_{\rho_1}^{\rho} \pm F^{-1/2} d\rho, \quad R_3 = \int_{\rho_1}^{\rho} \pm (\rho^2 + c^2)^{-1} F^{-1/2} d\rho \quad (12)$$

and the η -integrals by

$$N_1 = \int_{\eta_1}^{\eta} \pm \eta^2 G^{-1/2} d\eta \quad (12.1)$$

$$N_2 = \int_{\eta_1}^{\eta} \pm G^{-1/2} d\eta \quad (12.2)$$

$$N_3 = \int_{\eta_1}^{\eta} \pm (1 - \eta^2)^{-1} G^{-1/2} d\eta. \quad (12.3)$$

With η_1 as lower limit in the η -integrals, rather than 0 as before, we shall find that in the Keplerian limit ($J_2 \rightarrow 0, J_3 \rightarrow 0$) the constant $-\beta_1$ will again reduce to the time of passage through perigee. The constant β'_2 , however, will not reduce to ω , the argument of perigee, but to $\omega + \pi/2$. Similarly β'_3 will not reduce to Ω , the right ascension of the ascending node, but to $\Omega - \frac{1}{2} \pi \operatorname{sgn} \alpha_3$.

The ρ -integrals have already been worked out [V1961b]. The new η -integrals, however, will contain δ , involving J_3 , and will thus have to be reevaluated. The task of the present paper is thus to reevaluate these η -integrals N_1, N_2 , and N_3 , insert them into (11), invert (11.1) and (11.2) to find ρ and η as functions of time, and then insert the results into (11.3) to find φ as a function of time. We shall then have the new reference orbit, corresponding to the generalized potential (4) that allows for exact inclusion of the third harmonic.

2. Factoring the Quartics

In order to evaluate the integrals, we first have to factor the quartics $F(\rho)$ and $G(\eta)$. The factoring of $F(\rho)$ was essentially done in V1961b and the only new task that we have in connection with it is to relate associated quantities to the new orbital elements, which will depend on the new factoring of $G(\eta)$. To factor $G(\eta)$ note that by (7.2)

$$G(\eta) = \alpha_2^2 - \alpha_3^2 + 2\mu\eta\delta + (2\alpha_1 c^2 - \alpha_2^2)\eta^2 - 2\mu\eta^3\delta - 2\alpha_1 c^2 \eta^4. \quad (13)$$

By (6.3) and (13), if the orbit ever reaches $\eta = 0$, we should have $\alpha_2^2 - \alpha_3^2 \geq 0$. If indeed the odd harmonics are all put equal to zero, we should have $\delta = \frac{1}{2} r_e J_2^{-1} |J_3| = 0$ and by (1.2) a motion $z = 0$ in the equatorial plane would be possible, with η remaining zero and $\alpha_2^2 - \alpha_3^2 \geq 0$.

As soon as we insert the third harmonic, however, there arises a doubt that $\eta = 0$ will always be reached. Accordingly, we must leave the sign of $\alpha_2^2 - \alpha_3^2$ open. For bounded orbits we now have $\alpha_1 < 0$ and the table

η	$G(\eta)$
$-\infty$	$+\infty$
-1	$-\alpha_3^2 \leq 0$
0	$\alpha_2^2 - \alpha_3^2$, sign either \pm
$+1$	$-\alpha_3^2 \leq 0$
$+\infty$	$+\infty$

There are thus two non-physical zeros, between $-\infty$ and -1 and between $+1$ and $+\infty$, and two physical zeros, between -1 and $+1$. Call these latter η_1 and η_0 , with $\eta_1 \leq \eta_0$. If $\delta = 0$, $G(\eta)$ becomes a quadratic in η^2 and $\eta_1 = -\eta_0$, where $\eta_0 \geq 0$, the equality sign holding when $\alpha_2^2 - \alpha_3^2 = 0$.

To see what happens when $\delta \neq 0$, make a plot of $G(\eta)$ versus η , noting that $G'(0) = 2\mu\delta > 0$. In order to avoid more than two zeros between $\eta = -1$ and $\eta = +1$, we must then have

if	that	and
$\alpha_2^2 - \alpha_3^2 > 0$	$\eta_1 < 0 < \eta_0$	$-\eta_0\eta_1 > 0$
$\alpha_2^2 - \alpha_3^2 = 0$	$0 = \eta_1 < \eta_0$	$-\eta_0\eta_1 = 0$
$\alpha_2^2 - \alpha_3^2 < 0$	$0 < \eta_1 \leq \eta_0$	$-\eta_0\eta_1 < 0$

The paper V1961b indicated the following form for factoring $G(\eta)$:

$$G(\eta) = (\alpha_2^2 - \alpha_3^2) \left(1 - \frac{\eta}{\eta_0}\right) \left(1 - \frac{\eta}{\eta_1}\right) (1 + C_1\eta - C_2\eta^2). \quad (14)$$

Here, since $\eta_1 \leq \eta \leq \eta_0$, we have

$$(\eta - \eta_1)(\eta_0 - \eta) \geq 0. \quad (15)$$

For $\alpha_2^2 - \alpha_3^2 > 0$, we have $-\eta_0\eta_1 > 0$ and division gives

$$\left(1 - \frac{\eta}{\eta_0}\right) \left(1 - \frac{\eta}{\eta_1}\right) \geq 0. \quad (16)$$

For $\alpha_2^2 - \alpha_3^2 < 0$, we have $\eta_0\eta_1 > 0$ and division gives

$$\left(1 - \frac{\eta}{\eta_0}\right) \left(1 - \frac{\eta}{\eta_1}\right) < 0. \quad (17)$$

In either case, since $G(\eta) \geq 0$, we find that the factor $1 + C_1\eta - C_2\eta^2 > 0$, where C_1 and C_2 will both be small compared to unity.

On comparing coefficients in (13) and (14), with

$$q_0 \equiv \mu^{-1}(\alpha_2^2 - \alpha_3^2) \quad (18)$$

$$P \equiv \frac{1}{2}(\eta_0 + \eta_1) \quad (19)$$

$$S \equiv -\eta_0\eta_1, \quad (20)$$

and with the relations from V1961b, viz,

$$a_0 \equiv -\frac{1}{2} \frac{\mu}{\alpha_1}, \quad e_0 \equiv \left(1 + \frac{2\alpha_1\alpha_2^2}{\mu^2}\right)^{1/2}, \quad p_0 \equiv a_0(1 - e_0^2) \quad (21.1)$$

$$\alpha_2^2 = \mu p_0, \quad (21.2)$$

we have

$$2P + C_1S = \frac{2\delta}{q_0} S \quad (22.1)$$

$$1 + C_2S - 2C_1P = \frac{p_0 + c^2/a_0}{q_0} S \quad (22.2)$$

$$C_1 + 2C_2P = \frac{2\delta}{q_0} S \quad (22.3)$$

$$C_2 = \frac{c^2}{a_0 q_0} S. \quad (22.4)$$

Along with three constants, to be derived later from β_1 , β'_2 , and β'_3 , we shall choose as orbital elements

$$a = \frac{1}{2} (\rho_1 + \rho_2) \quad (23.1)$$

$$e = \frac{\rho_2 - \rho_1}{\rho_2 + \rho_1} \quad (23.2)$$

$$p \equiv a(1 - e^2) \quad (23.3)$$

as in V1961b, and

$$S \equiv -\eta_0 \eta_1. \quad (23.4)$$

If $\delta=0$, η_1 becomes $-\eta_0$, and S reduces to the old $\eta_0^2 = \sin^2 I$ of V1961b. Note that now, however, the new orbital element S can sometimes become negative, viz, for orbits sufficiently close to equatorial. Note that for a polar orbit $\eta_0 = +1$ and $\eta_1 = -1$, so that S then equals unity. In any case, since $-1 \leq \eta_1 \leq \eta_0 \leq 1$, we have

$$S^2 \leq 1. \quad (24)$$

Our next task is to express the factorization of the quartics $F(\rho)$ and $G(\eta)$ completely in terms of the elements a , e , and the new S . To do so, we must first express C_1 , C_2 , and P in terms of c , δ , a_0 , p_0 , and S ; that is, we must eliminate q_0 .

On inserting (22.4) into (22.3) and solving the resulting equation simultaneously with (22.1), we find

$$C_1 = \left(1 - \frac{c^2}{a_0 q_0} S^2\right)^{-1} \frac{2\delta}{q_0} S \left(1 - \frac{c^2}{a_0 q_0} S\right) \quad (25.1)$$

$$P = \left(1 - \frac{c^2}{a_0 q_0} S^2\right)^{-1} \frac{\delta}{q_0} S(1 - S). \quad (25.2)$$

If we now insert (25.1), (25.2), and (22.4) into (22.2) and call

$$\frac{p_0}{q_0} S \equiv u, \quad (26)$$

we find

$$\frac{1}{u} = 1 + \frac{c^2}{a_0 p_0} (1 - S) + \frac{\left(\frac{2\delta}{p_0}\right)^2 (1 - S) \left(\frac{1}{u} - \frac{c^2}{a_0 p_0}\right)}{\left(\frac{1}{u} - \frac{c^2}{a_0 p_0} S\right)^2}. \quad (27)$$

For $\delta=0$, S becomes $\sin^2 I$, u becomes $\sin^2 I / \sin^2 i_0$, where

$$i_0 \equiv \cos^{-1} \frac{\alpha_3}{\alpha_2}, \quad (28)$$

from V1961b, and (27) reduces to

$$\sin^2 i_0 = \sin^2 I + \frac{c^2}{a_0 \rho_0} \sin^2 I \cos^2 I, \quad (29)$$

in agreement with (4.8) of V1961b. Equation (27) is a cubic in u^{-1} , but we may solve it by successive approximations. To do so, note first that

$$\frac{1}{u} = 1 + \frac{c^2}{a_0 \rho_0} (1 - S) + O(\delta^2). \quad (30)$$

If we insert (30) into the right side of (27), the error will be of order δ^4 , i.e., of order J_2^4 . Thus

$$\frac{1}{u} = 1 + \frac{c^2}{a_0 \rho_0} (1 - S) + \epsilon, \quad (31.1)$$

where

$$\epsilon = \frac{\left(\frac{2\delta}{\rho_0}\right)^2 (1 - S) \left(1 - \frac{c^2}{a_0 \rho_0} S\right)}{\left[1 + \frac{c^2}{a_0 \rho_0} (1 - 2S)\right]^2} + O(J_2^4). \quad (31.2)$$

(Insertion of (31) in the right side of (27) would reduce the error to $O(J_2^6)$, but such accuracy is not needed.) Equation (31) frees u from q_0 and thus also C_1 , C_2 , and P , since by (22.4), (25), and (26)

$$C_2 = \frac{c^2}{a_0 \rho_0} u \quad (32.1)$$

$$C_1 = \left(1 - \frac{c^2}{a_0 \rho_0} S u\right)^{-1} \frac{2\delta}{\rho_0} u \left(1 - \frac{c^2}{a_0 \rho_0} u\right) \quad (32.2)$$

$$P = \left(1 - \frac{c^2}{a_0 \rho_0} S u\right)^{-1} \frac{\delta}{\rho_0} u (1 - S). \quad (32.3)$$

These quantities C_1 , C_2 , and P are then correct through order J_2^4 .

Since $|S| \leq 1$, we find from (31) that $\epsilon \geq 0$ and $u^{-1} \geq 1$. Thus

$$0 < u \leq 1 \quad (33.1)$$

and then, by (32)

$$C_2 > 0, \quad C_1 > 0, \quad P \geq 0. \quad (33.2)$$

In order to obtain all the required quantities as functions of a , e , and S , we must now summarize the factoring of $F(\rho)$, given in V1961b. We have

$$F(\rho) = -2\alpha_1(\rho - \rho_1)(\rho_2 - \rho)(\rho^2 + A\rho + B) \quad (34.1)$$

$$a \equiv \frac{1}{2}(\rho_1 + \rho_2), \quad e \equiv \frac{\rho_2 - \rho_1}{\rho_2 + \rho_1}, \quad p \equiv a(1 - e^2). \quad (34.2)$$

Then, with q_0 defined by (18) and a_0 and p_0 by (21), comparison of coefficients in (34.1) and (7.1) gives

$$2a - A = 2a_0 \quad (35.1)$$

$$B + ap - 2Aa = c^2 + a_0p_0 \quad (35.2)$$

$$2aB - Aap = 2a_0c^2 \quad (35.3)$$

$$Bap = a_0p_0c^2q_0/p_0. \quad (35.4)$$

We now regard a , p , S , c , and δ as known and have to solve the eight equations in (35), (32), and (26) for the eight unknowns a_0 , p_0 , q_0 , A , B , C_1 , C_2 , and P . Here u is an auxiliary parameter given by (31).

From (35.4), (35.2), (26), and (31), we find

$$Bap = c^2S \left[B + ap - 2Aa - c^2S + \frac{4 \frac{a_0}{p_0} \delta^2 (1-S) \left(1 - \frac{c^2}{a_0p_0} S \right)}{\left[1 + \frac{c^2}{a_0p_0} (1-2S) \right]^2} + O(J_2^4) \right]. \quad (36)$$

Now since A and B are both of order J_2 , we have from (35.1) and (35.2)

$$\frac{c^2}{a_0p_0} = \frac{c^2}{ap} + O(J_2^2) \quad (37.1)$$

$$\frac{a_0}{p_0} = \frac{a}{p} \left(1 + \frac{c^2}{ap} + \frac{2A}{p} - \frac{A}{a} - \frac{B}{ap} \right) + O(J_2^2) \quad (37.2)$$

Because of the factor δ^2 , of order J_2^2 , insertion of (37) into (36) will not change the order of the error. Thus

$$Bap = c^2S \left[B + ap - 2Aa - c^2S + \frac{4a}{p} (1-S)\delta^2 \left\{ 1 + \frac{c^2}{ap} (3S-1) + A \left(\frac{2}{p} - \frac{1}{a} \right) - \frac{B}{ap} \right\} \right] + O(J_2^3). \quad (38)$$

This is an equation involving only A , B , and "known" quantities. Another follows by combination of (35.1) and (35.3). It is

$$2aB - Aap = (2a - A)c^2. \quad (39)$$

The solution of (38) and (39) for A and B is

$$A = \frac{-2ac^2(ap - c^2S)(1-S) + \frac{8a^2c^2}{p} \delta^2 \left\{ 1 + \frac{c^2}{ap} (3S-2) \right\} S(1-S)}{(ap - c^2)(ap - c^2S) + 4a^2c^2S + \frac{4c^2}{p^2} \delta^2 (3ap - 4a^2 - c^2)S(1-S)} \quad (40.1)$$

$$B = (2a)^{-1}(ap - c^2)A + c^2. \quad (40.2)$$

For a polar orbit, for which $S = 1$, we have from (40) and (35)

$$A = 0 \quad B = c^2 \quad a_0 = a \quad p_0 = p. \quad (41)$$

For such a polar orbit, if $e = 0$, we then have

$$\rho = a = a_0 = p = p_0, \quad e_0 = 0. \quad (42)$$

Equations (40) give us A and B in terms of known quantities. To find the others, note that a_0 then follows from (35.1), p_0 from (35.2), q_0 from (26) and (31), and C_2 , C_1 , and P from (32). With use of these equations, the errors arising from the approximate solution of the cubic (27) are of order J_2^5 for all the unknowns except q_0 . For q_0 the error is of order J_2^4 . The "known" quantities are c , δ , a , p , and S . Here $c = r_e J_2^{1/2} \left(1 - \frac{1}{4} J_3^2 J_2^{-3}\right)^{1/2}$, $\delta = \frac{1}{2} r_e J_2^{-1} |J_3|$, a is a constant "semi-major axis," p a constant "semi-latus rectum," and S a constant quantity analogous to the squared sine of the inclination for inclinations not too close to zero. For inclinations close to zero, however, S may be negative for $\delta \neq 0$.

3. The η -Integrals

From (14), (18), (19), and (20), we find

$$G(\eta) = \mu q_0 S^{-1} (S + 2P\eta - \eta^2)(1 + C_1\eta - C_2\eta^2). \quad (43)$$

Now

$$S + 2P\eta - \eta^2 \equiv P^2 + S - (\eta - P)^2. \quad (44)$$

The physical zeros η_1 and η_0 of $G(\eta)$ are the zeros of (44), viz,

$$\eta_0 = P + (P^2 + S)^{1/2} \quad (45.1)$$

$$\eta_1 = P - (P^2 + S)^{1/2}. \quad (45.2)$$

For $\delta \neq 0$, S may be negative, with smallest possible value $S = -P^2$, corresponding to

$$\eta_0 = \eta_1 = \eta' = \frac{\delta}{p_0 + c^2/a_0} \left(1 - \frac{3\delta^2}{p_0^2}\right) + O(J_2^4), \quad (46)$$

from V1966.

With

$$Q \equiv (P^2 + S)^{1/2} \geq 0, \quad (47)$$

we may now introduce a variable ψ , such that

$$\eta = P + Q \sin \psi. \quad (48)$$

If ψ continually increases, so that $\dot{\psi} > 0$ always, η will then vary between the limits η_1 and η_0 and the double sign in eqs (12) for the η -integrals will disappear, with ψ as integration variable. Thus ψ is a uniformizing variable.

Insertion of (48) into (43) then gives

$$G(\eta) = \mu q_0 Q^2 S^{-1} (1 + C_1 \eta - C_2 \eta^2) \cos^2 \psi. \quad (49)$$

and the factor $\pm G^{-1/2} d\eta$ in the integrals (12) becomes

$$\pm G^{-1/2} d\eta = \left(\frac{S}{\mu q_0 Q} \right)^{1/2} (1 + C_1 \eta - C_2 \eta^2)^{-1/2} d\psi \quad (50.1)$$

$$= \alpha_2^{-1} u^{1/2} (1 + C_1 \eta - C_2 \eta^2)^{-1/2} d\psi \quad (50.2)$$

by (26) and (21.2).

4. The Integral N_2

By (12.2), (50), and (48) we then find

$$\alpha_2 u^{-1/2} N_2 = \int_{-\pi/2}^{\psi} (1 + C_1 \eta - C_2 \eta^2)^{-1/2} d\psi. \quad (51)$$

Here C_1 and C_2 are both positive, of order J_2 . To evaluate (51), we place

$$h \equiv i C_2^{1/2} \eta = O(J_2), \quad \lambda \equiv \frac{1}{2} i C_1 C_2^{-1/2} = O(J_2^{1/2}), \quad (52)$$

so that

$$1 + C_1 \eta - C_2 \eta^2 \equiv 1 - 2\lambda h + h^2. \quad (53)$$

Since $\lambda = O(J_2^{1/2})$, it follows that $|\lambda \pm \sqrt{\lambda^2 - 1}| = O(1)$ and

$$|h| \ll \text{smaller of } |\lambda \pm \sqrt{\lambda^2 - 1}|. \quad (54)$$

Thus a Legendre expansion is legitimate, so that

$$(1 + C_1 \eta - C_2 \eta^2)^{-1/2} = (1 - 2\lambda h + h^2)^{-1/2} = \sum_{n=0}^{\infty} h^n P_n(\lambda) = \sum_{n=0}^{\infty} i^n C_2^{n/2} \eta^n P_n(\lambda). \quad (55)$$

Then (51) and (55) lead to

$$\alpha_2 u^{-1/2} N_2 = \sum_{n=0}^{\infty} F_n(C_1, C_2) \int_{-\pi/2}^{\psi} \eta^n d\psi, \quad (56.1)$$

where

$$F_n(C_1, C_2) \equiv i^n C_2^{n/2} P_n\left(\frac{1}{2} i C_1 C_2^{-1/2}\right), \quad (56.2)$$

are real polynomials in C_1 and C_2 . Through F_8 they are

$$F_0 = 1, \quad F_1 = -\frac{1}{2} C_1, \quad F_2 = \frac{3}{8} C_1^2 + \frac{1}{2} C_2, \quad F_3 = -\frac{5}{16} C_1^3 - \frac{3}{4} C_1 C_2,$$

$$\begin{aligned}
F_4 &= \frac{35}{128} C_1^4 + \frac{15}{16} C_1^2 C_2 + \frac{3}{8} C_2^2, & F_5 &= -\frac{63}{256} C_1^5 - \frac{35}{32} C_1^3 C_2 - \frac{15}{16} C_1 C_2^2, \\
F_6 &= \frac{231}{1024} C_1^6 + \frac{315}{256} C_1^4 C_2 + \frac{105}{64} C_1^2 C_2^2 + \frac{5}{16} C_2^3, & F_7 &= -\frac{429}{2048} C_1^7 - \frac{693}{512} C_1^5 C_2 - \frac{315}{128} C_1^3 C_2^2 - \frac{35}{32} C_1 C_2^3, \\
F_8 &= \frac{6435}{128} C_1^8 + \frac{3003}{2048} C_1^6 C_2 + \frac{3465}{1024} C_1^4 C_2^2 + \frac{315}{128} C_1^2 C_2^3 + \frac{35}{128} C_2^4.
\end{aligned} \tag{57}$$

Higher F 's are of order $J^{\frac{5}{2}}$ or smaller.

We now have to evaluate

$$\int_{-\frac{\pi}{2}}^{\psi} \eta^n d\psi = \int_{-\frac{\pi}{2}}^{\psi} (P - Q \sin x)^n dx = \int_0^w (P + Q \cos y)^n dy \equiv f_n(w), \tag{58.1}$$

where

$$w \equiv \pi/2 + \psi. \tag{58.2}$$

As in V1961b, p. 178, we now readily show that

$$f_n(w) = \pi^{-1} w f_n(\pi) + g_n(w), \tag{59}$$

where $g_n(w)$ is an odd function of w , of period 2π . From that same paper

$$f_n(\pi) \equiv \int_0^{\pi} (P - Q \cos y)^n dy = \pi P^n (1 - e_1^2)^{n/2} P_n[(1 - e_1^2)^{-1/2}], \tag{60.1}$$

where $P_n(x)$ is a Legendre polynomial in x and where

$$e_1 \equiv Q/P \tag{60.2}$$

$$0 < e_1 < 1 \text{ for } S < 0 \tag{60.3}$$

$$e_1 > 1 \text{ for } S > 0. \tag{60.4}$$

For $S < 0$, the factors $(1 - e_1^2)^{n/2}$ and $P_n[(1 - e_1^2)^{-1/2}]$ are both real, for all n . For $S > 0$, they are both real if n is even and both imaginary if n is odd. In any case their product is real.

If we now introduce, as in V1961b, p. 178,

$$R_n(x) \equiv x^n P_n(1/x), \tag{61}$$

a polynomial of degree $[n/2]$ in x^2 , we find

$$\pi^{-1} f_n(\pi) = G_n(P, Q) \tag{62.1}$$

$$G_n(P, Q) \equiv P^n R_n(\sqrt{1 - e_1^2}), \tag{62.2}$$

polynomials of degree n in P and Q . Through $n=8$ they are given by

$$G_0 = 1, G_1 = P, G_2 = P^2 + \frac{1}{2} Q^2, G_3 = P^3 + \frac{3}{2} P Q^2,$$

$$G_4 = P^4 + 3P^2Q^2 + \frac{3}{8}Q^4, G_5 = P^5 + 5P^3Q^2 + \frac{15}{8}PQ^4,$$

$$16 G_6 = 16P^6 + 120 P^4Q^2 + 90P^2Q^4 + 5Q^6, \quad 16G_7 = 16P^7 + 168P^5Q^2 + 210P^3Q^4 + 35PQ^6$$

$$128 G_8 = 128P^8 + 1972P^6Q^2 + 3360P^4Q^4 + 1120P^2Q^6 + 35Q^8. \quad (63)$$

On inserting (58), (59), and (62) into (56.1), we find

$$\alpha_2 u^{-1/2} N_2 = \left(\frac{\pi}{2} + \psi\right) B_2 + \sum_{n=0}^{\infty} F_n(C_1, C_2) \sum_{j=1}^{\infty} c_{nj} \cdot \sin jw, \quad (64.1)$$

where

$$B_2 = \sum_{n=0}^{\infty} F_n(C_1, C_2) G_n(P, Q). \quad (64.2)$$

Remembering that C_1 , C_2 , and P are all of order J_2 and that Q is of order J_2^0 , we have from (64.2), (57), and (63), through order J_2^3 :

$$\begin{aligned} B_2 = & 1 - \frac{1}{2}C_1P + \left(\frac{3}{8}C_1^2 + \frac{1}{2}C_2\right) \left(P^2 + \frac{1}{2}Q^2\right) + \frac{9}{64}C_2^2Q^4 - \frac{9}{8}C_1C_2PQ^2 \\ & - \frac{15}{32}C_1^3PQ^2 + \frac{45}{128}C_1^2C_2Q^4 + \frac{25}{256}C_2^3Q^6 + \frac{105}{1024}C_1^4Q^4 \\ & + \frac{9}{8}C_2^2P^2Q^2 - \frac{225}{128}C_1C_2^2PQ^4 + \frac{525}{1024}C_1^2C_2^2Q^6 + \left(\frac{35}{128}\right)^2 C_2^3Q^8 + O(J_2^5). \end{aligned} \quad (65)$$

Note that B_2 contains $-\frac{1}{2}C_1P + \frac{3}{16}C_1^2Q^2$, proportional to $J_3^2/J_2^2 = O(J_2^2)$, so that inclusion of J_3 in the separable problem apparently leads to new second-order secular terms. We shall say more about this point later.

Let us check B_2 for $\delta=0$, in which case it should reduce to

$$B_2 = \pi^{-1} K(q) = 1 + \frac{1}{4}q^2 + \frac{9}{64}q^4 + \frac{25}{256}q^6 + \left(\frac{35}{128}\right)^2 q^8 + \dots, \quad (66)$$

given on p. 184 of V1961b. For $\delta=0$, we have $C_1=0$, $P=0$, $\eta_1 = -\eta_0$, $S = \eta_0^2 = \sin^2 I$, and $Q = \eta_0 = \sin I$. For C_2 we then find from (22.4)

$$C_2 = \frac{c^2}{a_0 p_0} \frac{p_0}{q_0} \eta_0^2 = \frac{c^2}{a_0 p_0} \frac{\alpha_2^2}{\alpha_2^2 - \alpha_3^2} \eta_0^2, \quad (67)$$

with use of (18) and (21.2). But in the notation of V1961b, $(\alpha_2^2 - \alpha_3^2)/\alpha_2^2 = \sin^2 i_0$, so that we find

$$C_2 = \frac{c^2}{a_0 p_0} \frac{\eta_0^2}{\sin^2 i_0} \quad (68.1)$$

$$= \eta_2^2, \quad (68.2)$$

from p. 173 of V1961b. On inserting these values into (65) and using $q \equiv \eta_0/\eta_2$, we find

$$B_2 = 1 + \frac{1}{4}q^2 + \frac{9}{64}q^4 + \frac{25}{256}q^6 + \left(\frac{35}{128}\right)^2 q^8 + \dots \quad (69)$$

This agrees with (66), so that B_2 reduces correctly for $\delta=0$.

We now have to find the periodic terms in (64.1), i.e., the coefficients c_{nj} . Since $F_5(c_1, c_2)$ is of order J_2^3 , we need take n only through the value 4. By expressing $(P + Q \cos y)^k$, $k=0, 1, 2, 3, 4$, as trigonometric polynomials, evaluating $\int_0^\omega (P - Q \cos y)^k dy$, picking out the periodic terms, and keeping results only through order J_2^3 , we find from (56.1) and (57)

$$\begin{aligned} \alpha_2 u^{-1/2} N_2 &= \left(\frac{\pi}{2} + \psi\right) B_2 + \left(P - \frac{3}{16} C_1 Q^2\right) C_2 Q \sin w \\ &+ \frac{1}{32} [(4C_2 + 3C_1^2) Q^2 + 3C_2^2 Q^4] \sin 2w + \frac{C_1 C_2}{16} Q^3 \sin 3w \\ &+ \frac{3}{256} C_2^2 Q^4 \sin 4w + \text{periodic terms of order } J_2^3. \end{aligned} \quad (70)$$

In terms of $\psi = w - \pi/2$, this becomes

$$\begin{aligned} \alpha_2 u^{-1/2} N_2 &= \left(\frac{\pi}{2} + \psi\right) B_2 + \left[C_2 Q \left(P + \frac{9}{16} C_1 Q_2\right) + \frac{1}{2} C_1 Q \right] \cos \psi \\ &- \frac{1}{32} [(4C_2 + 3C_1^2) Q^2 + 3C_2^2 Q^4] \sin 2\psi - \frac{C_1 C_2}{16} Q^3 \cos 3\psi \\ &+ \frac{3}{256} C_2^2 Q^4 \sin 4\psi + \text{periodic terms of } O(J_2^3). \end{aligned} \quad (71)$$

For $\delta=0$, we find that the periodic terms in (71) agree with those on p. 184 of V1961b, as they should.

5. The Integral N_1

By (12.1), (50), and (48), we have

$$\alpha_2 u^{-1/2} N_1 = \int_{-\pi/2}^{\psi} (1 + C_1 \eta - C_2 \eta^2)^{-1/2} \eta^2 d\psi. \quad (72)$$

Then, from (55), (56.2), and (58.1)

$$\alpha_2 u^{-1/2} N_1 = \sum_{n=0}^{\infty} F_n(C_1, C_2) f_{n+2}(w). \quad (73)$$

As before,

$$f_{n+2}(w) = w G_{n+2}(P, Q) + g_{n+2}(w), \quad (74)$$

where $g_{n+2}(w)$ is odd in w , with period 2π . Thus

$$\alpha_2 u^{-1/2} N_1 = \left(\frac{\pi}{2} + \psi \right) B'_1 + \sum_{n=0}^{\infty} F_n(C_1, C_2) g_{n+2}(w), \quad (75.1)$$

where

$$B'_1 \equiv \sum_{n=0}^{\infty} F_n(C_1, C_2) G_{n+2}(P, Q). \quad (75.2)$$

Since the integral N_1 has a factor $c^2 = O(J_2)$ in the kinetic equations (11), we need each $F_n G_{n+2}$ correct only through order J_2^3 , in order to obtain secular terms correct through order J_2^4 . From (75.2), (57), and (63), we then find

$$B'_1 = \frac{1}{2} Q^2 + P^2 - \frac{3}{4} C_1 P Q^2 + \frac{3}{2} C_2 P^2 Q^2 + \frac{3}{64} (4C_2 + 3C_1^2) Q^4 \\ - \frac{45}{32} C_1 C_2 P Q^4 + \frac{5}{256} (6C_2^2 + 15C_1^2 C_2) Q^6 + \frac{175}{2048} C_2^3 Q^8 + O(J_2^4). \quad (76)$$

For $\delta=0$, this reduces to

$$B'_1 = \frac{1}{2} \eta_0^2 + \frac{3}{16} \eta_2^{-2} \eta_0^4 + \frac{15}{128} \eta_2^{-4} \eta_0^6 + \frac{175}{2048} \eta_2^{-6} \eta_0^8 + \dots$$

or

$$B'_1 \eta_0^{-2} = \frac{1}{2} + \frac{3}{16} q^2 + \frac{15}{128} q^4 + \frac{175}{2048} q^6 + \dots \quad (77)$$

in agreement with the value of B_1 on p. 184 of V1961b.

To find the periodic terms in (75.1), we proceed just as for N_2 , obtaining

$$\alpha_2 u^{-1/2} N_1 = \left(\frac{\pi}{2} + \psi \right) B'_1 + \left(2PQ + \frac{3}{8} C_1 Q^3 \right) \cos \psi \\ - \frac{1}{8} (2Q^2 + C_2 Q^4) \sin 2\psi - \frac{1}{24} C_1 Q^3 \cos 3\psi \\ + \frac{1}{64} C_2 Q^4 \sin 4\psi + \text{periodic terms of } O(J_2^3). \quad (78)$$

For $\delta=0$, the periodic terms agree with those appearing in (6.38) of V1961b.

6. The Integral N_3

By (12.3), (50), and (48), we have

$$\alpha_2 u^{-1/2} N_3 = \int_{-\pi/2}^{\psi} (1 - \eta^2)^{-1} (1 + C_1 \eta - C_2 \eta^2)^{-1/2} d\psi. \quad (79)$$

Then, from (55) and (56.6),

$$\alpha_2 u^{-1/2} N_3 = \sum_{n=0}^{\infty} F_n(C_1, C_2) L_n \quad (80.1)$$

$$L_n \equiv \int_{-\frac{\pi}{2}}^{\psi} (1 - \eta^2) \eta^n d\psi. \quad (80.2)$$

In particular

$$L_0 \equiv \int_{-\frac{\pi}{2}}^{\psi} (1 - \eta^2)^{-1} d\psi \quad (81.1) \quad L_1 \equiv \int_{-\frac{\pi}{2}}^{\psi} (1 - \eta^2)^{-1} \eta d\psi. \quad (81.2)$$

To evaluate the L_n 's we first prove the lemmas

$$L_{2\sigma} = L_0 - \sum_{\tau=0}^{\sigma-1} \int_{-\frac{\pi}{2}}^{\psi} \eta^{2\tau} d\psi \quad (\sigma \geq 1) \quad (82.1)$$

$$L_{2\sigma+1} = L_1 - \sum_{\tau=0}^{\sigma-1} \int_{-\frac{\pi}{2}}^{\psi} \eta^{2\tau+1} d\psi \quad (\sigma \geq 1). \quad (82.2)$$

To prove them, note first that

$$\frac{\eta^n}{1 - \eta} = \frac{1}{1 - \eta} - \sum_{k=0}^{n-1} \eta^k \quad (83)$$

and

$$\frac{\eta^n}{1 + \eta} = \frac{(-1)^n}{1 + \eta} - \sum_{k=0}^{n-1} (-1)^{n+k} \eta^k. \quad (84)$$

Addition of (83) and (84) then gives

$$\frac{\eta^n}{1 - \eta} + \frac{\eta^n}{1 + \eta} = \frac{1}{1 - \eta} + \frac{(-1)^n}{1 + \eta} - \sum_{k=0}^{n-1} [1 + (-1)^{n+k}] \eta^k, \quad (85)$$

reducing to the two cases $n = \text{odd}$ and $n = \text{even}$, viz.,

$$\frac{\eta^{2\sigma}}{1 - \eta^2} = \frac{1}{1 - \eta^2} - \sum_{\tau=0}^{\sigma-1} \eta^{2\tau} \quad (\sigma \geq 1) \quad (86.1)$$

$$\frac{\eta^{2\sigma+1}}{1 - \eta^2} = \frac{\eta}{1 - \eta^2} - \sum_{\tau=0}^{\sigma-1} \eta^{2\tau+1} \quad (\sigma \geq 1). \quad (86.2)$$

The lemmas (82) then follow directly from (80.2) and (86).

With application of (58.1), (59), and (62), we then find from them

$$L_{2\sigma} = L_0 - w \sum_{\tau=0}^{\sigma-1} G_{2\tau}(P, Q) - \sum_{\tau=0}^{\sigma-1} g_{2\tau}(w), \quad \sigma \geq 1 \quad (87.1)$$

$$L_{2\sigma+1} = L_1 - w \sum_{\tau=0}^{\sigma-1} G_{2\tau+1}(P, Q) - \sum_{\tau=0}^{\sigma-1} g_{2\tau+1}(w), \quad \sigma \geq 1, \quad (87.2)$$

where $w = \psi + \pi/2$. Insertion of (87) into (80.1) then gives

$$\begin{aligned} \alpha_2 u^{-1/2} N_3 = & L_0 \sum_{\sigma=0}^{\infty} F_{2\sigma} + L_1 \sum_{\sigma=0}^{\infty} F_{2\sigma+1} + B_3 w \\ & - \sum_{\sigma=1}^{\infty} F_{2\sigma} \sum_{\tau=0}^{\sigma-1} g_{2\tau}(w) - \sum_{\sigma=1}^{\infty} F_{2\sigma+1} \sum_{\tau=0}^{\sigma-1} g_{2\tau+1}(w), \quad (88.1) \end{aligned}$$

$$B_3 \equiv - \sum_{\sigma=1}^{\infty} F_{2\sigma} \sum_{\tau=0}^{\sigma-1} G_{2\tau} - \sum_{\sigma=1}^{\infty} F_{2\sigma+1} \sum_{\tau=0}^{\sigma-1} G_{2\tau+1}. \quad (88.2)$$

To simplify the first terms in (88.1), note that by (56.2) and (52), if

$$h_0 \equiv iC_2^{1/2}, \quad (89)$$

the value of h for $\eta=1$, we obtain

$$F_{2\sigma} = h_0^{2\sigma} P_{2\sigma}(\lambda) \quad (90.1) \quad F_{2\sigma+1} = h_0^{2\sigma+1} P_{2\sigma+1}(\lambda). \quad (90.2)$$

To evaluate the sums $\sum_0^{\infty} F_{2\sigma}$ and $\sum_0^{\infty} F_{2\sigma+1}$, note that

$$(1 - 2\lambda h_0 + h_0^2)^{-1/2} = \sum_0^{\infty} h_0^{2\sigma} P_{2\sigma}(\lambda) + \sum_0^{\infty} h_0^{2\sigma+1} P_{2\sigma+1}(\lambda) \quad (91.1)$$

and, since $P_{2\sigma}(-\lambda) = P_{2\sigma}(\lambda)$ and $P_{2\sigma+1}(-\lambda) = -P_{2\sigma+1}(\lambda)$, that

$$(1 + 2\lambda h_0 + h_0^2)^{-1/2} = \sum_0^{\infty} h_0^{2\sigma} P_{2\sigma}(\lambda) - \sum_0^{\infty} h_0^{2\sigma+1} P_{2\sigma+1}(\lambda). \quad (91.2)$$

Solving eqs (91) for each sum, with use of (90), we find

$$2 \sum_0^{\infty} F_{2\sigma} = (1 - 2\lambda h_0 + h_0^2)^{-1/2} + (1 + 2\lambda h_0 + h_0^2)^{-1/2} \quad (92.1)$$

$$2 \sum_0^{\infty} F_{2\sigma+1} = (1 - 2\lambda h_0 + h_0^2)^{-1/2} - (1 + 2\lambda h_0 + h_0^2)^{-1/2}. \quad (92.2)$$

Since $h = h_0$ when $\eta=1$, we have from (53)

$$1 \mp 2\lambda h_0 + h_0^2 = 1 \pm C_1 - C_2. \quad (93)$$

Then from (92) and (93)

$$\begin{aligned} L_0 \sum_0^{\infty} F_{2\sigma} + L_1 \sum_0^{\infty} F_{2\sigma+1} &= \frac{1}{2} L_0 [(1 + C_1 - C_2)^{-1/2} + (1 - C_1 - C_2)^{-1/2}] \\ &\quad + \frac{1}{2} L_1 [(1 + C_1 - C_2)^{-1/2} - (1 - C_1 - C_2)^{-1/2}]. \end{aligned} \quad (94)$$

For B_3 , the coefficient of a secular term, we find from (88.2), (57), and (63), carrying terms through order J_2^4 ,

$$\begin{aligned} B_3 &= -\frac{1}{2} C_2 - \frac{3}{8} C_1^2 - \left(\frac{35}{128} C_1^4 + \frac{15}{16} C_1^2 C_2 + \frac{3}{8} C_2^2 \right) \left(1 + \frac{1}{2} Q^2 \right) \\ &\quad - \frac{3}{8} C_2^2 P^2 - \left(\frac{105}{64} C_1^2 C_2^2 + \frac{5}{16} C_2^3 \right) \left(1 + \frac{1}{2} Q^2 + \frac{3}{8} Q^4 \right) \end{aligned}$$

$$-\frac{35}{128} C_2^4 \left(1 + \frac{1}{2} Q^2 + \frac{3}{8} Q^4 + \frac{5}{16} Q^6\right) - \left(\frac{5}{16} C_1^3 + \frac{3}{4} C_1 C_2\right) P \\ + \frac{15}{16} C_1 C_2^2 \left(P + \frac{3}{2} P Q^2\right) + O(J_2^5). \quad (95)$$

Here the term $-\frac{3}{8} C_1^2$ is proportional to $J_3^2/J_2^2 = O(J_2^5)$, so that we apparently again find a second-order secular term produced by inclusion of J_3 in the separable problem.

To check (95) for $\delta=0$, put $C_1=P=0$, $C_2=\eta_2^{-2}$, and $Q=\eta_0$. It becomes

$$B_3 = \frac{1}{2} \eta_2^{-2} + \frac{3}{8} \eta_2^{-4} \left(1 + \frac{1}{2} \eta_0^2\right) + \frac{5}{16} \eta_2^{-6} \left(1 + \frac{1}{2} \eta_0^2 + \frac{3}{8} \eta_0^4\right) \\ + \frac{35}{128} \eta_2^{-8} \left(1 + \frac{1}{2} \eta_0^2 + \frac{3}{8} \eta_0^4 + \frac{5}{16} \eta_0^6\right) + O(J_2^5), \quad (95.1)$$

agreeing with (6.65) of V1961b through terms of order J_2^4 .

To evaluate the periodic terms in (88.1) through order J_2^2 , note that all coefficients F_n are of order J_2^3 or higher for $n \geq 5$. Thus we need only g_0 , g_1 , and g_2 , where g_n is the periodic part of

$$f_n \equiv \int_0^w (P - Q \cos y)^n dy.$$

Then

$$g_0 = 0, \quad g_1 = Q \sin w, \quad g_2 = -2PQ \sin w + \frac{Q^2}{4} \sin 2w. \quad (96)$$

From (88.1) the relevant periodic terms are then $-F_3 g_1$ and $-F_4 g_2$. With the aid of (96) and (57), their sum then becomes, through order J_2^2 ,

$$\text{Periodic terms} = -\frac{3}{4} C_1 C_2 Q \sin w - \frac{3}{32} C_2^2 Q^2 \sin 2w \quad (97.1)$$

$$= -\frac{3}{4} C_1 C_2 Q \cos \psi + \frac{3}{32} C_2^2 Q^2 \sin 2\psi. \quad (97.2)$$

We come next to the evaluation of L_0 and L_1 . From (81), we have by expansion in partial fractions

$$2L_0 = M'_1 + M'_2$$

$$2L_1 = M'_1 - M'_2, \quad (98.1)$$

where

$$M'_1 = \int_{-\pi/2}^{\psi} (1 - \eta)^{-1} d\psi \quad (99.1) \quad M'_2 = \int_{-\pi/2}^{\psi} (1 + \eta)^{-1} d\psi. \quad (99.2)$$

With the aid of (48) and the definitions

$$e_2 \equiv \frac{Q}{1-P} \quad (100.1) \quad e_3 \equiv \frac{Q}{1+P}, \quad (100.2)$$

we find

$$M'_1 = (1 - P)^{-1} \int_{-\pi/2}^{\psi} (1 - e_2 \sin x)^{-1} dx \quad (101.1)$$

$$M'_2 = (1 + P)^{-1} \int_{-\pi/2}^{\psi} (1 + e_3 \sin x)^{-1} dx. \quad (101.2)$$

Here $e_2 \leq 1$ and $e_3 \leq 1$, as we shall now prove. Since $|S| \leq 1$, we have by (31) that $u \approx 1$ and then by (32.3) that

$$P = \zeta(1 - S) \quad (102.1) \quad \zeta = \left(1 - \frac{c^2}{a_0 p_0} S u\right)^{-1} \frac{\delta}{p_0} u \quad (102.2)$$

where $0 < \zeta < 1$. Now by (47)

$$(1 - P)^2 - Q^2 = 1 - 2P + P^2 - P^2 - S = 1 - 2P - S = (1 - S)(1 - 2\zeta),$$

by (102). But $0 < \zeta < 1$, so that $1 - 2\zeta > 0$ and $1 - S \geq 0$, so that $(1 - P)^2 - Q^2 \geq 0$ and $1 - P \geq Q$, since neither $1 - P$ nor Q is negative. Thus $e_2 \leq 1$. Also $(1 + P)^2 - Q^2 = 1 + 2P - S \geq 0$, since $P \geq 0$ and $S \leq 1$.

Thus $1 + P \geq Q$ and $e_3 \leq 1$.

If in (101.1) we place $x = y - \pi/2$ and in (101.2) we place $x = y + \pi/2$, we now find

$$M'_1 = (1 - P)^{-1} \int_0^{\psi + \pi/2} (1 + e_2 \cos y)^{-1} dy \quad (103.1)$$

$$M'_2 = (1 + P)^{-1} \int_{-\pi}^{\psi - \pi/2} (1 + e_3 \cos y)^{-1} dy. \quad (103.2)$$

To evaluate the M' 's we define a function $E'_e(y)$ by

$$\cos E'_e = \frac{e + \cos y}{1 + e \cos y} \quad \sin E'_e = \frac{(1 - e^2)^{1/2} \sin y}{1 + e \cos y} \quad (e < 1). \quad (104)$$

Then when $y = n\pi$, we find $\sin E'_e = 0$ and $\cos E'_e = (-1)^n$. We may therefore impose the further requirement that $E'_e(y) = y$ whenever $y = n\pi$. The relation between E'_e and y is then the same as that between the eccentric and true anomalies, respectively, of an ellipse with eccentricity e .

We next prove that

$$\int_0^w (1 + e \cos y)^{-1} dy = (1 - e^2)^{-1/2} E'_e(w). \quad (105)$$

To prove it, we first derive from (104) the relations

$$\sin y = \frac{(1 - e^2)^{1/2} \sin E'}{1 - e \cos E'} \quad (106)$$

$$\frac{1 + e^2}{1 + e \cos y} = 1 - e \cos E'. \quad (107)$$

Differentiation of (107) gives

$$\frac{(1-e^2) \sin y \, dy}{(1+e \cos y)^2} = \sin E' \, dE'. \quad (108)$$

Division of (108) by (107) then shows that

$$\frac{\sin y \, dy}{1+e \cos y} = \frac{\sin E' \, dE'}{1-e \cos E'} = (1-e^2)^{-1/2} \sin y \, dE', \quad (109)$$

by (106). Thus

$$\frac{dy}{1+e \cos y} = (1-e^2)^{-1/2} \, dE', \quad (110)$$

from which theorem (105) follows at once, since $E' = 0$ when $y = 0$. Then, since $E'_e(0) = 0$ and $E'_e(-\pi) = -\pi$, we find on applying (105) to (103),

$$M'_1 = (1-P)^{-1} (1-e_2^2)^{-1/2} E'_2(\psi + \pi/2) \quad (111.1)$$

$$M'_2 = (1+P)^{-1} (1-e_3^2)^{-1/2} [E'_3(\psi - \pi/2) + \pi]. \quad (111.2)$$

From (111), (100), and $Q^2 = P^2 + S$, it then follows that

$$M_1 = (1-S-2P)^{-1/2} E'_2(\psi + \pi/2) \quad (112.1)$$

$$M_2 = (1-S+2P)^{-1/2} [E'_3(\psi - \pi/2) + \pi]. \quad (112.2)$$

By (112), (98), and (102), we then find

$$(1-S)^{1/2} L_0 = \frac{\pi/2}{\sqrt{1+2\zeta}} + \chi_0 \quad (113.1)$$

$$(1-S)^{1/2} L_1 = \frac{-\pi/2}{\sqrt{1+2\zeta}} + \chi_1, \quad (113.2)$$

where

$$2\chi_0 = \frac{E'_2(\psi + \pi/2)}{\sqrt{1-2\zeta}} + \frac{E'_3(\psi - \pi/2)}{\sqrt{1+2\zeta}} \quad (114.1)$$

$$2\chi_1 = \frac{E'_2(\psi + \pi/2)}{\sqrt{1-2\zeta}} - \frac{E'_3(\psi - \pi/2)}{\sqrt{1+2\zeta}}. \quad (114.2)$$

From (88.1), (94), (97), and (113), there follows the complete expression for N_3

$$\begin{aligned} \alpha_2 u^{-1/2} N_3 = & \frac{1}{2} [(1+C_1-C_2)^{-1/2} + (1-C_1-C_2)^{-1/2}] (1-S)^{-1/2} \left[\frac{\pi/2}{\sqrt{1+2\zeta}} + \chi_0 \right] \\ & + \frac{1}{2} [(1+C_1-C_2)^{-1/2} - (1-C_1-C_2)^{-1/2}] (1-S)^{-1/2} \left[\frac{-\pi/2}{\sqrt{1+2\zeta}} + \chi_1 \right] \\ & + (\psi + \pi/2) B_3 - \frac{3}{4} C_1 C_2 Q \cos \psi + \frac{3}{32} C_3^2 Q^2 \sin 2\psi + \text{periodic terms of } O(J_3^3). \end{aligned} \quad (115)$$

Here B_3 is given by (95) and χ_0 and χ_1 are given by (114).

To check this expression for $\delta=0$, note that then $C_1=0$, $\zeta=0$, $S=\eta_0^2$, $u=(\alpha_2^2-\alpha_3^2)^{-1}\alpha_2^2\eta_0^2$, so that $\alpha_2 u^{-1/2}$ becomes $(\alpha_2^2-\alpha_3^2)^{1/2}\eta_0^{-1}$. We have to compare with eq (6.64) of V1961b. Now we have already checked B_3 and the periodic terms in (115). The check will thus consist only in showing that the terms involving χ_0 and χ_1 in (115) reduce to the expression $(1-\eta_0^2)^{-1/2}(1-\eta_2^2)^{-1/2}\chi$ of V1961b.

Since $C_1=0$ for $\delta=0$, the term involving χ_1 disappears, because of the minus sign inside the brackets. Since C_2 reduces to η_2^{-2} , the first line becomes

$$(1-\eta_0^2)^{-1/2}(1-\eta_2^2)^{-1/2}\left(\frac{\pi}{2}+\chi_0\right).$$

Now the $\pi/2$ is an additive constant, arising only because we now have the lower limit η_1 in the η -integrals instead of zero. We shall later get rid of it on replacing the constant β_2' by another constant β_2 . Thus we now have only to verify that χ_0 reduces to χ when $\delta=0$. When $\delta=0$, $e_2=(1-P)^{-1}Q$ reduces to η_0 , and we find from (114.1) and (104), with $\zeta=0$, that

$$\cos 2\chi_0 = \frac{\cos^2 \psi - (1-\eta_0^2) \sin^2 \psi}{1-\eta_0^2 \sin^2 \psi} \quad \sin 2\chi_0 = \frac{2\sqrt{1-\eta_0^2} \sin \psi \cos \psi}{1-\eta_0^2 \sin^2 \psi}.$$

But the χ of V1961b, by p. 516 of V1961c, satisfies

$$\cos \chi = \frac{\cos \psi}{\sqrt{1-\eta_0^2 \sin^2 \psi}} \quad \sin \chi = \frac{\sqrt{1-\eta_0^2} \sin \psi}{\sqrt{1-\eta_0^2 \sin^2 \psi}}.$$

One verifies easily that $\cos 2\chi_0 = \cos 2\chi$ and $\sin 2\chi_0 = \sin 2\chi$. Thus in the limit $\delta=0$, the quantity χ_0 and thus the integral N_3 both reduce correctly.

7. Assembly of Equations for the Inversion

From (11), (67), (78), and (115) of the present paper and (5.30), (5.35), and (5.60) of V1961b, we have

$$t + \beta_1 = (-2\alpha_1)^{-1/2} [b_1 E + a(E - e \sin E) + A_1 v + A_{11} \sin v + A_{12} \sin 2v] \\ + c^2 \alpha_2^{-1} u^{1/2} \left[\left(\psi + \frac{\pi}{2} \right) B_1 + B_{11} \cos \psi + B_{12} \sin 2\psi + B_{13} \cos 3\psi + B_{14} \sin 4\psi \right] \quad (116.1)$$

$$\beta_2' \alpha_2^{-1} = -(-2\alpha_1)^{-1/2} \left[A_2 v + \sum_{k=1}^4 A_{2k} \sin kv \right] + \alpha_2^{-1} u^{1/2} \left[\left(\psi + \frac{\pi}{2} \right) B_2 + B_{21} \cos \psi \right. \\ \left. + B_{22} \sin 2\psi + B_{23} \cos 3\psi + B_{24} \sin 4\psi \right] \quad (116.2)$$

where

$$B_{11} = 2PQ - \frac{3}{8} C_1 Q^3, \quad B_{12} = -\left(\frac{Q^2}{4} + \frac{1}{8} C_2 Q^4 \right) \\ B_{13} = -C_1 Q^3 / 24, \quad B_{14} = C_2 Q^4 / 64 \quad (116.3)$$

$$\begin{aligned}
B_{21} &= C_2 P Q - \frac{9}{16} C_1 C_2 Q^3 + \frac{1}{2} C_1 Q, & B_{22} &= -\frac{1}{32} [(4C_2 + 3C_1^2)Q^2 + 3C_2^2 Q^4] \\
B_{23} &= -\frac{1}{16} C_1 C_2 Q^3, & B_{24} &= \frac{3}{256} C_2^2 Q^4
\end{aligned} \tag{116.4}$$

and where B'_2 and B_2 are given by (76) and (65).

Note that by V1962 the restrictions on p. 176 of V1961b, concerning orbital inclinations, have been removed.

Before proceeding further, it is convenient to eliminate the terms above involving $\pi/2$. If we lump the term $-c^2\alpha_2^{-1}u^{1/2}B'_1\pi/2$ into β_1 , the physical meaning of β_1 will not be essentially changed. In the limit $J_2 \rightarrow 0$, this change will vanish, since $c^2 = O(J_2)$, and $-\beta_1$ will still reduce in the elliptic limit to τ , the time of passage through perigee. Also, for $J_2 = 0$ and $J_3 = 0$, $u = 1$ and $B_2 = 1$, so that the term $-\alpha_2^{-1}u^{1/2}B_2\pi/2$ reduces to $-\frac{1}{2}\pi\alpha_2^{-1}$. Since $\beta'_2 \rightarrow \omega + \pi/2$ in the elliptic limit, if we put

$$\beta_2 = \beta'_2 - u^{1/2}B_2\pi/2, \tag{117}$$

the new constant β_2 will reduce in the elliptic limit to ω , the argument of perigee.

With these changes, we then have

$$\begin{aligned}
t + \beta_1 &= (-2\alpha_1)^{-1/2} [b_1 E + a(E - e \sin E) + A_1 v + A_{11} \sin v + A_{12} \sin 2v] \\
&\quad + c^2 \alpha_2^{-1} u^{1/2} [B'_1 \psi + B_{11} \cos \psi + B_{12} \sin 2\psi + B_{13} \cos 3\psi + B_{14} \sin 4\psi]
\end{aligned} \tag{118.1}$$

$$\begin{aligned}
\beta_2 \alpha_2^{-1} &= -(-2\alpha)^{-1/2} [A_2 v + \sum_{k=1}^4 A_{2k} \sin kv] \\
&\quad + \alpha_2^{-1} u^{1/2} [B_2 \psi + B_{21} \cos \psi + B_{22} \sin 2\psi + B_{23} \cos 3\psi + B_{24} \sin 4\psi].
\end{aligned} \tag{118.2}$$

To carry out the inversion, we write

$$E = M_s + E_p, \quad v = M_s + E_p, \quad \psi = \psi_s + \psi_p, \tag{119}$$

where the subscript s means "secular" and the subscript p "periodic."

8. The Secular Solution

To obtain the secular terms M_s and ψ_s , we put $E = v = M_s$ in (118) and discard all the sines and cosines, thereby obtaining for M_s and ψ_s the pair of linear algebraic equations:

$$(-2\alpha_1)^{-1/2} (a + b_1 + A_1) M_s + c^2 \alpha_2^{-1} u^{1/2} B'_1 \psi_s = t + \beta_1 \tag{120.1}$$

$$-(-2\alpha_1)^{1/2} A_2 M_s + \alpha_2^{-1/2} u^{1/2} B_2 \psi_s = \beta_2 \alpha_2^{-1}. \tag{120.2}$$

Their solution is

$$M_s = l_0 + 2\pi\nu_1 t \quad (121.1), \quad \psi_s = l_0 + g_0 + 2\pi\nu_2 t, \tag{121.2}$$

where

$$2\pi\nu_1 = (-2\alpha_1)^{1/2} (a + b_1 + A_1 + c^2 A_2 B'_1 B_2^{-1})^{-1} \tag{122.1}$$

$$2\pi\nu_2 = \alpha_2 u^{-1/2} A_2 B_2^{-1} (a + b_1 + A_1 + c^2 A_2 B_1' B_2^{-1})^{-1} \quad (122.2)$$

$$l_0 = 2\pi\nu_1 (\beta_1 - c^2 \beta_2 \alpha_2^{-1} B_1' B_2^{-1}) \quad (123.1)$$

$$l_0 + g_0 = 2\pi\nu_2 [\beta_1 + \beta_2 \alpha_2^{-1} (a + b_1 + A_1) A_2^{-1}]. \quad (123.2)$$

9. The Fundamental Frequencies

By V1961a and pp. 188 and 189 of V1961b, the above ν 's should satisfy

$$\nu_1 = \frac{\partial \alpha_1}{\partial j_1} \quad (124.1) \quad \nu_2 = \frac{\partial \alpha_1}{\partial j_2} \quad (124.2)$$

where j_1 and j_2 are the action variables corresponding to ρ and η , respectively. From V1961b, they should thus satisfy

$$\nu_1 = (j_{11} j_{22} - j_{12} j_{21})^{-1} j_{22}, \quad \nu_2 = - (j_{11} j_{22} - j_{12} j_{21})^{-1} j_{12} \quad (125)$$

where

$$j_{11} = 2\pi(-2\alpha_1)^{-1/2} (a + b_1 + A_1), \quad j_{12} = -2\pi\alpha_2(-2\alpha_1)^{-1/2} A_2 \quad (126)$$

$$j_{21} = 2c^2 N_1(\eta_0), \quad j_{22} = 2\alpha_2 N_2(\eta_0). \quad (127)$$

Now when $\eta = \eta_0$, we have $\psi = \pi/2$, by (45.1), (47), and (48). Then, by (71), (78), and (127)

$$j_{21} = 2\pi c^2 \alpha_2^{-1} u^{1/2} B_1', \quad j_{22} = 2\pi u^{1/2} B_2. \quad (128)$$

By means of (125) through (128), we readily verify that the fundamental frequencies $\partial \alpha_1 / \partial j_1$ and $\partial \alpha_2 / \partial j_2$ agree with the expressions given in (122).

10. Periodic Terms

In calculating the periodic terms through the second order, we first set

$$E_\rho = E_0 + E_1 + E_2, \quad v_\rho = v_0 + v_1 + v_2, \quad \psi_\rho = \psi_0 + \psi_1 + \psi_2, \quad (129)$$

where the subscript denotes the order of a term in J_2 . We shall calculate the zeroth, first, and second order terms in succession.

Before we can do so, however, we need a table of the order of the various coefficients and constant quantities. Including the quantities necessary for the right ascension φ also, we find

Quantity	Order	Quantity	Order
A_2	J_2^0	B_3	J_2
B_1'	J_2^0	A_{11}	J_2^2
B_2	J_2^0	A_{12}	J_2^2
B_{12}	J_2^0	A_{23}	J_2^2
b_1	J_2	A_{24}	J_2^2
A_1	J_2	B_{21}	J_2^2
B_{11}	J_2	B_{23}'	J_2^2
A_{21}	J_2	B_{24}	J_2^2
A_{22}	J_2		
B_{13}	J_2		
B_{14}	J_2		
B_{22}	J_2		

10.1. Zeroth Order Periodic Terms

To find E_0 , v_0 , and ψ_0 , place $E = M_s + E_0$, $v = M_s + v_0$, $\psi = \psi_s + \psi_0$ in (118), retain secular terms with full accuracy, and discard all periodic quantities of order higher than J_2^0 . However, as cautioned in V1961b, keep all terms that are retained to as high an accuracy as possible; such a procedure simplifies the resulting formulas. Then

$$t + \beta_1 = (-2\alpha_1)^{-1/2} [(a + b_1)(M_s + E_0) - ae \sin(M_s + E_0) + A_1 M_s] + c^2 \alpha_2^{-1} u^{1/2} B'_1 \psi_s \quad (129.1)$$

$$\beta_2 \alpha_2^{-1} = -(2\alpha_1)^{-1/2} A_2 (M_s + v_0) + \alpha_2^{-1} u^{1/2} B_2 (\psi_s + \psi_0). \quad (129.2)$$

Subtraction of (120.1) from (129.1) then gives

$$(a + b_1)E_0 - ae \sin(M_s + E_0) = 0. \quad (130)$$

On dividing (120) by $a + b_1$ and adding M_s to both sides of the result, we find a Kepler equation for $M_s + E_0$, viz,

$$M_s + E_0 - e' \sin(M_s + E_0) = M_s \quad (131.1)$$

$$e' \equiv \frac{ae}{a + b_1} < e < 1. \quad (131.2)$$

Solve this Kepler equation for $E = M_s + E_0$. Then find $v = M_s + v_0$ by the anomaly connections

$$\cos v = \frac{\cos E - e}{1 - e \cos E} \quad \sin v = \frac{(1 - e^2)^{1/2} \sin E}{1 - e \cos E}. \quad (132)$$

To find ψ_0 , subtract (120.2) from (129.2) and solve for ψ_0 . The result is

$$\psi_0 = \alpha_2 (-2\alpha_1)^{-1/2} u^{-1/2} A_2 B_2^{-1} v_0. \quad (133)$$

As in V1961b, this gives $\psi_0 = v_0 + O(J_2)$, but it is important to retain the coefficient with full accuracy, for later simplifications.

10.2. First Order Periodic Terms

We now place $E = M_s + E_0 + E_1$, $v = M_s + v_0 + v_1$, $\psi = \psi_s + \psi_0 + \psi_1$ in (118), discarding periodic terms of order J_2^2 , and use our knowledge of M_s , ψ_s , E_0 , v_0 , and ψ_0 to find E_1 , v_1 , and ψ_1 . From (118) we obtain

$$t + \beta_1 = (-2\alpha_1)^{-1/2} [(a + b_1)(M_s + E_0 + E_1) - ae \sin(M_s + E_0 + E_1) + A_1(M_s + v_0)] + c^2 \alpha_2^{-1} u^{1/2} [B'_1(\psi_s + \psi_0) + B_{12} \sin(2\psi_s + 2\psi_0)] \quad (134.1)$$

$$\beta_2 \alpha_2^{-1} = -(-2\alpha_1)^{-1/2} [A_2(M_s + v_0 + v_1) + A_{21} \sin(M_s + v_0) + A_{22} \sin(2M_s + 2v_0)] + \alpha_2^{-1} u^{1/2} [B_2(\psi_s + \psi_0 + \psi_1) + B_{22} \sin(2\psi_s + 2\psi_0)]. \quad (134.2)$$

On subtracting (120.1) from (134.1), transposing some terms, and dividing through by $(-2\alpha_1)^{-1/2}(a+b_1)$, we find:

$$E_0 + E_1 - e' \sin(M_s + E_0 + E_1) = -(a+b_1)^{-1}[A_1 v_0 + c^2 \alpha_2^{-1}(-2\alpha_1)^{1/2} u^{1/2} B'_1 \psi_0 + c^2 \alpha_2^{-1}(-2\alpha_1)^{1/2} u^{1/2} B_{12} \sin(2\psi_s + 2\psi_0)], \quad (135)$$

where e' is given by (131.2). By (133) the terms in v_0 and ψ_0 consolidate into $(A_1 + c^2 A_2 B'_1 B_2^{-1})v_0$ and (135) becomes

$$M_s + E_0 + E_1 - e' \sin(M_s + E_0 + E_1) = M_s + M_1, \quad (136.1)$$

where

$$M_1 = -(a+b_1)^{-1}[(A_1 + c^2 A_2 B'_1 B_2^{-1})v_0 + c^2 \alpha_2^{-1}(-2\alpha_1)^{1/2} u^{1/2} B_{12} \sin(2\psi_s + 2\psi_0)]. \quad (136.2)$$

By p. 194 of V1961b, the solution for E_1 is then given by

$$E_1 = \frac{M_1}{1 - e' \cos(M_s + E_0)} - \frac{e'}{2} \frac{M_1^2 \sin(M_s + E_0)}{[1 - e' \cos(M_s + E_0)]^3} + O(J_3^3). \quad (137)$$

To find v_1 , insert the known values of $E = M_s + E_0 + E_1$ and e into the anomaly connections (132) and solve for $v = M_s + v_0 + v_1$.

To find ψ_1 , subtract (120.2) from (134.2). The result is

$$\alpha_2^{-1} u^{1/2} [B_2(\psi_0 + \psi_1) + B_{22} \sin(2\psi_s + 2\psi_0)] = (-2\alpha_1)^{-1/2} [A_2(v_0 + v_1) + A_{21} \sin(M_s + v_0) + A_{22} \sin(2M_s + 2v_0)]. \quad (138)$$

By virtue of (133) the terms proportional to ψ_0 and v_0 cancel out. On solving for ψ_1 , we then find

$$\psi_1 = -B_{22} B_2^{-1} \sin(2\psi_s + 2\psi_0) + \alpha_2 (-2\alpha_1)^{-1/2} u^{-1/2} B_2^{-1} [A_2 v_1 + A_{21} \sin(M_s + v_0) + A_{22} \sin(2M_s + 2v_0)]. \quad (139)$$

10.3. Second Order Periodic Terms

To find E_2 , v_2 , and ψ_2 , insert $E = M_s + E_0 + E_1 + E_2$, $v = M_s + v_0 + v_1 + v_2$, and $\psi = \psi_s + \psi_0 + \psi_1 + \psi_2$ into (118), retaining all periodic terms of order J_2^2 and discarding all those of higher order. We obtain

$$\begin{aligned} t + \beta_1 = & (-2\alpha_1)^{-1/2} [(a+b_1)(M_s + E_0 + E_1 + E_2) - ae \sin(M_s + E_0 + E_1 + E_2) \\ & + A_1(M_s + v_0 + v_1) + A_{11} \sin(M_s + v_0) + A_{12} \sin(2M_s + 2v_0)] \\ & + c^2 \alpha_2^{-1} u^{1/2} [B'_1(\psi_s + \psi_0 + \psi_1) + B_{11} \cos(\psi_s + \psi_0) + B_{12} \sin(2\psi_s + 2\psi_0 + 2\psi_1) \\ & + B_{13} \cos(3\psi_s + 3\psi_0) + B_{14} \sin(4\psi_s + 4\psi_0)]. \end{aligned} \quad (140.1)$$

$$\begin{aligned}
\beta_2 \alpha_2^{-1} = & -(-2\alpha_1)^{-1/2} [A_2(M_s + v_0 + v_1 + v_2) + A_{21} \sin(M_s + v_0 + v_1) + A_{22} \sin(2M_s + 2v_0 + 2v_1) \\
& + A_{23} \sin(3M_s + 3v_0) + A_{24} \sin(4M_s + 4v_0)] + \alpha_2^{-1} u^{1/2} [B_2(\psi_s + \psi_0 + \psi_1 + \psi_2) + B_{21} \cos(\psi_s + \psi_0) \\
& + B_{22} \sin(2\psi_s + 2\psi_0 + 2\psi_1) + B_{23} \cos(3\psi_s + 3\psi_0) + B_{24} \sin(4\psi_s + 4\psi_0)]. \quad (140.2)
\end{aligned}$$

To solve for E_2 , subtract (134.1) from (140.1). We find

$$\begin{aligned}
(-2\alpha_1)^{-1/2} [(a + b_1)E_2 - ae \sin(M_s + E_0 + E_1 + E_2) + ae \sin(M_s + E_0 + E_1) + A_1 v_1 + A_{11} \sin(M_s + v_0) \\
+ A_{12} \sin(2M_s + 2v_0)] + c^2 \alpha_2^{-1} u^{1/2} [B'_1 \psi_1 + B_{11} \cos(\psi_s + \psi_0) + B_{12} \sin(2\psi_s + 2\psi_0 + 2\psi_1) \\
- B_{12} \sin(2\psi_s + 2\psi_0) + B_{13} \cos(3\psi_s + 3\psi_0) + B_{14} \sin(4\psi_s + 4\psi_0)], \quad (141)
\end{aligned}$$

Now

$$\sin(M_s + E_0 + E_1 + E_2) - \sin(M_s + E_0 + E_1) = E_2 \cos(M_s + E_0 + E_1) + O(J_2^4) \quad (142.1)$$

and

$$c^2 \sin(2\psi_s + 2\psi_0 + 2\psi_1) - c^2 \sin(2\psi_s + 2\psi_0) = 2c^2 \psi_1 \cos(2\psi_s + 2\psi_0) + O(J_3^3), \quad (142.2)$$

where E_2 and ψ_1 are in radians. On inserting (142) into (141) and solving for E_2 , we find

$$E_2 = \frac{M_2}{1 - e' \cos(M_s + E_0 + E_1)} \quad (143.1)$$

$$\begin{aligned}
M_2 \equiv & -(a + b_1)^{-1} [A_1 v_1 + A_{11} \sin(M_s + v_0) + A_{12} \sin(2M_s + 2v_0) \\
& + c^2 \alpha_2^{-1} (-2\alpha_1)^{1/2} u^{1/2} \{B'_1 \psi_1 + B_{11} \cos(\psi_s + \psi_0) + 2B_{12} \psi_1 \cos(2\psi_s + 2\psi_0) \\
& + B_{13} \cos(3\psi_s + 3\psi_0) + B_{14} \sin(4\psi_s + 4\psi_0)\}]. \quad (143.2)
\end{aligned}$$

Comparison with (8.45) of V1961b shows that a term in $\sin(2\psi_s + 2\psi_0)$ is now missing. The reason is that, by using the full accuracy for B_{12} in the first order periodic terms, we have already included it in M_1 .

To find v_2 , insert $E = M_s + E_0 + E_1 + E_2$ in the anomaly connections (132) and solve for $v = M_s + v_0 + v_1 + v_2$.

To find ψ_2 , subtract (134.2) from (140.2) to obtain

$$\begin{aligned}
-(-2\alpha_1)^{-1/2} [A_2 v_2 + A_{21} \sin(M_s + v_0 + v_1) - A_{21} \sin(M_s + v_0) \\
+ A_{22} \sin(2M_s + 2v_0 + 2v_1) - A_{22} \sin(2M_s + 2v_0) \\
+ A_{23} \sin(3M_s + 3v_0) + A_{24} \sin(4M_s + 4v_0)] \\
+ \alpha_2^{-1} u^{1/2} [B_2 \psi_2 + B_{21} \cos(\psi_s + \psi_0) + B_{22} \sin(2\psi_s + 2\psi_0 + 2\psi_1) \\
- B_{22} \sin(2\psi_s + 2\psi_0) + B_{23} \cos(3\psi_s + 3\psi_0) + B_{24} \cos(4\psi_s + 4\psi_0)] = 0. \quad (144)
\end{aligned}$$

On inserting

$$A_{21} \sin (M_s + v_0 + v_1) - A_{21} \sin (M_s + v_0) = A_{21} v_1 \cos (M_s + v_0) + O(J_2^3)$$

$$A_{22} \sin (2M_s + 2v_0 + 2v_1) - A_{22} \sin (2M_s + 2v_0) = 2A_{22} v_1 \cos (2M_s + 2v_0) + O(J_2^3)$$

$$B_{22} \sin (2\psi_s + 2\psi_0 + 2\psi_1) - B_{22} \sin (2\psi_s + 2\psi_0) = 2B_{22} \psi_1 \cos (2\psi_s + 2\psi_0) + O(J_2^3)$$

into (144) and solving for ψ_2 , we find

$$\begin{aligned} \psi_2 = & -B_2^{-1} [B_{21} \cos (\psi_s + \psi_0) + 2B_{22} \psi_1 \cos (2\psi_s + 2\psi_0) + B_{23} \cos (3\psi_s + 3\psi_0) + B_{24} \sin (4\psi_s + 4\psi_0)] \\ & + \alpha_2 u^{-1/2} (-2\alpha_1)^{-1/2} B_2^{-1} [A_2 v_2 + A_{21} v_1 \cos (M_s + v_0) + 2A_{22} v_1 \cos (2M_s + 2v_0) \\ & + A_{23} \sin (3M_s + 3v_0) + A_{24} \sin (4M_s + 4v_0)]. \end{aligned} \quad (145)$$

This completes the solution for the E , v , and ψ and thus for the spheroidal coordinates ρ and η .

11. The Right Ascension

To find φ we have to insert (115) for N_3 and (5.60) of V1961b for R_3 into (11.3). Then

$$\begin{aligned} \varphi = & \beta_3' - c^2 \alpha_3 (-2\alpha_1)^{-1/2} \left[A_3 v + \sum_{j=1}^4 A_{3j} \sin jv \right] \\ & + \alpha_3 \alpha_2^{-1} u^{1/2} \left\{ (1-S)^{-1/2} \left[(h_1 + h_2) \left\{ \frac{\pi}{2} (1+2\zeta)^{-1/2} + \chi_0 \right\} \right. \right. \\ & \left. \left. + (h_1 - h_2) \left\{ -\frac{\pi}{2} (1+2\zeta)^{-1/2} + \chi_1 \right\} \right] + \left(\frac{\pi}{2} + \psi \right) B_3 - \frac{3}{4} C_1 C_2 Q \cos \psi + \frac{3}{32} C_2^2 Q^2 \sin 2\psi \right\}. \end{aligned} \quad (146)$$

Here A_3 is given by (5.61) through (5.65) in V1961b. Because of typographical errors there, I include here the correct expressions for A_{32} and A_{33} :

$$\begin{aligned} A_{32} = & (1 - e^2)^{1/2} p^{-3} \left[\frac{e^2}{4} + \frac{3}{4} \frac{b_1}{p} e^2 - p^{-2} \left(\frac{e^4}{4} + \frac{3}{2} e^2 \right) \left(\frac{b_2^2}{2} + c^2 \right) \right] \\ A_{33} = & (1 - e^2)^{1/2} p^{-3} \left[\frac{e^3 b_1}{12p} - \frac{e^3}{3p^2} \left(\frac{b_2^2}{2} + c^2 \right) \right]. \end{aligned} \quad (146.1)$$

Here b_2^2 is simply the B in (34.1) of this paper.

The symbols h_1 and h_2 are abbreviations:

$$h_1 \equiv \frac{1}{2} (1 + C_1 - C_2)^{-1/2} \quad (147.1) \quad h_2 \equiv \frac{1}{2} (1 - C_1 - C_2)^{-1/2}. \quad (147.2)$$

By (18), (21.2), (26), and (31), we find for α_3

$$\alpha_3 = [\mu p_0 (1-S)]^{1/2} \operatorname{sgn} \alpha_3 \left[1 - \frac{c^2 S}{a_0 p_0} \frac{\left(\frac{2\delta}{p_0} \right)^2 \left(1 - \frac{c^2 S}{a_0 p_0} \right) S}{\left[1 + \frac{c^2}{a_0 p_0} (1-2S) \right]^2} \right]^{1/2}. \quad (148)$$

The error here is of order J_2^4 . With the same error, we can simplify this to

$$\alpha_3 = [\mu p_0 (1-S)]^{1/2} \operatorname{sgn} \alpha_3 \left[1 - S \left\{ \frac{c^2}{a_0 p_0} + \left(\frac{2\delta}{p_0} \right)^2 \left(1 - \frac{2c^2}{ap} + \frac{3c^2 S}{ap} \right) \right\} \right]^{1/2} \quad (149)$$

Alternatively, when α_2 is known, we may write

$$\alpha_3 = \alpha_2 \left(1 - \frac{S}{u}\right)^{1/2} \operatorname{sgn} \alpha_3. \quad (149.1)$$

In (146) the constant β'_3 would reduce to $\Omega - \frac{1}{2}\pi \operatorname{sgn} \alpha_3$ in the Keplerian case, as one may see by solving the two body problem with lower limit $\eta_1 (= -\eta_0)$ in the η -integrals. Also in (146), consider the constant term:

$$\frac{1}{2} \pi \alpha_3 \alpha_2^{-1} u^{1/2} (1-S)^{-1/2} (1+2\zeta)^{-1/2} [h_1 + h_2 - (h_1 - h_2)] + \frac{1}{2} \pi B_3.$$

For $J_2=0$ and $J_3=0$, we have $u=1$, $\zeta=0$, $h_1+h_2=1$, $h_1-h_2=0$, $B_3=0$, $S=\sin^2 I$, and $\alpha_3\alpha_2^{-1}=\cos I$. Thus it then becomes $\frac{1}{2} \pi \cos I |\sec I| = \frac{1}{2} \pi \operatorname{sgn} \alpha_3$. If we lump it into the constant β'_3 and call the sum β_3 , we then find that β_3 reduces, in the Keplerian case, to Ω , the right ascension of the node.

Our amended equation thus takes the form

$$\begin{aligned} \varphi = \beta_3 - c^2 \alpha_3 (-2\alpha_1)^{-1/2} \left[A_3 v + \sum_{j=1}^4 A_{3j} \sin jv \right] + \alpha_3 \alpha_2^{-1} u^{1/2} \left\{ (1-S)^{-1/2} [(h_1+h_2)\chi_0 \right. \\ \left. + (h_1-h_2)\chi_1] + B_3 \psi - \frac{3}{4} C_1 C_2 Q \cos \psi + \frac{3}{32} C_2^2 Q^2 \sin 2\psi \right\}, \quad (150) \end{aligned}$$

where χ_0 and χ_1 are given by (114) and E'_2 and E'_3 in (114) by (104), with $e_2 = Q(1-P)^{-1}$ and $e_3 = Q(1+P)^{-1}$.

It is not difficult to verify that (150) reduces to (8.50) of V1961b in the limit $\delta=0$. A much more delicate check is to show that (150) makes φ a constant in a polar orbit, with jumps of π as the satellite goes over a pole.

As an orbit approaches being a polar orbit, $\alpha_3 \rightarrow 0$ and $S \rightarrow 1$, so that

$$\varphi \rightarrow \beta_3 + \alpha_3 \alpha_2^{-1} u^{1/2} [(h_1+h_2)\chi_0 + (h_1-h_2)\chi_1] (1-S)^{-1/2}. \quad (151)$$

Now by (148), since $(\mu p_0)^{1/2} = \alpha_2$, we have for $S \rightarrow 1$

$$\alpha_3 \alpha_2^{-1} u^{1/2} (1-S)^{-1/2} = \operatorname{sgn} \alpha_3 \left[1 - \frac{c^2}{a_0 p_0} - \frac{(2\delta/p_0)^2}{1 - \frac{c^2}{a_0 p_0}} \right]^{1/2}, \quad (152)$$

since by (31) $u \rightarrow 1$ as $S \rightarrow 1$. Also, for $S=1$, we have by (32) that (152) becomes

$$\alpha_3 \alpha_2^{-1} u^{1/2} (1-S)^{-1/2} = \operatorname{sgn} \alpha_3 [(1-C_2)^2 - C_1^2]^{1/2} (1-C_2)^{-1/2}. \quad (153)$$

To simplify the term in (151) involving the χ 's, note that by (102.2), (32.1), and (32.2)

$$2\zeta = (1-C_2)^{-1} C_1. \quad (154)$$

Then, by (147) and (154)

$$(1-2\zeta)^{-1/2} = 2h_2(1-C_2)^{1/2} \quad (155.1), \quad (1+2\zeta)^{1/2} = 2h_1(1-C_2)^{1/2}. \quad (155.2)$$

In (114) let us now abbreviate $E'_2(\psi + \pi/2)$ to E'_2 and $E'_3(\psi - \pi/2)$ to E'_3 . Equations (114) and (155) then yield

$$\chi_0 = (1 - C_2)^{1/2}(h_2 E'_2 + h_1 E'_3) \quad (156.1)$$

$$\chi_1 = (1 - C_2)^{1/2}(h_2 E'_2 - h_1 E'_3), \quad (156.2)$$

from which it follows that

$$(h_1 + h_2)\chi_0 + (h_1 - h_2)\chi_1 = 2(1 - C_2)^{1/2}h_1 h_2(E'_2 + E'_3) \quad (157)$$

$$= \frac{1}{2}(1 - C_2)^{1/2}[(1 - C_2)^2 - C_1^2]^{-1/2}(E'_2 + E'_3), \quad (158)$$

after insertion into (157) of the expressions (147) for h_1 and h_2 . On inserting (153) and (158) into (151), we then find that for a polar orbit

$$\varphi \rightarrow \beta_3 + \frac{1}{2}(E'_2 + E'_3) \operatorname{sgn} \alpha_3. \quad (159)$$

To discuss this expression, note that $e_2 = Q(1 - P)^{-1}$ and $e_3 = Q(1 + P)^{-1}$ are both unity for a polar orbit, since by (32.3) $P = 0$ when $S = 1$ and since by (47) Q then equals $S = 1$.

To see how $E'_2 + E'_3$ behaves as $S \rightarrow 1$, we have to discuss the function $E'(f)$, where the argument f plays the role of true anomaly and the function E' the role of eccentric anomaly.

Now by their definitions E' and f agree in value at all multiples of π . Also

$$\frac{dE'}{df} = \frac{\sqrt{1 - e^2}}{1 + e \cos f}. \quad (160)$$

Thus

$$\frac{dE'}{df} = \left(\frac{1 - e}{1 + e}\right)^{1/2} \rightarrow 0 \text{ as } e \rightarrow 1, \quad f = 2\tau\pi \quad (161.1)$$

$$= \left(\frac{1 + e}{1 - e}\right)^{1/2} \rightarrow \infty \text{ as } e \rightarrow 1, \quad f = (2\tau + 1)\pi \quad (161.2)$$

The plot of E' versus f for $e = 1$ is thus as follows

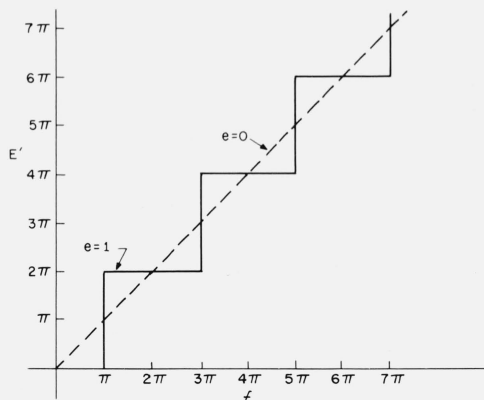


FIGURE 1. Plot of eccentric anomaly versus true anomaly for $e = 1$.

For $e=1$, one sees that $E'(f)$ remains constant from $f=(2\tau+1)\pi$ to $(2\tau+3)\pi$, but then jumps by 2π .

Let us assume $\text{sgn } \alpha_3=1$ and consider what happens as ψ increases from $-\frac{\pi}{2} + \epsilon'$ to $\frac{\pi}{2} - \epsilon'$. In this range $\psi + \pi/2$ increases from ϵ' to $\pi - \epsilon'$ and $\psi - \pi/2$ from $-\pi + \epsilon'$ to $-\epsilon'$. In this range, therefore, both $E'(\psi + \pi/2)$ and $E'(\psi - \pi/2)$ remain constant, so that φ remains constant, by (159). As ψ increases beyond $\pi/2$, however, $\psi + \pi/2$ passes through an odd multiple of π and $\psi - \pi/2$ through an even multiple of π . Thus $E'(\psi + \pi/2)$ jumps by 2π and $E'(\psi - \pi/2)$ remains constant, so that, by (159), φ jumps by π . This is at the north pole. A similar jump takes place at the south pole. These results are of course trivial for a polar orbit, but they constitute an important test that the equation (150) for the right ascension had to pass.

12. The Final Algorithm

To summarize results, assume that μ , r_e , J_2 , J_3 , and the orbital elements a , e , S , l_0 , g_0 , and β_3 are given. (Here S is an element that would reduce to $\sin^2 I = \eta_0^2$ of V1961b, but with J_3 incorporated into the reference orbit it may go negative for orbits close to equatorial.)

To calculate the reference orbit, compute

$$c^2 = r_e^2 J_2 \left(1 - \frac{1}{4} J_3^2 J_2^{-3} \right), \quad \delta = -\frac{1}{2} r_e J_2^{-1} J_3 > 0 \text{ for } J_3 < 0 \quad p = a(1 - e^2),$$

A by (40.1), and B by (40.2). Then

$a_0 = a - \frac{1}{2} A$, $p_0 = a_0^{-1}(B + ap - 2Aa - c^2)$, $\alpha_2 = (\mu p_0)^{1/2}$, u by (31), C_2 by (32.2), C_1 by (31.1), P by (32.3), $b_1 = -\frac{1}{2} A$, $b_2 = B^{1/2}$, $\alpha_1 = -\frac{1}{2} \mu a_0^{-1}$. Then compute

$A_1, A_2, A_3, A_{11}, A_{12}, A_{21}, A_{22}, A_{23}, A_{24}, A_{31}, A_{32}, A_{33}, A_{34}$ by p. 198 of V1961b.

Now compute $Q = (P^2 + S)^{1/2}$, B_2 by (65), B'_1 by (76), B_3 by (95), ζ by (102.2),

$$e_2 = Q(1 - P)^{-1} \quad e_3 = Q(1 + P)^{-1}$$

$2\pi\nu_1$ by (122.1), $2\pi\nu_2$ by (122.2), and l_0 and g_0 by (123). Then

$$M_s = l_0 + 2\pi\nu_1 t, \quad \psi_s = l_0 + g_0 + 2\pi\nu_2 t$$

$$e' = \frac{ae}{a + b_1} < e < 1.$$

Find $M_s + E_0$ by solving

$$M_s + E_0 - e' \sin(M_s + E_0) = M_s.$$

Solve (132) for $v = M_s + v_0$ and (133) for ψ_0 . Then find M_1 by (136.2) and E_1 by (137) and solve (132) for $v = M_s + v_0 + v_1$. Find ψ_1 by (139) and then M_2 by (143.2) and E_2 by (143.1).

Solve (132) for $v = M_s + v_0 + v_1 + v_2$ and find ψ_2 by (145). Then

$$E = M_s + E_0 + E_1 + E_2, \quad v = M_s + v_0 + v_1 + v_2, \quad \psi = \psi_s + \psi_0 + \psi_1 + \psi_2.$$

The spheroidal coordinates ρ and η are then given by

$$\rho = a(1 - e \cos E) = (1 + e \cos v)^{-1} p$$

$$\eta = P + Q \sin \psi.$$

To find the right ascension φ , note first the correct formulas for A_{32} and A_{33} in (146.1) of this paper or on p. 198 of V1961b. Then compute h_1 and h_2 by (147) and α_3 by (148). Define $E'_2(y)$ by

$$\cos E'_2(y) = \frac{e_2 + \cos y}{1 + e_2 \cos y} \quad \sin E'_2(y) = \frac{(1 - e_2^2)^{1/2} \sin y}{1 + e_2 \cos y}$$

and the requirement that E'_2 and y shall be equal whenever y is a multiple of π . Define $E'_3(y)$ in the same way, with e_3 replacing e_2 .

Then calculate χ_0 and χ_1 by (114) and φ by (150). Having ρ , η , and φ , we may now find the rectangular and spheroidal coordinates by means of (1.1) and (1.2).

13. Further Remarks

The present inclusion of J_3 into the separable orbit has apparently led to new secular terms of order J_2^2 . These terms, $-\frac{1}{2} C_1 P + \frac{3}{16} C_1^2 Q^2$ in B_2 and $-\frac{3}{8} C_1^2$ in B_3 , are proportional to $J_3^2 J_2^{-2} = O(J_2^2)$ and lead to terms in M_s , ψ_s , and $\dot{\varphi}$. Since no previous treatment of third harmonic perturbations has led to such secular terms, the result requires checking.

To do so, note that by (2) and (3) the present c^2 is given by

$$c^2 = c_0^2 - \delta^2$$

where $c_0^2 = r_0^2 J_2$. It turns out that the δ^2 terms implicit in the new c^2 are just sufficient to cancel the new terms in δ^2 .

Since $c^4 = c_0^4 - 2c_0^2 \delta^2 + \delta^4$, terms of order c^4 cannot cancel terms in δ^2 . In expanding the fundamental frequencies it is thus sufficient to carry terms only through c^2 and δ^2 , in checking this point. A rather long calculation then shows that

$$\begin{aligned} \dot{M}_s &= 2\pi\nu_1 = n_0 + O(c^4) \\ \dot{\psi}_s &= 2\pi\nu_2 = n_0 \left[1 + \frac{3}{4} \frac{c_0^2}{p^2} (4 - 5S) \right] + O(c^4) \\ \dot{\varphi} &= 2\pi\nu_3 = n_0 \operatorname{sgn} \alpha_3 \left[1 + \frac{3}{4} \frac{c_0^2}{p^2} (4 - 5S) - \frac{3}{2} \frac{c_0^2}{p^2} (1 - S)^{1/2} \right] + O(c^4). \end{aligned}$$

Here $n_0 = \mu^{1/2} a_0^{-3/2}$, the mean motion corresponding to $a_0 = -\mu(2\alpha_1)^{-1}$, where α_1 is the total energy. This means that, if we solve the problem for given initial conditions without using the third harmonic and then re-solve it with inclusion of the third harmonic, there will arise no secular terms of order $J_3^2 J_2^{-2}$, provided that we use the same initial total energy. In turn, this means that if we use the same initial coordinates, we must change the initial velocities slightly or that if we use the same initial velocities we must change the initial coordinates slightly.

It does not appear worth while to do the extra calculation that would give us final formulas manifestly free of secular terms of order $J_3^2 J_2^{-2}$. If we use the present results, we shall obtain secular terms correct through order J_3^2 ; this limitation comes from the error of order J_2^4 in (31.2). To obtain this secular accuracy, we find from (122) and (150) that we should keep terms through order

J_3^2 in B_2 and B_3 and through order J_2^2 in B'_1 . By improving the solution of the cubic (27), we could calculate secular terms to any order of accuracy. Periodic terms are correct through order J_2^2 ; they also may be carried to a higher order if the need ever arises.

To obtain the orbital elements one may proceed as follows. From initial values determine α_1 , α_2 , and α_3 by the appropriate modifications of eqs (3.8), (3.9), and (3.10) of V1961. They are

$$\alpha_1 = \frac{1}{2} u_i^2 - \mu(\rho_i + \eta_i \delta) (\rho_i^2 + c^2 \eta_i^2)^{-1}$$

$$\alpha_3 = r_i^2 \cos^2 \theta_i \dot{\varphi}_i = x_i \dot{y}_i - y_i \dot{x}_i$$

$$\alpha_2^2 = (1 - \eta_i^2)^{-1} [(\rho_i^2 + c^2 \eta_i^2)^2 \dot{\eta}_i^2 + \alpha_3^2 - (1 - \eta_i^2) (2\alpha_1 c^2 \eta_i^2 + 2\mu \eta_i \delta)].$$

Here the equation for α_2 follows from (10.2), (13.2), (50), and (59.1) of V1959. The subscript i means initial value and u_i is the initial speed.

From the α 's one can determine a_0 , e_0 , p_0 , and i_0 from eqs (21) and (28) of the present paper and then determine a , e , and $\eta_0 = \sin I$ by means of the equations on pp. (172) and (173) of V1961b. Knowing a , e , and I to this approximation, one can then determine the values of β_1 , β_2 , and β_3 or of l_0 , g_0 , and β_3 by fitting the values of ρ , η , and φ at various times, still using the equations in V1961b.

To improve these elements, now calculate the orbit, with elements a , e , $S = \sin^2 I$, l_0 , g_0 , and β_3 by means of the present paper. Observe the residuals and use them, with the aid of least squares, to obtain a differential correction to improve the orbital elements. Here a word of caution is necessary. If the orbit is close to equatorial, S may go negative. This is all right, since it is only $Q^2 = P^2 + S$ that cannot become negative, with the present method.

Then repeat the process, until the elements converge to fixed values. One or two iterations should suffice, according to Bonavito (1964).

The present method of accounting for J_3 ought to be an improvement over the perturbation theory of V1963, for two reasons. There appear no small denominators, such as those containing e or $\sin I$ in the perturbation theory. All periodic terms are correct through order J_2^2 .

14. References

- Bonavito, N. L. (1964), NASA Report D2312.
 Vinti, J. P. (1959), J. Res. NBS **63B** (Math. and Math. Phys.), No. 2, 105-116.
 Vinti, J. P. (1961a), J. Res. NBS **65B** (Math. and Math. Phys.), No. 2, 131-135.
 Vinti, J. P. (1961b), J. Res. NBS **65B** (Math. and Math. Phys.), No. 3, 169-201.
 Vinti, J. P. (1961c), Astron. J. **66**, 514-516.
 Vinti, J. P. (1962), J. Res. NBS **66B** (Math. and Math. Phys.), No. 1, 5-13.
 Vinti, J. P. (1963), J. Res. NBS **67B** (Math. and Math. Phys.), No. 4, 191-222.
 Vinti, J. P. (1966), J. Res. NBS **70B** (Math. and Math. Phys.), No. 1, 1-16.

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